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Maximal k-Degenerate Graphs with Diameter 2

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Abstract: A graph is k-degenerate if its vertices can be successively deleted so that when deleted, they have degree at most k. A k-tree is a graph that can be formed by starting with K_{k+1} and iterating the operation of making a new vertex adjacent to all the vertices of a k-clique of the existing graph. A structural characterization of maximal 2-degenerate graphs with diameter 2, containing 45 distinct infinite classes of graphs, is proven. A forbidden subgraph characterization of k-trees with diameter 2 is proven.

Key Words: Degeneracy, diameter, k-tree, k-path.

AMS(2010): 05C25.

§1. Introduction

In this paper, we work toward a characterization of the maximal k-degenerate graphs with diameter 2.

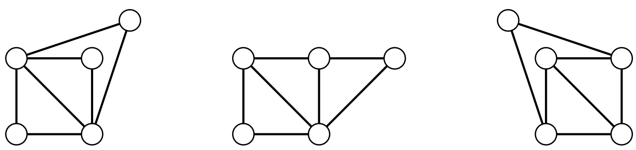
Definition 1.1 *Let k be a positive integer. A graph is k-degenerate if its vertices can be successively deleted so that when deleted, they have degree at most k. A graph is maximal k-degenerate if no edges can be added without violating this condition.*

A k-tree is a graph that can be formed by starting with K_{k+1} and iterating the operation of making a new vertex adjacent to all the vertices of a k-clique of the existing graph.

A k-leaf is a degree k vertex of a maximal k-degenerate graph.

Lick and White introduced k-degenerate graphs in 1970 [13], and their properties have been studied by many authors [2, 7, 8, 9, 10, 11, 12, 14, 16, 19]. For $n \geq k+1$, a maximal k-degenerate graph has at least one k-leaf, and a k-tree has at least 2.

The three maximal 2-degenerate graphs of order 5 are shown below [3]. The two on the left are 2-trees.



$K_2 + \overline{K}_3$ $\qquad$ $P_4 + K_1$ $\qquad$ W_5^-

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Undefined notation and terminology will generally follow [3]. In particular, the join of graphs G and H is denoted $G+H$, and the distance between vertices u and v is $d(u,v)$. The eccentricity $e_G(v)$ of a vertex v is the maximum distance between v and any other vertex of G. If G is a graph, the square G^2 is formed by adding all edges between pairs of vertices with distance 2 in G.

We solve two special cases of the problem of characterizing the maximal k-degenerate graphs with diameter 2. One restricts the problem to maximal 2-degenerate graphs, the other restricts it to k-trees (which are all maximal k-degenerate). The first provides a structural characterization, and the latter provides a forbidden subgraph characterization.

This work is inspired by a previous paper [6]. I coauthored with Zhongyuan Che on the Wiener index of maximal k-degenerate graphs. We showed that the Wiener index is minimized when these graphs have diameter 2. We also characterized 2-trees with diameter at most 2.

Proposition 1.2([6]) *The following are equivalent for a 2-tree G:*

(1) *G has diameter at most 2;*

(2) *G does not contain P_6^2;*

(3) *G is $T+K_1$ for any tree T, or any graph formed by adding any number of vertices adjacent to pairs of vertices of K_3.*

§2. Maximal 2-Degenerate Graphs with Diameter 2

In this section, we provide a structural characterization of maximal 2-degenerate graphs with diameter 2.

Definition 2.1 *A dominating vertex of a graph is a vertex adjacent to all other vertices. A fan is the graph $P_{n-1}+K_1$.*

Lemma 2.2 *If G is a maximal 2-degenerate graph with order $n \geq 3$ containing a dominating vertex, then G is a 2-tree that can be represented as $T+K_1$ for some tree T. If G has exactly two 2-leaves, then it is a fan.*

Proof We use induction on n. When $n=3$, $G=K_3$ and the result holds. Let G be a maximal 2-degenerate graph with order n containing dominating vertex u, and assume the result holds for all graphs with order $n-1$. Then G has a 2-leaf v, which is adjacent to u. Now $G-v$ is maximal 2-degenerate with order $n-1$ [13], so it is a 2-tree that can be represented as $T+K_1$. Then the other neighbor of v is a neighbor of u, so G is a 2-tree that can be represented as $T+K_1$.

If G has exactly two 2-leaves, then deleting its dominating vertex produces a tree with exactly two leaves, a path. Thus G is a fan. □

Definition 2.3 *When constructing a maximal 2-degenerate graph, we duplicate a 2-leaf by adding another 2-leaf with the same neighborhood. The inside graph H of a maximal 2-degenerate graph G is formed by deleting all the 2-leaves. The stem set of G is the set of* *neighbors of 2-leaves.*

Note that in a maximal 2-degenerate graph with diameter 2, any 2-leaf can be duplicated arbitrarily many times. The new 2-leaf is distance two from its duplicate, and hence at most two from every other vertex. Thus the result is a maximal 2-degenerate graph with diameter 2.

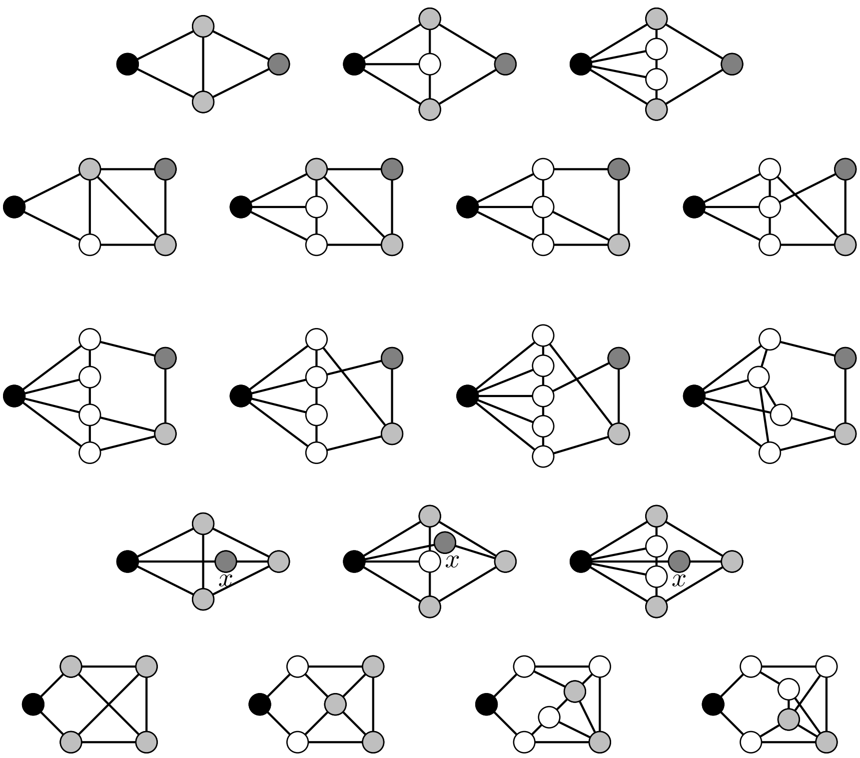
Lemma 2.4 *In any maximal 2-degenerate graph with diameter 2 and order $n > 3$, either*

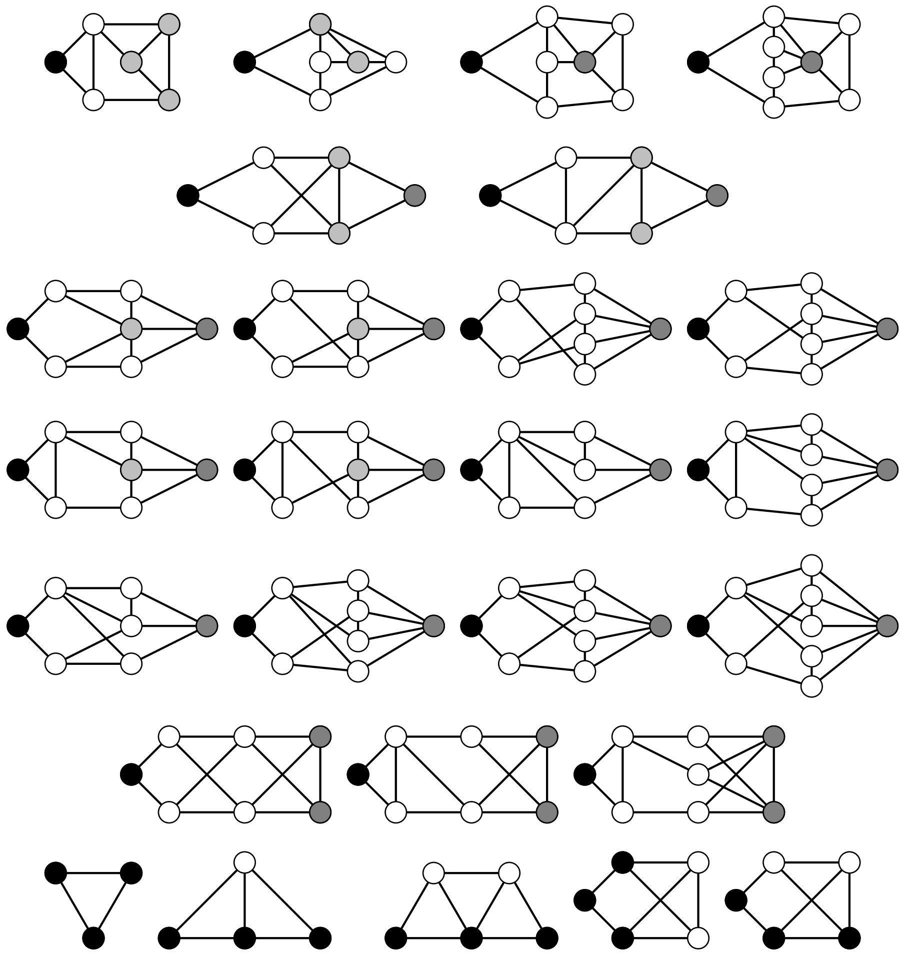
(A) all 2-leaves have a single common neighbor, or

B) the stem set is $S = \{u, v, w\}$, and there are 2-leaves with neighborhoods $\{u, v\}$, $\{u, w\}$, and $\{v, w\}$.

Proof Any maximal 2-degenerate graph with diameter 2 has at least one 2-leaf. No 2-leaves can have disjoint neighborhoods, since then they would be at least distance 3 apart. If all 2-leaves have the same neighborhood, the result follows. If two 2-leaves have distinct neighborhoods, we may call them $\{a, b\}$ and $\{a, c\}$. Any other 2-leaf must have neighborhood $\{b, c\}$ or $\{a, x\}$ for some x. □

Theorem 2.5 *Let G be a maximal 2-degenerate graph with diameter 2. Then G is a 2-tree that can be represented as $T + K_1$ for some tree T, or the inside graph of G is one of the 44 possibilities shown below. (Vertices labeled x may be duplicated arbitrarily many times.) There must be at least one 2-leaf of G neighboring any pair of black vertices or pair of black and gray vertices, and there may be at least one 2-leaf of G neighboring any pair of black and lightgray vertices.*





The proof of this theorem has many cases. We use Case A.2.1 to mean case A, Subcase 2, Subsubcase 1, and similarly for the other cases. Figures are referenced in parentheses, with labels beginning with their main case (A or B). We say an inside graph is valid if it is the inside graph of a maximal 2-degenerate graph with diameter 2.

Proof Let G be a maximal 2-degenerate graph with diameter 2 with inside graph H. By Lemma 2.4, there are two possibilities for the positions of the 2-leaves.

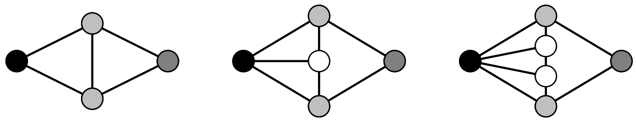
Case A. All 2-leaves of G have a single common neighbor u.

Case A.1 If u is a dominating vertex of H, it does the same for G, so by Lemma 2.2, G is a 2-tree that can be represented as $T + K_1$ for some tree T.

Case A.2 If u has eccentricity 2 in H, let v_1, ... v_j be distance 1 from u, $w_1, \cdots, w_k$ be distance 2 from u. Now no 2-leaf of H has neighborhood $\{u, v_i\}$ since a 2-leaf of G that neighbors it and u is more than 2 from w_1.

Case A.2.1 If w_1 is a 2-leaf of H, there is a 2-leaf of G that neighbors it and u. Then w_1 neighbors all other w_i, and since w_1 neighbors some v_i, $k \leq 2$. If $k = 1$, then u is a dominating vertex of $H - w_1$. By Lemma 2.2, $H - w_1$ is a 2-tree. Now its 2-leaves are not 2-leaves of H, aside from possibly u. Then w_1 is adjacent to all (two) of them, and $H - w_1$ is a fan with at most five vertices (A1, A2, A3).

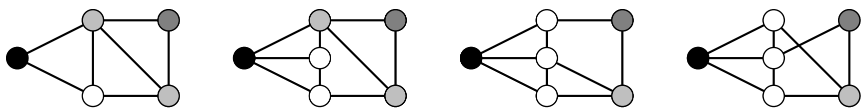
Since all 2-leaves of G have a single common neighbor u, it is colored black (uniquely, in Case A). Any 2-leaf of H must be black or gray, and any vertex distance 3 from u will be gray. If $\{u, u'\}$ is a dominating set of H, then u' will be lightgray if not already colored. Since these statuses are trivial to check, verification will be left to the reader for the other figures.



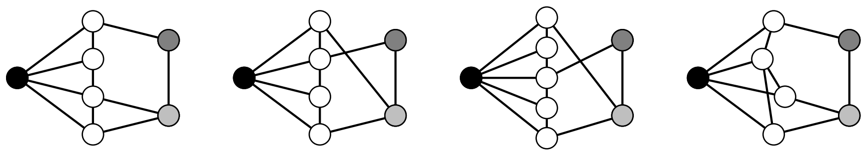
*A*1 *A*2 *A*3

If $k = 2$, there is no 2-leaf of H with neighborhood $\{u, w_2\}$, since a 2-leaf of G neighboring it is not within 2 of w_1. Then w_2 is a 2-leaf of $H - w_1$. As before, $H - w_1 - w_2$ is a 2-tree, and w_1 and w_2 have two or three neighbors in it, including all its 2-leaves. Now $T = H - w_1 - w_2 - u$ is a tree with all vertices either neighbors of w_2 or within 2 of w_1.

If T a path, its length is at most 5. If $T = P_2$, there is one possibility (A4). If $T = P_3$, w_1 may neighbor a leaf and w_2 may or may not neighbor the nonleaf, or w_1 may neighbor the nonleaf (A5, A6, A7). If $T = P_4$, w_1 may neighbor a leaf or nonleaf (A8, A9). If $T = P_5$, w_1 must neighbor the middle vertex, and w_2 neighbors the leaves (A10). If T has three leaves, w_2 neighbors two, and w_1 neighbors the third, so $T = K_{1,3}$ (A11).

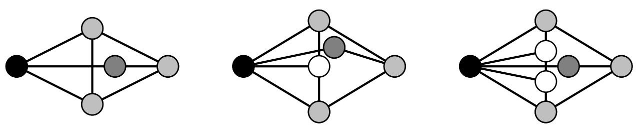


*A*4 *A*5 *A*6 *A*7



*A*8 *A*9 *A*10 *A*11

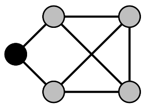
Case A.2.2 Suppose there is a 2-leaf v_1 of H neighboring u and w_1. Then there is a 2-leaf of G neighboring u and v_1. Then there is no w_2, but v_1 may be duplicated arbitrarily many times. Let K be the inside graph of H (delete v_1 and all its duplicates). Then w_1 is a 2-leaf of K. Then u is a dominating vertex of $K - w_1$, so by Lemma 2.2, $K - w_1$ is a fan. This fan must have order 3, 4, or 5 (A12, A13, A14).



*A*12 *A*13 *A*14

Case A.2.3 If u is a 2-leaf of H and no w_i is, $j = 2$. If both v_1 and v_2 are 2-leaves of

$H-u$, then $H-u-v_1-v_2$, has order at most 4, so it is K_2 (A15), K_3 (A16), or K_4-e. In the latter case, there are two ways to attach v_1 and v_2 to K_4-e (A17, A18).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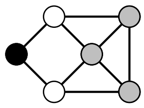
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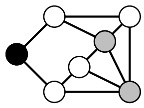
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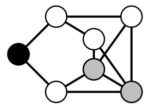
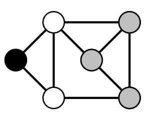

*A*18

Assume v_1 is a 2-leaf of $H-u$ and v_2 is not. If $v_1 \leftrightarrow v_2$, say $w_1 \leftrightarrow v_1$. Then v_2 is adjacent to all other w's. If $v_2 \leftrightarrow w_1$, v_2 is adjacent to all vertices, so by Lemma 2.2, H is a 2-tree, and some w_i is a 2-leaf, contrary to assumption. If $v_2 \nleftrightarrow w_1$, then w_1 is a 2-leaf of $H-u-v_1$. By Lemma 2.2, $H-u-v_1-w_1$ is a fan. Now some 2-leaf of G has neighborhood $\{u, w_i\}$, so all ws must be adjacent, and $k=3$ (A19).

Assume $v_1 \nleftrightarrow v_2$. Since v_1 is a 2-leaf of $H-u$, its neighbors are (say) w_1 and w_2. Now v_2 is adjacent to all other w's, and $k>2$. Now some 2-leaf of G has either v_2 or w_i as a neighbor, so one of these vertices neighbors all w's (excluding itself). Then $H-u-v_1$ has a dominating vertex, so by Lemma 2.2, it is a fan with 2-leaves w_1 and w_2. If v_2 is the dominating vertex, the fan has order at most 5, due to v_1. Order 5 duplicates A14, but order 4 yields a new case (A20). If (say) w_3 is the dominating vertex, the fan has order 5 or 6 (A21, A22).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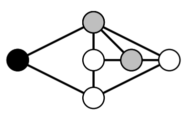
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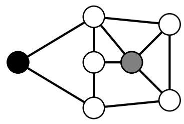
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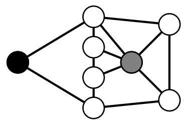
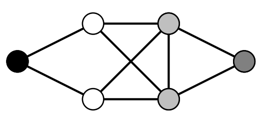

*A*22

Case A.3 If $e_H(u) > 2$ and vertex y is at least 3 from u, then $\{u, y\}$ is the neighborhood of a 2-leaf a of G. If $d_H(u,y) \geq 4$, there is a vertex z with $d_H(u,z) = 2$ and $d_H(a,z) > 2$, so this is impossible. Thus $e_H(u) = 3$. Let v_1, ... v_j be distance 1 from u, w_1, ... w_k be distance 2 from u, and x_1, ... x_l be distance 3 from u. Note $j, k \geq 2$ since H has no cut-vertex [13].

Now all vertices in the stem set other than u must be adjacent to each w_i and x_i (else a 2-leaf has eccentricity more than 2). No v_i is in the stem set, since it cannot be adjacent to an x_i. Since K_4 is not 2-degenerate, there are at most 3 stems excluding u, and $l \leq 2$. No w_i is a 2-leaf of H, since if there were, it would be adjacent to a v_i, and all w_i and x_i. Now x_1 is a 2-leaf only if there is no x_2, so H has at most two 2-leaves.

Case A.3.1 Assume u and x_1 are 2-leaves of H. Then $j=k=2$, and there is no x_2. Thus H has order 6, and $H-u-x_1 = K_4-e$. There are three ways it can be arranged, but the case where $w_1 \nleftrightarrow w_2$ combines into the case where $v_1 \nleftrightarrow v_2$. In the third case, $H = P_6^2$ (A23, A24).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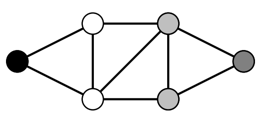
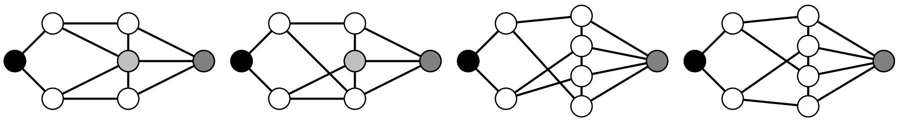
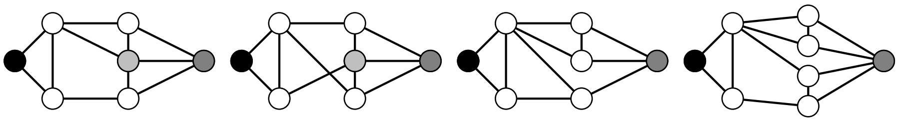

*A*24

Case A.3.2 Assume u is the only 2-leaf of H, and $l = 1$ (there is no x_2). Then at least one of v_1 and v_2 are 2-leaves of $H - u$. If both are 2-leaves, then $3 \leq k \leq 4$ since each w_i is adjacent to some v_i. If $k = 3$, then $H - u - v_1 - v_2 = K_4 - e$ by Lemma 2.2. Then v_1 and v_2 have one common neighbor, and there are two choices (A25, A26). If $k = 4$, then $H - u - v_1 - v_2$ is $P_4 + K_1$ or $K_{1,3} + K_1$ by Lemma 2.2. If it is $P_4 + K_1$, there are three choices for the adjacencies between the v's and w's, two of which produce valid inside graphs (A27, A28). If it is $K_{1,3} + K_1$, some v and w have distance more than 2.



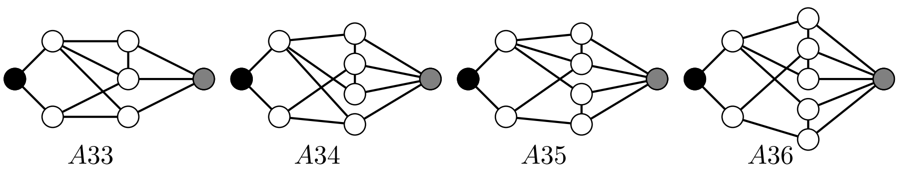
A25 *A26* *A27* *A28*

Assume only v_1 is a 2-leaf of $H - u$. If its neighbors are v_2 and (say) w_1, at least one of which are 2-leaves of $H - u - v_1$. If v_2 is a 2-leaf of $H - u - v_1$, $k = 3$, and its neighbors are either adjacent or not (A29, A30). If v_2 is a not 2-leaf of $H - u - v_1$, w_1 is, with neighbors x_1 and v_2 or (say) w_2. If $w_1 \leftrightarrow v_2$, x_1 and v_2 are adjacent to all remaining w's. Thus w_2 is the only 2-leaf of this graph, which is W_5^- (A31). If $w_1 \leftrightarrow w_2$, x_1 and v_2 are adjacent to all w's of $H - u - v_1 - w_1$. Thus w_2 is the only 2-leaf of this graph, which is W_5^- (A32).



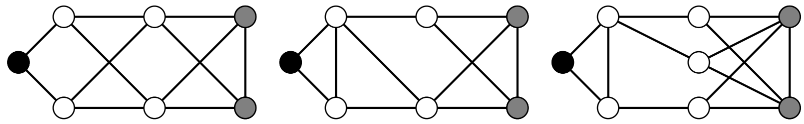
A29 *A30* *A31* *A32*

Suppose v_1 is the only 2-leaf of $H - u$ with neighbors (say) w_1 and w_2, and w_1 is a 2-leaf of $H - u - v_1$. If w_1 has neighbors x_1 and v_2, then $H - u - v_1 - w_1$ has order at least 4. Now w_2 is adjacent to all other w's (so v_1 is distance 2 from them) and v_2 is adjacent to all w's, except perhaps w_2. Since x_1 is adjacent to all w's, $H - u - v_1 - w_1$ contains $K_{3,k-2}$, so $k \leq 4$. There are two possibilities (A33, A34). If w_1 has neighbors w_3 and x_1, then w_3 neighbors v_2 and x_1. As before, $H - u - v_1 - w_1 - w_3$ contains $K_{3,k-3}$, so $k \leq 5$. There are two possibilities (A35, A36).



A33 *A34* *A35* *A36*

Case A.3.3 Assume u is the only 2-leaf of H, and $l = 2$. Then $2 \leq k \leq 4$. Now one or both of v_1 and v_2 are 2-leaves of $H - u$. If $k = 2$, there are two cases, both leading to valid graphs (A37, A38). If $k = 3$, there is one way to make both v_1 and v_2 2-leaves of $H - u$. However, some v and w will have distance more than 2, so this is not to a valid graph. If only v_1 is a 2-leaf this leads to a valid graph (A39). If $k = 4$, there is one way to connect each w to a v, but this does not lead to a valid graph.



$A37$ $A38$ $A39$

Case A.3.4 Assume u is not a 2-leaf. Then x_1 is the only 2-leaf of H, so there is no x_2. Then essentially the same argument as in Case A.3.2 repeats, with u and x_1 switching roles, and the same graphs are found.

Case B. The stem set is $S = \{u, v, w\}$, and there are 2-leaves with neighborhoods $\{u, v\}$, $\{u, w\}$, and $\{v, w\}$. Thus u, v, and w will be colored black.

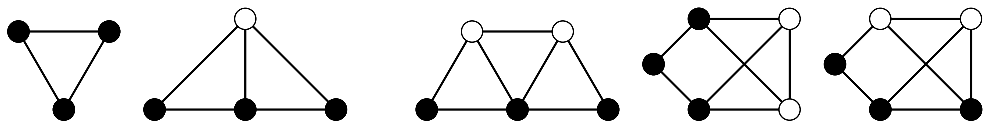
Each 2-leaf of the inside graph H is in S, so H has at most three 2-leaves.

Case B.1 If H has three 2-leaves, it may be K_3 (B1). If not, it has order at least 4, so none of the 2-leaves of H are neighbors. Then each 2-leaf of G has distance more than 2 from a 2-leaf of H, which is impossible.

Case B.2 If H has two 2-leaves, the third vertex in S must be in both of their neighborhoods. Thus H has order at most 5. Thus H is $K_4 - e$ or $P_4 + K_1$ (B2, B3).

Case B.3 If H has one 2-leaf v, then u must be one of its neighbors. If u is a 2-leaf of $H - v$, H has order 5, so it is W_5^-. There are two distinct choices for which vertex is w (B4, B5). If u is not a 2-leaf of $H - v$, v has another neighbor, x, that is. Then u is adjacent to every vertex of $H - v - x$. If u is adjacent to x, then by Lemma 2.2, H is a 2-tree, so it has at least two 2-leaves, a contradiction. If u is not adjacent to x, then by Lemma 2.2, $H - v - x$ is a 2-tree.

Now x is adjacent to all 2-leaves of $H - v - x$, so $H - v - x$ is a fan. Now w must be one of the 2-leaves of $H - v - x$, but it cannot neighbor all vertices of the fan unless the fan is K_3 and $H = W_5^-$, a previous case.



$B1$ $B2$ $B3$ $B4$ $B5$

This completes the proof. □

A structural characterization of maximal 2-degenerate graphs with diameter 2 allows us to evaluate or bound parameters on this class, which would otherwise be difficult. Sharp bounds have been proved for the maximum degree of maximal planar graphs with diameter 2 [18, 20]. We state sharp bounds on the maximum degree Δ of maximal 2-degenerate graph with diameter 2. A maximal 2-degenerate graph with $\Delta = n - 1$ must have diameter at most 2. A maximal 2-degenerate graph with $\Delta = n - 2$ need not have diameter at most 2 (for example, add one vertex to a fan with at least 5 vertices). Proposition 1.2 implies 2-trees with diameter 2 have $\Delta \geq \frac{2}{3}n$, and this bound is sharp.

Corollary 2.6 *A maximal 2-degenerate graph G with order n and diameter at most 2 has*

$$\Delta(G) \geq \begin{cases} n-1 & 1 \leq n \leq 4 \\ 3 & n = 5 \\ 4 & 6 \leq n \leq 8 \\ n-5 & 9 \leq n \leq 11 \\ n-6 & 12 \leq n \leq 16 \\ \left\lceil \frac{2}{3}(n-1) \right\rceil & n \geq 16 \end{cases},$$

and this bound is sharp for all n.

Proof For $1 \leq n \leq 4$, there is only one maximal 2-degenerate graph, which has a dominating vertex. For $n = 5$, there are three such graphs, one (W_5^-) of which has no dominating vertex. The fact that maximal 2-degenerate graphs have size $m = 2n-3$ and minimum degree 2 implies $\Delta \geq 4$ for $n \geq 6$. For $6 \leq n \leq 8$, this is attained by adding 2-leaves to A4 and A23.

Let G be a graph found under Case A, and H its inside graph. Then H has a stem that is adjacent to all 2-leaves of G with at most 5 vertices not adjacent to it, and only A39 attains this. Adding the 2-leaves of G to A39 as evenly as possible produces vertices with degree $n-6$ and $n-4-\left\lfloor \frac{n-8}{2} \right\rfloor$. Thus $\Delta \geq n-6$ for A39 when $n \geq 12$. Otherwise, $\Delta \geq n-5$ for graphs in Case A, and this is attained by graphs constructed from A37 when $n \geq 9$.

Let G be a graph found under Case B, and H its inside graph with stem set $\{u, v, w\}$. Consider summing the degrees of u, v, and w. There are $n-3$ other vertices, each of which is adjacent to at least two of u, v, and w. The graph induced by u, v, and w has at least two edges. Thus $2n-2 = 2(n-3)+4 \leq d(u)+d(v)+d(w) \leq 3\Delta$, so $\Delta \geq \left\lceil \frac{2}{3}(n-1) \right\rceil$. This is attained by graphs constructed from B3. For $n \geq 16$, $\left\lceil \frac{2}{3}(n-1) \right\rceil \leq n-6$, so the bound is as stated. □

We have seen that some maximal 2-degenerate graphs with diameter 3 are contained in a maximal 2-degenerate graph with diameter 2 (graphs A23-A39 above). The smallest maximal 2-degenerate graphs not contained in a maximal 2-degenerate graph with diameter 2 have order 7. They are all those with order 7 and diameter 3, excluding those listed in Theorem 2.5 (A25, A26, A29-A31, A33, A37, A38).

Proposition 2.7 *Let G be a maximal 2-degenerate graph. Then G is contained in a maximal 2-degenerate graph with diameter at most 3.*

Proof If G has diameter at most 3, we are done. If not, consider a vertex v with maximum eccentricity. Let S be the set of all vertices with distance more than 2 from v. Add 2-leaves adjacent to v and each vertex in S, and call the set vertices added S'. Now the distance between v and any other vertex is at most 2. Vertices in S' are all distance 2 from each other. A vertex in S' and a vertex in G have distance at most 3, since there is now a path through v. Thus no new pairs with distance more than 3 are created. This process can be repeated with other vertices until a graph is constructed that contains G and has diameter at most 3. □

§3. Diameter 2 k-Trees

In this section, we prove a forbidden subgraph characterization of k-trees with diameter 2.

Definition 3.1 *A k-path graph G is an alternating sequence of distinct k- and $k+1$-cliques $e_0, t_1, e_1, t_2, \cdots, t_p, e_p$, starting and ending with a k-clique and such that t_i contains exactly two k-cliques e_{i-1} and e_i.*

Note that k-paths are also known a linear k-trees [1]. They are closely related to pathwidth [17]; in particular, they are the maximal graphs with proper pathwidth k. I have have further examined k-paths in two forthcoming papers [4, 5]. There is a simple characterization of k-paths.

Theorem 3.2([15]) *Let G be a k-tree with $n > k+1$ vertices. Then G is a k-path graph if and only if G has exactly two k-leaves.*

A k-path with a dominating vertex has nice structure.

Lemma 3.3 *A k-path has diameter at most 2 if and only if it has a dominating vertex. When $k \geq 2$, a k-path with a dominating vertex can be represented as $P + K_1$, where P is a $k-1$-path.*

Proof Every k-path with order $n \leq k+2$ has diameter at most 2 and a dominating vertex. Consider constructing the k-path from $K_k + \overline{K}_2$, which has k-leaves u and v_1, and $N(u) = S_1 = N(v_1)$. Iteratively add vertex v_i with neighborhood S_i, so that S_i replaces one vertex of S_{i-1} with v_{i-1}. As long as S_1 and S_i contain a common vertex, the graph has diameter 2 and a dominating vertex. Once S_1 and S_i do not contain a common vertex, the graph has diameter more than 2 and no dominating vertex.

For the second claim, we use induction on order n. When $n = k$, $G = K_k$ and the result holds. Let G be a k-path with order $n > k$ containing a dominating vertex u, and assume the result holds for all graphs with order $n-1$. Then G has a k-leaf v, which is adjacent to u. Now $G - v$ is a k-path with a dominating vertex, so it can be represented as $P' + K_1$, where P' is a $k-1$-path. Then the other neighbors of v induce a clique in P', so G can be represented as $P + K_1$. □

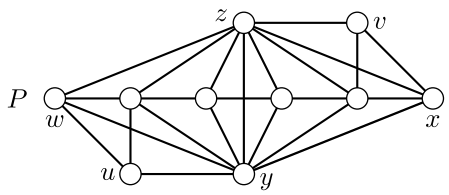
Note for $k \geq 2$, a k-tree with diameter 2 need not have a dominating vertex.

Adding a k-leaf to a k-tree cannot change any existing distances. Thus when constructing a k-tree, the diameter can increase, but it cannot decrease, as it can in a maximal k-degenerate graph.

Definition 3.4 *A k-tree is minimal with respect to diameter 3 if deleting any k-leaf results in a k-tree with diameter 2.*

We can characterize these graphs. A tree is minimal with respect to diameter 3 if and only if it is P_4. We have seen in Proposition 1.2 that a 2-tree is minimal with respect to diameter 3 if and only if it is P_6^2. In general, P_{2k+2}^k is the smallest k-tree with diameter 3, but for $k \geq 3$ it is not the only one.

Algorithm 3.5 Let P be a $k-2$-path, $k \geq 3$, of order $n-4$ with k-leaves w and x. Join dominating vertices y and z to P, forming $P+K_2$. Add u with neighborhood $N_P(w) \cup \{w, y\}$, and v with neighborhood $N_P(x) \cup \{x, z\}$. Let $\mathbb{G}_k$ be the class of all graphs formed this way.

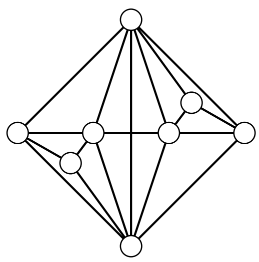


Theorem 3.6 *A graph G is a k-tree minimal with respect to diameter 3 if and only if $G \in \mathbb{G}_k$.*

Proof ($\Leftarrow$) Let G be a graph in $\mathbb{G}_k$ constructed using the algorithm. Then G is a k-tree, $d(u, v) = 3$, and u and v are the only pair with distance more than 2.

($\Rightarrow$) Let G be as stated. A k-tree with diameter 3 must contain a pair of vertices distance 3 apart. Thus in a minimal k-tree with diameter 3, the vertices at distance 3 must be k-leaves, and no other vertices are k-leaves. Thus G is a k-path with leaves (say) u and v. Since G is minimal, $G-u$ has diameter 2. By Lemma 3.3, it has a dominating vertex y, so $G-u-y$ is a $k-1$-path. Similarly, $G-v$ has a dominating vertex z. Thus $G-\{u, v, y, z\}$ is a $k-2$-path. Then u and v must each neighbor one of y and z, and one of the k-leaves of the $k-2$-path. Thus G can be constructed using the algorithm, so $G \in \mathbb{G}_k$. □

Equivalently, a k-tree has diameter at most 2 if and only if it does not contain any graph in $\mathbb{G}_k$. When $k=3$ and $n \geq 8$, the algorithm produces a unique 3-tree of order n minimal with respect to diameter 3 (shown below for $n=8$).



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