International J.Math. Combin. Vol.2(2020), 101-108

On Ideals of Torian Algebras

Ilojide Emmanuel

(Department of Mathematics, Federal University of Agriculture, Abeokuta 110101, Nigeria)

E-mail: emmailojide@yahoo.com, ilojidee@funaab.edu.ng

Abstract: In this paper, the notion of ideals in torian algebras is introduced. Their properties are investigated. Moreover, the dual and nuclei of ideals as well as congruences developed on ideals of torian algebras are studied.

Key Words: Ideals, torian algebras, dual, congruences, Smarandachely torian algebra.

AMS(2010): 20N02, 20N05, 06F35.

§1. Introduction

An algebra of type (2,0) is a well known type of algebraic structures. It comprises a non-empty set, some constant element together with a binary operation and interesting behaviors. In [1], Kim and Kim introduced the notion of BE-algebras. Ahn and So, in [2] and [3] introduced the notions of ideals and upper sets in BE-algebras and investigated related properties.

In [6], obic algebras were introduced. Homomorphisms and krib maps as well as monics of obic algebras were studied in this paper. Some properties of a class of obic algebras were studied in [7]. In this paper, the notion of ideals in torian algebras is introduced. Their properties are investigated. Moreover, the dual and nuclei of ideals as well as congruences developed on ideals of torian algebras are studied.

§2. Preliminaries

Definition 2.1([6]) *A triple $(X;*,0)$; where X is a non-empty set, $*$ a binary operation on X, and 0 a constant element of X is called an obic algebra if the following axioms hold for all $x, y, z \in X$:*

(1) $x * 0 = x$;
(2) $[x * (y * z)] * x = x * [y * (z * x)]$;
(3) $x * x = 0$.

Lemma 2.1([6]) *Let X be an obic algebra. Then, for all $x, y \in X$ hold with*

$$x * y = [x * (y * x)] * x.$$

[1]Received February 22, 2020, Accepted June 12, 2020.

Definition 2.2([6]) *Let $(X;*,0)$ and $(Y;\circ,0')$ be obic algebras. A function $f: X \to Y$ is called an obic homomorphism if $f(a*b) = f(a) \circ f(b)$ for all $a, b \in X$.*

Definition 2.3([6]) *Let $f: X \to Y$ be an obic homomorphism. The set $\{x \in X : f(x) = 0'\}$ is called the kernel of f. It is denoted by $Ker(f)$.*

Let $f: X \to Y$ be an obic homomorphism. If f is injective, then it is called a monomorphism. If f is surjective, then it is called an epimorphism. If f is both injective and surjective, then it is called an isomorphism.

Definition 2.4([6]) *An obic algebra X is said to have the weak property (WP) if $x*y = 0$ and $y*x = 0$ imply that $x = y$.*

Theorem 2.1 *Let $\phi: X \to X$ be an obic homomorphism; where X has the weak property. Then ϕ is injective if and only if $ker(\phi) = \{0\}$.*

Definition 2.5([6]) *An equivalence relation $\sim^*$ on an obic algebra X is called a congruence if $(x \sim^* y)$ and $(u \sim^* v) \Rightarrow (x*u) \sim^* (y*v)$.*

Lemma 2.2([6]) *Let $f: X \to Y$ be an obic homomorphism. The equivalence relation $\sim^*$ defined by $(x \sim^* y) \Rightarrow f(x) = f(y)$ is a congruence.*

Definition 2.6([7]) *An obic algebra X is called torian if $[(x*y)*(x*z)]*(z*y) = 0$ for all $x, y, z \in X$. Otherwise, if there are $x, y, z \in X$ such that $[(x*y)*(x*z)]*(z*y) \neq 0$, such an obic algebra X is called Smarandachely torian.*

Example 2.1 Let $X = \{0, 1\}$. Define a binary operation $*$ on X by the multiplication table below

$*$	0	1
0	0	1
1	1	0

Then, $(X, *, 0)$ is torian algebra.

Example 2.2 Consider the multiplicative group $G = \{1, -1, i, -i\}$. Define a binary operation $*$ on G by $a*b = ab^{-1}$. Then $(G; *, 1)$ is torian.

From now on, X will denote a torian algebra equipped with the weak property.

Lemma 2.3([7]) *Let X be a torian algebra. Then, for all $x, y, z \in X$ hold with*

$$(x*y)*z = (x*z)*y.$$

Proposition 2.1([7]) *Let X be a torian algebra. Then, for all $x, y \in X$ hold with*

$$(0 * x) * (0 * y) = 0 * (x * y).$$

Definition 2.7([7]) *Let X be a torian algebra. An element $x \in X$ is said to fix 0 if $0 * x = 0$. If every element in X fixes 0, then X is said to fix 0.*

The set of all elements of X which fix 0 is denoted by 0^*.

Lemma 2.4([7]) *Let X be a torian algebra. Define the relation $\sim$ on X by $x \sim y \Leftrightarrow x * y = 0$ for all $x, y \in X$. Then $(X; \sim)$ is a partially ordered set.*

The following Lemma follows from definition.

Lemma 2.4([7]) *Let X be a torian algebra with the partial ordering $\sim$. Then,*

$$[(x * y) * (z * y)] \sim (x * z)$$

for all $x, y, z \in X$.

§3. Main Results

Definition 3.1 *Let X be a torian algebra. A non-empty set S of X is called a left ideal of X if the following hold*

(1) $0 \in S$;

(2) *if $x, y \in X$ such that $x, [[y * (x * y)] * y] \in S$, then $y \in S$.*

Definition 3.2 *Let X be a torian algebra. A non-empty set S of X is called a right ideal of X if the following hold*

(1) $0 \in S$;

(2) *if $x, y \in X$ such that $x, [[x * (y * x)] * x] \in S$, then $y \in S$.*

Remark 3.1 If S is both a left ideal and a right ideal of X, then S is called an ideal of X.

Example 3.1 Every torian algebra X has at least two left ideals, namely, $\{0\}$ and X.

Example 3.2 The subset $S = \{1, -1\}$ is a left ideal of the torian algebra in example 2.2.

Proposition 3.1 *Let X be a torian algebra. The collection of all elements in X which fix 0 is a left ideal of X.*

Proof Now, $0 \in 0^*$. Let $x, y \in X$ such that $x, [[y * (x * y)] * y] \in 0^*$. Then we show that

$y \in 0^*$. Since $[y * (x * y)] * y \in 0^*$, we have $0 * [[y * (x * y)] * y] = 0$. But

$$\begin{aligned} 0 * y &= (0 * y) * 0 = (0 * y) * (0 * x) = 0 * (y * x) \quad \text{(by Proposition 2.1)} \\ &= 0 * [[y * (x * y)] * y] = 0 \end{aligned}$$

as required. □

Remark 3.2 If a left ideal S of X is such that $[[x * (y * x)] * x] \in S$ for all $x, y \in X$, then S is said to be a complete left ideal of X or that S is complete in X.

Proposition 3.2 *Let S be a left ideal of a torian algebra X. If $0 * x \in S$ for all $x \in S$, then S is a complete left ideal.*

Proof Let $x, y \in S$. Then notice that

$$[[x * (y * x)] * x] * x = (x * y) * x = 0 * y \in S.$$

And since S is a left ideal, we have that $[[x * (y * x)] * x] \in S$. This completes the proof. □

The following theorem follows from Propositions 3.1 and 3.2.

Theorem 3.1 *Let X be a torian algebra. Then 0^* is a complete left ideal of X if and only if*

$$0 * x \in 0^*$$

for all $x \in 0^*$.

Corollary 3.1 *Let X be a torian algebra which fixes 0. Then every left ideal of X is complete.*

The following proposition follows from definition.

Proposition 3.3 *Let $(X; *, 0)$ and $(Y; \odot, 0')$ be torian algebras. Let $f : X \to Y$ be a homomorphism. Then $Ker(f)$ is a complete left ideal of X.*

Theorem 3.2 *Let $(X; *, 0)$ and $(Y; \odot, 0')$ be torian algebras such that $[[x*[(x*y)*z]]*y]*z = 0$ for all $x, y, z \in X$. Let $f : X \to Y$ be an epimorphism. If S is a left ideal of X, then $f(S)$ is left ideal of Y.*

Proof Let $x', y' \in Y$ such that $x', [[y' \odot (x' \odot y')] \odot y'] \in f(S)$. Now, there exist $x, y, z \in S$ such that $f(x) = x', f(y) = [[y' \odot (x' \odot y')] \odot y'], f(z) = y'$. Clearly, $z * [(z * x) * y] \in X$. Let $z * [(z * x) * y] = w$; so that $[[z * [(z * x) * y] * x] * y = 0 \in S; \Rightarrow w \in S$. Hence, $f(w) \in f(S)$. We show that $y' = f(w)$. Notice that $f[(z * x) * y] = [f(z * x) \odot f(y)] = [f(z) \odot f(x)] \odot f(y) = (y' \odot x') \odot (y' \odot x') = 0'$.

Notice that $f(w) = f[z * [(z * x) * y]] = f(z) \odot f[(z * x) * y] = y' \odot 0' = y'$ as required. □

Corollary 3.2 *Let $(X; *, 0)$ and $(Y; \odot, 0')$ be torian algebras such that $[[x*[(x*y)*z]]*y]*z = 0$ for all $x, y, z \in X$. Let $f : X \to Y$ be an epimorphism. If S is a complete left ideal of X, then*

$f(S)$ is a complete left ideal of Y.

Theorem 3.3 *A left ideal S of a torian algebra X is complete in X if and only if the following hold*

(1) $0 \in S$;

(2) $[[x*(z*x)]*x], [[y*(z*y)]*y], z \in S \Rightarrow x*y \in S$.

Proof Suppose S is a left ideal of X satisfying (1) and (2). Now, $[[0*(0*0)]*0], [[x*(0*x)]*x], 0 \in S$. So, $0*x \in S$; and by Proposition 3.2, S is complete.

Conversely, suppose S is complete in X. Clearly, $0 \in S$. Let $[[x*(z*x)]*x], [[y*(z*y)]*y], z \in S$. Then $x, y \in S$. So, $x*y \in S$ as required. □

Definition 3.3 *Let S be a left ideal of a torian algebra X. The set $S^* = \{x \in S : [[0*(x*0)]*0] \in S\}$ is called the dual of S.*

The following proposition follows from definition.

Proposition 3.4 *Let S be a left ideal of a torian algebra X. Then the dual of S is a complete left ideal of X.*

Proposition 3.5 *Let X be a torian algebra. Let S be a left ideal of X. If T is a complete left ideal of X such that $T \subseteq S$, then $T \subseteq S^*$.*

Proof Let $x \in T$. Then $0*x \in T$. Since $T \subseteq S$, then $x, 0*x = [[0*(x*0)]*0] \in S$. Therefore, $x \in S^*$ as required. The proof is complete. □

By Propositions 3.4 and 3.5, we have the following Theorem.

Theorem 3.4 *Let S be a left ideal of a torian algebra X. Then the dual of S is complete in X. Moreover, S^* is the largest complete left ideal of X that contains S.*

Definition 3.4 *Let X be a torian algebra. The set $x_\lambda = \{x \in X : (x*y)*z = 0;\ y, z \in X\}$ is called a left nucleus of X.*

Proposition 3.6 *Let S be a left ideal of a torian algebra X. Then S is the union of left nuclei of X.*

Proof Let $x \in X$. Notice that $(x*0)*x = 0$. So, x belongs to a left nucleus of X. Now, let x be in the union of left nuclei of X. There exist $y, z \in S$ such that $(x*y)*z = 0 \in S$. It follows that $x \in S$ as required. The proof is complete. □

The following proposition is straightforward.

Proposition 3.7 *Let S be a non-empty subset of a torian algebra X with $0 \in S$ such that S is the union of left nuclei of X. Then S is a left ideal of X.*

By Propositions 3.6 and 3.7, we have the following Theorem.

Theorem 3.5 *Let S be a non-empty subset of a torian algebra X with $0 \in S$. Then S is a left ideal of X if and only if S is the union of left nuclei of X.*

Definition 3.5 *Let X be a torian algebra. An element $a \in X$ is said to be palindromic if there exists an element $x \in X$ such that $a * x = a$. The element x is then said to be palindromic to a.*

The collection of all elements in X that are palindromic to a is denoted by a^*.

Proposition 3.8 *Let a be a fixed element of a torian algebra X. Then $0 * x = 0$ for all $x \in a^*$.*

Proof Clearly, $0 \in a^*$. So, a^* is not empty. Now, let $x \in a^*$. Notice that $0 * x = (a * a) * x = (a * x) * a = 0$ as required. □

Theorem 3.6 *Let a be a fixed element of a torian algebra X with the partial ordering $\sim$. Then a^* is a complete left ideal of X.*

Proof Let $x, y \in X$ such that $x, [[y * (x * y)] * y] \in a^*$. Notice that

$$\begin{aligned} (a * y) * a &= (a * a) * y \\ &= 0 * y = (0 * y) * 0 \\ &= (0 * y) * (0 * x) = 0 * (y * x) \\ &= 0 * [[y * (x * y)] * y] = 0. \end{aligned}$$

So, $(a * y) \sim a$.

Notice also that

$$\begin{aligned} a &= a * [[y * (x * y)] * y] = a * (y * x) \\ &= (a * x) * (y * x) \sim (a * y) \quad \text{(by Lemma 2.4)}. \end{aligned}$$

So, $y \in a^*$. The completeness of a^* follows from Proposition 3.2. □

Proposition 3.9 *Let X be a torian algebra. Let S and T be left ideals of X. If $[[x*(y*x)]*x = x$ for all $x \in S, y \in T$, then $S \cap T = \{0\}$.*

Proof Let $x \in S \cap T$. Then $x \in S, x \in T$. Now, notice that

$$x = [x * (x * x)] * x = x * x = 0$$

as required. □

Theorem 3.7 *Let X be a torian algebra equipped with a congruence $\sim^*$. Then $\overline{0} = \{x \in X : x \sim^* 0\}$ is a complete left ideal of X.*

Proof Clearly, $0 \in \overline{0}$. Now, let $x, y \in X$ such that $x, [[y * (x * y)] * y] \in \overline{0}$. Then, $x \sim^* 0$ and $[[y * (x * y)] * y] = y * x \sim^* 0$. Also, $y \sim^* y$. We therefore have that $y * x \sim^* y$. So $y \sim^* y * x$. Hence, $y \sim^* 0$; giving us that $y \in \overline{0}$ as required.

Now, let $x, y \in \overline{0}$. We show that $[[x*(y*x)]*x] \in \overline{0}$. Notice that $x \sim^* 0$ and $y \sim^* 0$. So, $x*y \sim^* 0$; which gives $[[x*(y*x)]*x] = x*y \sim^* 0$ as required. The proof is complete. □

Theorem 3.8 *Let S be a left ideal of a torian algebra X. Let $\sim^1$ be a relation on X defined by X by $x \sim^1 y \Leftrightarrow [[x*(y*x)]*x] \in S$ and $[[y*(x*y)]*y] \in S$ for all $x, y \in X$. Then $\sim^1$ is a congruence on X.*

Proof We first show that $\sim^1$ is an equivalence relation. Clearly, $[[x*(x*x)]*x] = 0 \in S$. so, $\sim^1$ is reflexive. Let $x, y \in X$ such that $x \sim^1 y$. Then $[[x*(y*x)]*x] \in S$ and $[[y*(x*y)]*y] \in S$; which implies that $y \sim^1 x$. So, $\sim^1$ is symmetric. Let $x, y, z \in X$ such that $x \sim^1 y$ and $y \sim^1 z$. Then, $[[x*(y*x)]*x], [[y*(x*y)]*y], [[y*(z*y)]*y], [[z*(y*z)]*z] \in S$. Now, since X is torian, we have

$$\begin{aligned}&[[[x*(z*x)]*x]*[[x*(y*x)]*x]]*[[y*(z*y)]*y]\\&=[(x*z)*(x*y)]*(y*z)=0\in S.\end{aligned}$$

So, $[[x*(z*x)]*x] \in S$ by virtue of S being a left ideal.

Also,

$$\begin{aligned}&[[[z*(x*z)]*z]*[[z*(y*z)]*z]]*[[y*(x*y)]*y]\\&=[(z*x)*(z*y)]*(y*x)=0\in S.\end{aligned}$$

Hence, $x \sim^1 z$. So, $\sim^1$ is transitive.

Now let $x, y, u, v \in X$ such that $x \sim^1 y$ and $u \sim^1 v$. Then,

$$[[x*(y*x)]*x], [[y*(x*y)]*y], [[u*(v*u)]*u], [[v*(u*v)]*v] \in S.$$

Notice that by Lemma 2.5, we have

$$\begin{aligned}&[[[x*(u*x)]*x]*[[y*(u*y)]*y]]*[[x*(y*x)]*x]\\&=[(x*u)*(y*u)]*(x*y)=0\in S\end{aligned}$$

and

$$\begin{aligned}&[[[y*(u*y)]*y]*[[x*(u*x)]*x]]*[[y*(x*y)]*y]\\&=[(y*u)*(x*u)]*(y*x)=0\in S.\end{aligned}$$

So, $[(x*u)*(y*u)] \in S$ and $[(y*u)*(x*u)] \in S$. Hence, $(x*u) \sim^1 (y*u)$.

Similar argument gives $(y*u) \sim^1 (y*v)$. Since, $(x*u) \sim^1 (y*u)$ and $(y*u) \sim^1 (y*v)$, then $(x*u) \sim^1 (y*v)$ as required. The proof is complete. □

Corollary 3.3 *Let S be a left ideal of a torian algebra X. Let $\sim^2$ be a relation on X defined by $x \sim^2 y$ if and only if S is complete in X and $[[y*(x*y)]*y] \in S$ for all $x, y \in X$. Then $\sim^2$ is a congruence on X.*

References

[1] H. S. Kim and Y. H. Kim, On BE-algebras, *Sci. Math. Jpn.*, 66(2007),113–116.

[2] S.S. Ahn and K. S. So, On ideals and upper sets in BE-algebras, *Sci. Math. Jpn.*, 68(2008), 351–357.

[3] S.S. Ahn and K. S. So, On generalized upper sets in BE-algebras, *Bull. Korean Math. Soc.*, 46(2009), 281–287.

[4] R. H. Bruck, *A Survey of Binary Systems*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1966, 185pp.

[5] J. Dene and A. D. Keedwell, *Latin Squares and Their Applications*, the English University press Ltd, 1974, 549pp.

[6] E. Ilojide, On obic algebras, *International J. Math. Combin.*, 4(2019), 80–88.

[7] E. Ilojide, A note on Torian algebras, *International J. Math. Combin.*, 2(2020), 80–87.