On Solutions of Second-Order Fuzzy Initial Value Problem by Fuzzy Laplace Transform

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Abstract: In this paper, we investigate the solutions of second-order fuzzy initial value problem with positive coefficient using the fuzzy Laplace transform under the approach of generalized differentiability. Several theorems are given on the studied problem. The problem is shown on an example.

Key Words: Fuzzy initial value problems, second-order fuzzy differential equation, generalized differentiability, fuzzy Laplace transform.

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§1. Introduction

Fuzzy differential equations are important topic. Especially, fuzzy initial value problems and its applications. For example, real-word problems, mathematical models in science and technology, population models, civil engineering. So, many researchers have studied fuzzy differential equations.

There are several approach solving the fuzzy differential equations. The first is Hukuhara differentiability [7,14]. The second approach is generalized differentiability [9,15]. The third generate the fuzzy solution from the crips solution. These are extension principle [7,8], the concept of differential inclusion [13] and the fuzzy problem to be a set of crips problem [11]. But, many fuzzy initial and boundary value problems can not be solved as analytically. Therefore, the another approach is to find approximate solutions. The numeric methods are introduced and studied [1-4,12]. The another approach is the fuzzy Laplace transform. The solutions of fuzzy differential equation is studied by fuzzy Laplace transform [5,18,19,21]. One of the most important applications fuzzy Laplace transform is to solve fuzzy initial value problems.

In this paper, the solutions of second-order fuzzy initial value problem with positive coefficient are investigated by fuzzy Laplace transform. Generalized differentiability, fuzzy Laplace transform, Hukuhara difference and fuzzy arithmetic are used. The aim of this study is to investigate solutions using the properties fuzzy Laplace transform by generalized differentiability for second-order fuzzy initial value problem.

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§2. **Preliminaries**

Definition 2.1([17])　*A fuzzy number is a mapping* $u : \mathbb{R} \to [0,1]$ *satisfying the following properties:*

(1) *u is normal:* $\exists x_0 \in \mathbb{R}$ *for which* $u(x_0) = 1$;

(u) *u is convex fuzzy set:* $u(\lambda x + (1-\lambda)y) \geqslant \min\{u(x), u(y)\}$ *for all* $x, y \in \mathbb{R}, \lambda \in [0,1]$;

(3) *u is upper semi-continuous on* $\mathbb{R}$,

(4) $cl\{x \in \mathbb{R} \mid u(x) > 0\}$ *is compact, where cl denotes the closure of a subset.*

Let $\mathbb{R}_F$ denote the set of all fuzzy numbers.

Definition 2.2([15])　*Let* $u \in \mathbb{R}_F$. *The* α*-level set of u, denoted ,* $[u]^\alpha$, $0 < \alpha \leq 1$, *is* $[u]^\alpha = \{x \in \mathbb{R} \mid u(x) \geq \alpha\}$. *If* $\alpha = 0$, $[u]^0 = cl\{suppu\} = cl\{x \in \mathbb{R} \mid u(x) > 0\}$. *The notation,* $[u]^\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$ *denotes explicitly the* α*-level set of u, where* $\underline{u}_\alpha$ *and* $\overline{u}_\alpha$ *denote the left-hand endpoint and the right-hand endpoint of* $[u]^\alpha$, *respectively.*

The following remark shows when $[\underline{u}_\alpha, \overline{u}_\alpha]$ is a valid α-level set.

Remark 2.1([10,15])　The sufficient and necessary conditions for $[\underline{u}_\alpha, \overline{u}_\alpha]$ to define the parametric form of a fuzzy number as follows:

(1) $\underline{u}_\alpha$ is bounded monotonic increasing (nondecreasing) left-continuous function on $(0, 1]$ and right-continuous for $\alpha = 0$,

(2) $\overline{u}_\alpha$ is bounded monotonic decreasing (nonincreasing) left-continuous function on $(0, 1]$ and right-continuous for $\alpha = 0$,

(3) $\underline{u}_\alpha \leq \overline{u}_\alpha$, $0 \leq \alpha \leq 1$.

Definition 2.3([17])　*If A is a symmetric triangular fuzzy number with support* $[\underline{a}, \overline{a}]$, *the* $\alpha-$*level set of A is*

$$[A]^\alpha = [\underline{A}_\alpha, \overline{A}_\alpha] = \left[\underline{a} + \left(\frac{\overline{a} - \underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a} - \underline{a}}{2}\right)\alpha\right], \quad (\underline{A}_1 = \overline{A}_1, \ \underline{A}_1 - \underline{A}_\alpha = \overline{A}_\alpha - \overline{A}_1).$$

Definition 2.4([12,15,20])　*Let* $u, v \in \mathbb{R}_F$. *If there exists* $w \in \mathbb{R}_F$ *such that* $u = v + w$, *then w is called the Hukuhara difference of fuzzy numbers u and v, and it is denoted by* $w = u \ominus v$.

Definition 2.5([6,12,15])　*Let* $f : [a,b] \to \mathbb{R}_F$ *and* $t_0 \in [a,b]$. *We say that f is Hukuhara differentiable at* t_0, *if there exists an element* $f'(t_0) \in \mathbb{R}_F$ *such that for all* $h > 0$ *sufficiently small,* $\exists f(t_0 + h) \ominus f(t_0)$, $f(t_0) \ominus f(t_0 - h)$ *and the limits*

$$\lim_{h \to 0} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \to 0} \frac{f(t_0) \ominus f(t_0 - h)}{h} = f'(t_0).$$

Definition 2.6([15])　*Let* $f : [a,b] \to \mathbb{R}_F$ *and* $t_0 \in [a,b]$. *We say that f is (1)-differentiable at* t_0, *if there exists an element* $f'(t_0) \in \mathbb{R}_F$ *such that for all* $h > 0$ *sufficiently small near to 0,*

exist $f(t_0 + h) \ominus f(t_0)$, $f(t_0) \ominus f(t_0 - h)$ *and the limits*

$$\lim_{h \to 0} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \to 0} \frac{f(t_0) \ominus f(t_0 - h)}{h} = f'(t_0),$$

and f is (2)-differentiable if for all $h > 0$ sufficiently small near to 0, exist $f(t_0) \ominus f(t_0 + h)$, $f(t_0 - h) \ominus f(t_0)$ and the limits

$$\lim_{h \to 0} \frac{f(t_0) \ominus f(t_0 + h)}{-h} = \lim_{h \to 0} \frac{f(t_0 - h) \ominus f(t_0)}{-h} = f'(t_0).$$

Theorem 2.7([16]) *Let $f : [a, b] \to \mathbb{R}_F$ be fuzzy function, where $[f(t)]^\alpha = \left[\underline{f}_\alpha(t), \overline{f}_\alpha(t)\right]$, for each $\alpha \in [0, 1]$.*

(i) If f is (1)-differentiable then $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable functions and $\left[f'(t)\right]^\alpha = \left[\underline{f}'_\alpha(t), \overline{f}'_\alpha(t)\right]$,

(ii) If f is (2)-differentiable then $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable functions and $\left[f'(t)\right]^\alpha = \left[\overline{f}'_\alpha(t), \underline{f}'_\alpha(t)\right]$.

Theorem 2.8([16]) *Let $f' : [a, b] \to \mathbb{R}_F$ be fuzzy function, where $[f(t)]^\alpha = \left[\underline{f}_\alpha(t), \overline{f}_\alpha(t)\right]$, for each $\alpha \in [0, 1]$, f is (1)-differentiable or (2)-differentiable.*

(i) If f and f' are (1)-differentiable then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and $\left[f''(t)\right]^\alpha = \left[\underline{f}''_\alpha(t), \overline{f}''_\alpha(t)\right]$,

(ii) If f is (1)-differentiable and f' is (2)-differentiable then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and $\left[f''(t)\right]^\alpha = \left[\overline{f}''_\alpha(t), \underline{f}''_\alpha(t)\right]$,

(iii) If f is (2)-differentiable and f' is (1)-differentiable then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and $\left[f''(t)\right]^\alpha = \left[\overline{f}''_\alpha(t), \underline{f}''_\alpha(t)\right]$,

(iv) If f and f' are (2)-differentiable then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and $\left[f''(t)\right]^\alpha = \left[\underline{f}''_\alpha(t), \overline{f}''_\alpha(t)\right]$.

Definition 2.9([18,21]) *The fuzzy Laplace transform of fuzzy-valued function f is defined as follows:*

$$F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt = \lim_{\tau \to \infty} \int_0^\tau e^{-st} f(t)\, dt.$$

$$F(s) = L(f(t)) = \left[\lim_{\tau \to \infty} \int_0^\tau e^{-st} \underline{f}(t)\, dt, \lim_{\tau \to \infty} \int_0^\tau e^{-st} \overline{f}(t)\, dt\right],$$

$$F(s, \alpha) = L(f(t, \alpha)) = \left[L\left(\underline{f}(t, \alpha)\right), L\left(\overline{f}(t, \alpha)\right)\right]$$

where,

$$L\left(\underline{f}\left(t,\alpha\right)\right) = \int_0^\infty e^{-st}\underline{f}\left(t,\alpha\right)dt = \lim_{\tau\to\infty}\int_0^\tau e^{-st}\underline{f}\left(t,\alpha\right)dt,$$

$$L\left(\overline{f}\left(t,\alpha\right)\right) = \int_0^\infty e^{-st}\overline{f}\left(t,\alpha\right)dt = \lim_{\tau\to\infty}\int_0^\tau e^{-st}\overline{f}\left(t,\alpha\right)dt.$$

Theorem 2.10([5,18,21]) *Suppose that f is continuous fuzzy-valued function on $[0,\infty)$ and exponential order α and that f' is piecewise continuous fuzzy-valued function on $[0,\infty)$, then*

$$L\left(f'\left(t\right)\right) = sL\left(f\left(t\right)\right)\ominus f\left(0\right),$$

if f is (1)-differentiable,

$$L\left(f'\left(t\right)\right) = \left(-f\left(0\right)\right)\ominus\left(-sL\left(f\left(t\right)\right)\right),$$

if f is (2)-differentiable.

Theorem 2.11([18,21]) *Suppose that f and f' are continuous fuzzy-valued functions on $[0,\infty)$ and of exponential order α and that f'' is piecewise continuous fuzzy-valued function on $[0,\infty)$, then*

$$L\left(f''\left(t\right)\right) = s^2 L\left(f\left(t\right)\right)\ominus sf\left(0\right)\ominus f'\left(0\right)$$

if f and f' are (1)-differentiable,

$$L\left(f''\left(t\right)\right) = -f'\left(0\right)\ominus\left(-s^2\right)L\left(f\left(t\right)\right) - sf\left(0\right)$$

if f is (1)-differentiable and f' is (2)-differentiable,

$$L\left(f''\left(t\right)\right) = -sf\left(0\right)\ominus\left(-s^2\right)L\left(f\left(t\right)\right)\ominus f'\left(0\right)$$

if f is (2)-differentiable and f' is (1)-differentiable,

$$L\left(f''\left(t\right)\right) = s^2 L\left(f\left(t\right)\right)\ominus sf\left(0\right) - f'\left(0\right)$$

if f and f' are (2)-differentiable.

Theorem 2.12([5,18]) *Let $f\left(x\right)$, $g\left(x\right)$ be continuous fuzzy-valued functions suppose that c_1 and c_2 are constant, then*

$$L\left(c_1 f\left(x\right) + c_2 g\left(x\right)\right) = \left(c_1 L\left(f\left(x\right)\right)\right) + \left(c_2 L\left(g\left(x\right)\right)\right).$$

Theorem 2.13([5]) *Let $f(x)$ be continuous fuzzy-valued function on $[0, \infty)$ and $\lambda \geq 0$, then*

$$L(\lambda f(x)) = \lambda(L(f(x))).$$

§3. Main Results

In this section, we consider solutions of the fuzzy initial value problem

$$y''(t) = \lambda y(t), \tag{1}$$

$$y(0) = [A]^\alpha, \quad y'(0) = [B]^\alpha, \tag{2}$$

by Laplace transform, where $\lambda > 0$, A and B are symmetric triangular fuzzy numbers with supports $[\underline{a}, \overline{a}]$ and $[\underline{b}, \overline{b}]$,

$$[A]^\alpha = [\underline{A}_\alpha, \overline{A}_\alpha] = \left[\underline{a} + \left(\frac{\overline{a} - \underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a} - \underline{a}}{2}\right)\alpha\right],$$

$$[B]^\alpha = [\underline{B}_\alpha, \overline{B}_\alpha] = \left[\underline{b} + \left(\frac{\overline{b} - \underline{b}}{2}\right)\alpha, \overline{b} - \left(\frac{\overline{b} - \underline{b}}{2}\right)\alpha\right],$$

where, (i, j) solution $(i, j = 1, 2)$ means that y is (i)-differentiable, y' is (j)-differentiable.

(1,1) solution: Since y and y' are (1)-differentiable, taking the fuzzy Laplace transform of the equation (1),

$$s^2 L(y(t, \alpha)) \ominus sy(0, \alpha) \ominus y'(0, \alpha) = \lambda L(y(t, \alpha))$$

is obtained. From this, we have the equations

$$s^2 L(\underline{y}(t, \alpha)) - s\underline{y}(0, \alpha) - \underline{y}'(0, \alpha) = \lambda L(\underline{y}(t, \alpha)),$$

$$s^2 L(\overline{y}(t, \alpha)) - s\overline{y}(0, \alpha) - \overline{y}'(0, \alpha) = \lambda L(\overline{y}(t, \alpha)).$$

Using the initial values (2), we obtain

$$L(\underline{y}(t, \alpha)) = \frac{s}{s^2 - \lambda}\underline{A}_\alpha + \frac{1}{s^2 - \lambda}\underline{B}_\alpha,$$

$$L(\overline{y}(t, \alpha)) = \frac{s}{s^2 - \lambda}\overline{A}_\alpha + \frac{1}{s^2 - \lambda}\overline{B}_\alpha.$$

From here, taking the inverse Laplace transform of these equations, it gives

$$\underline{y}(t, \alpha) = L^{-1}\left(\frac{s}{s^2 - \lambda}\right)\underline{A}_\alpha + L^{-1}\left(\frac{1}{s^2 - \lambda}\right)\underline{B}_\alpha,$$

$$\overline{y}(t, \alpha) = L^{-1}\left(\frac{s}{s^2 - \lambda}\right)\overline{A}_\alpha + L^{-1}\left(\frac{1}{s^2 - \lambda}\right)\overline{B}_\alpha.$$

Thus, $(1, 1)$ solution is

$$\underline{y}(t, \alpha) = \frac{1}{2}\underline{A}_\alpha \left(e^{\sqrt{\lambda}t} + e^{-\sqrt{\lambda}t}\right) + \frac{1}{2\sqrt{\lambda}}\underline{B}_\alpha \left(e^{\sqrt{\lambda}t} - e^{-\sqrt{\lambda}t}\right), \tag{3}$$

$$\overline{y}(t, \alpha) = \frac{1}{2}\overline{A}_\alpha \left(e^{\sqrt{\lambda}t} + e^{-\sqrt{\lambda}t}\right) + \frac{1}{2\sqrt{\lambda}}\overline{B}_\alpha \left(e^{\sqrt{\lambda}t} - e^{-\sqrt{\lambda}t}\right), \tag{4}$$

$$[y(t)]^\alpha = \left[\underline{y}(t, \alpha), \overline{y}(t, \alpha)\right]. \tag{5}$$

(1,2) solution: Since y is (1)-differentiable and y' is (2)-differentiable, taking the fuzzy Laplace transform of the equation (1),

$$-y'(0, \alpha) \ominus \left(-s^2\right) L\left(y(t, \alpha)\right) - sy(0, \alpha) = \lambda L\left(y(t, \alpha)\right)$$

is obtained. Then, we have the equations

$$-\overline{y}'(0, \alpha) + s^2 L\left(\overline{y}(t, \alpha)\right) - s\overline{y}(0, \alpha) = \lambda L\left(\underline{y}(t, \alpha)\right), \tag{6}$$

$$-\underline{y}'(0, \alpha) + s^2 L\left(\underline{y}(t, \alpha)\right) - s\underline{y}(0, \alpha) = \lambda L\left(\overline{y}(t, \alpha)\right). \tag{7}$$

If $L\left(\overline{y}(t, \alpha)\right)$ in the equation (7) is replaced by the equation (6) and using the initial conditions, we have

$$L\left(\underline{y}(t, \alpha)\right) = \frac{s^2}{s^4 - \lambda^2}\underline{B}_\alpha + \frac{\lambda}{s^4 - \lambda^2}\overline{B}_\alpha + \frac{s^3}{s^4 - \lambda^2}\underline{A}_\alpha + \frac{\lambda s}{s^4 - \lambda^2}\overline{A}_\alpha.$$

From this, the lower solution is obtained as

$$
\begin{aligned}
\underline{y}(t, \alpha) \;\; = \;\; & \frac{e^{\sqrt{\lambda}t}}{4}\left(\frac{\underline{B}_\alpha + \overline{B}_\alpha}{\sqrt{\lambda}} + \underline{A}_\alpha + \overline{A}_\alpha\right) \\
& + \frac{e^{-\sqrt{\lambda}t}}{4}\left(-\left(\frac{\underline{B}_\alpha + \overline{B}_\alpha}{\sqrt{\lambda}}\right) + \underline{A}_\alpha + \overline{A}_\alpha\right) \\
& + \frac{\sin\left(\sqrt{\lambda}t\right)}{2\sqrt{\lambda}}\left(\underline{B}_\alpha - \overline{B}_\alpha\right) + \frac{\cos\left(\sqrt{\lambda}t\right)}{2}\left(\underline{A}_\alpha - \overline{A}_\alpha\right).
\end{aligned}
\tag{8}
$$

Similarly, the upper solution is obtained as

$$
\begin{aligned}
\overline{y}(t, \alpha) \;\; = \;\; & \frac{e^{\sqrt{\lambda}t}}{4}\left(\frac{\underline{B}_\alpha + \overline{B}_\alpha}{\sqrt{\lambda}} + \underline{A}_\alpha + \overline{A}_\alpha\right) \\
& + \frac{e^{-\sqrt{\lambda}t}}{4}\left(-\left(\frac{\underline{B}_\alpha + \overline{B}_\alpha}{\sqrt{\lambda}}\right) + \underline{A}_\alpha + \overline{A}_\alpha\right) \\
& + \frac{\sin\left(\sqrt{\lambda}t\right)}{2\sqrt{\lambda}}\left(\overline{B}_\alpha - \underline{B}_\alpha\right) + \frac{\cos\left(\sqrt{\lambda}t\right)}{2}\left(\overline{A}_\alpha - \underline{A}_\alpha\right).
\end{aligned}
\tag{9}
$$

That is, $(1, 2)$ solution is

$$[y(t)]^{\alpha} = \left[\underline{y}(t, \alpha), \overline{y}(t, \alpha)\right], \tag{10}$$

where $\underline{y}(t, \alpha)$ is the equation (8) and $\overline{y}(t, \alpha)$ is the equation (9).

(2,1) solution: Since y is (2)-differentiable and y' is (1)-differentiable, from the equation

$$-sy(0, \alpha) \ominus \left(-s^2\right) L\left(y(t, \alpha)\right) \ominus y'(0, \alpha) = \lambda L\left(y(t, \alpha)\right)$$

and $y'(0, \alpha) = \left[\overline{y}'(0, \alpha), \underline{y}'(0, \alpha)\right]$, we have the equations

$$-s\overline{y}(0, \alpha) + s^2 L\left(\overline{y}(t, \alpha)\right) - \overline{y}'(0, \alpha) = \lambda L\left(\underline{y}(t, \alpha)\right), \tag{11}$$

$$-s\underline{y}(0, \alpha) + s^2 L\left(\underline{y}(t, \alpha)\right) - \underline{y}'(0, \alpha) = \lambda L\left(\overline{y}(t, \alpha)\right). \tag{12}$$

If $L\left(\overline{y}(t, \alpha)\right)$ in the equation (12) is replaced by the equation (11) and using the initial conditions, it gives the equation

$$L\left(\underline{y}(t, \alpha)\right) = \frac{\lambda s}{s^4 - \lambda^2}\overline{A}_{\alpha} + \frac{s^3}{s^4 - \lambda^2}\underline{A}_{\alpha} + \frac{s^2}{s^4 - \lambda^2}\overline{B}_{\alpha} + \frac{\lambda}{s^4 - \lambda^2}\underline{B}_{\alpha}.$$

From this, we have the lower solution

$$
\begin{aligned}
\underline{y}(t, \alpha) \;=\; & \frac{e^{\sqrt{\lambda}t}}{4}\left(\frac{\underline{B}_{\alpha} + \overline{B}_{\alpha}}{\sqrt{\lambda}} + \underline{A}_{\alpha} + \overline{A}_{\alpha}\right) \\
& + \frac{e^{-\sqrt{\lambda}t}}{4}\left(-\left(\frac{\underline{B}_{\alpha} + \overline{B}_{\alpha}}{\sqrt{\lambda}}\right) + \underline{A}_{\alpha} + \overline{A}_{\alpha}\right) \\
& + \frac{\sin\left(\sqrt{\lambda}t\right)}{2\sqrt{\lambda}}\left(\overline{B}_{\alpha} - \underline{B}_{\alpha}\right) + \frac{\cos\left(\sqrt{\lambda}t\right)}{2}\left(\underline{A}_{\alpha} - \overline{A}_{\alpha}\right).
\end{aligned}
\tag{13}
$$

Similarly, the upper solution is obtained as

$$
\begin{aligned}
\overline{y}(t, \alpha) \;=\; & \frac{e^{\sqrt{\lambda}t}}{4}\left(\frac{\underline{B}_{\alpha} + \overline{B}_{\alpha}}{\sqrt{\lambda}} + \underline{A}_{\alpha} + \overline{A}_{\alpha}\right) \\
& + \frac{e^{-\sqrt{\lambda}t}}{4}\left(-\left(\frac{\underline{B}_{\alpha} + \overline{B}_{\alpha}}{\sqrt{\lambda}}\right) + \underline{A}_{\alpha} + \overline{A}_{\alpha}\right) \\
& + \frac{\sin\left(\sqrt{\lambda}t\right)}{2\sqrt{\lambda}}\left(\underline{B}_{\alpha} - \overline{B}_{\alpha}\right) + \frac{\cos\left(\sqrt{\lambda}t\right)}{2}\left(\overline{A}_{\alpha} - \underline{A}_{\alpha}\right).
\end{aligned}
\tag{14}
$$

That is, $(2, 1)$ solution is

$$[y(t)]^{\alpha} = \left[\underline{y}(t, \alpha), \overline{y}(t, \alpha)\right], \tag{15}$$

where $\underline{y}(t, \alpha)$ is the equation (13) and $\overline{y}(t, \alpha)$ is the equation (14).

(2,2) solution: Since y and y' are (2)-differentiable,

$$s^2 L\left(y\left(t,\alpha\right)\right) \ominus sy\left(0,\alpha\right) - y'\left(0,\alpha\right) = \lambda L\left(y\left(t,\alpha\right)\right).$$

From this, we have the equations

$$s^2 L\left(\underline{y}\left(t,\alpha\right)\right) - s\underline{y}\left(0,\alpha\right) - \underline{y}'\left(0,\alpha\right) = \lambda\underline{y}\left(t,\alpha\right),$$

$$s^2 L\left(\overline{y}\left(t,\alpha\right)\right) - s\overline{y}\left(0,\alpha\right) - \overline{y}'\left(0,\alpha\right) = \lambda\overline{y}\left(t,\alpha\right).$$

Then, (2,2) solution is

$$\underline{y}\left(t,\alpha\right) = \frac{1}{2}\underline{A}_\alpha\left(e^{\sqrt{\lambda}t} + e^{-\sqrt{\lambda}t}\right) + \frac{1}{2\sqrt{\lambda}}\overline{B}_\alpha\left(e^{\sqrt{\lambda}t} - e^{-\sqrt{\lambda}t}\right), \qquad (16)$$

$$\overline{y}\left(t,\alpha\right) = \frac{1}{2}\overline{A}_\alpha\left(e^{\sqrt{\lambda}t} + e^{-\sqrt{\lambda}t}\right) + \frac{1}{2\sqrt{\lambda}}\underline{B}_\alpha\left(e^{\sqrt{\lambda}t} - e^{-\sqrt{\lambda}t}\right), \qquad (17)$$

$$\left[y\left(t\right)\right]^\alpha = \left[\underline{y}\left(t,\alpha\right), \overline{y}\left(t,\alpha\right)\right]. \qquad (18)$$

Theorem 3.1 *The* $(1,1)$ *solution of the initial value problem* $(1) - (2)$ *is a valid* $\alpha-$*level set.*

Proof Since

$$\frac{\partial \underline{y}\left(t,\alpha\right)}{\partial \alpha} = \frac{1}{4}\left(\overline{a} - \underline{a}\right)\left(e^{\sqrt{\lambda}t} + e^{-\sqrt{\lambda}t}\right) + \frac{1}{4\sqrt{\lambda}}\left(\overline{b} - \underline{b}\right)\left(e^{\sqrt{\lambda}t} - e^{-\sqrt{\lambda}t}\right) \geq 0,$$

$$\frac{\partial \overline{y}\left(t,\alpha\right)}{\partial \alpha} = -\frac{1}{4}\left(\overline{a} - \underline{a}\right)\left(e^{\sqrt{\lambda}t} + e^{-\sqrt{\lambda}t}\right) - \frac{1}{4\sqrt{\lambda}}\left(\overline{b} - \underline{b}\right)\left(e^{\sqrt{\lambda}t} - e^{-\sqrt{\lambda}t}\right) \leq 0,$$

$$\begin{aligned}\overline{y}\left(t,\alpha\right) - \underline{y}\left(t,\alpha\right) &= \left(1-\alpha\right)\left(\frac{1}{2}\left(\overline{a} - \underline{a}\right)\left(e^{\sqrt{\lambda}t} + e^{-\sqrt{\lambda}t}\right)\right.\\ &\left. \frac{1}{2\sqrt{\lambda}}\left(\overline{b} - \underline{b}\right)\left(e^{\sqrt{\lambda}t} - e^{-\sqrt{\lambda}t}\right)\right) \geq 0,\end{aligned}$$

Thus,(1,1) solution of the initial value problem (1)-(2) is a valid $\alpha-$level set. $\qquad \square$

Theorem 3.2 *The* $(1,2)$ *solution of the initial value problem* $(1) - (2)$ *is valid* $\alpha-$*level set, when*

$$t \geq \frac{1}{\sqrt{\lambda}}\tan^{-1}\left(-\sqrt{\lambda}\left(\frac{\overline{a} - \underline{a}}{\overline{b} - \underline{b}}\right)\right)$$

for $t \in \left(0, \frac{\pi}{2\sqrt{\lambda}}\right).$

Proof If

$$\frac{\partial \underline{y}\left(t,\alpha\right)}{\partial \alpha} \geq 0, \quad \frac{\partial \overline{y}\left(t,\alpha\right)}{\partial \alpha} \leq 0, \quad \underline{y}\left(t,\alpha\right) \leq \overline{y}\left(t,\alpha\right),$$

the $(1,2)$ solution of the initial value problem (1)-(2) is valid $\alpha-$level set. Thus, it must be

$$\sin\left(\sqrt{\lambda}t\right)\left(\overline{b}-\underline{b}\right)+\sqrt{\lambda}\cos\left(\sqrt{\lambda}t\right)\left(\overline{a}-\underline{a}\right)\geq 0.$$

For $\sqrt{\lambda}t\in\left(0,\frac{\pi}{2}\right)\Rightarrow t\in\left(0,\frac{\pi}{2\sqrt{\lambda}}\right)$, we have

$$\sqrt{\lambda}t\geq\tan\left(-\sqrt{\lambda}\left(\frac{\overline{a}-\underline{a}}{\overline{b}-\underline{b}}\right)\right)\Rightarrow t\geq\frac{1}{\sqrt{\lambda}}\tan^{-1}\left(-\sqrt{\lambda}\left(\frac{\overline{a}-\underline{a}}{\overline{b}-\underline{b}}\right)\right).$$

This completes the proof.   □

Theorem 3.3   *The $(2,1)$ solution of the initial value problem $(1)-(2)$ is valid $\alpha-$level set, when*

$$t\leq\frac{1}{\sqrt{\lambda}}\tan^{-1}\left(\sqrt{\lambda}\left(\frac{\overline{a}-\underline{a}}{\overline{b}-\underline{b}}\right)\right)$$

for $t\in\left(0,\frac{\pi}{2\sqrt{\lambda}}\right)$.

    Proof The proof is similar to Theorem 3.1 and 3.2.   □

Theorem 3.4   *The $(2,2)$ solution of the initial value problem $(1)-(2)$ is a valid $\alpha-$level set for $t>0$ satisfying the inequality*

$$\frac{e^{\sqrt{\lambda}t}+e^{-\sqrt{\lambda}t}}{e^{\sqrt{\lambda}t}-e^{-\sqrt{\lambda}t}}\geq\frac{\overline{b}-\underline{b}}{\sqrt{\lambda}\left(\overline{a}-\underline{a}\right)}$$

    Proof The proof is similar to Theorem 3.1 and 3.2.   □

Theorem 3.5   *All of the solutions are symmetric triangular fuzzy numbers for any $t>0$.*

    Proof For (1,1) solution, since

$$\underline{y}\left(t,1\right)=\frac{1}{4}\left(\overline{a}+\underline{a}\right)\left(e^{\sqrt{\lambda}t}+e^{-\sqrt{\lambda}t}\right)+\frac{1}{4\sqrt{\lambda}}\left(\overline{b}+\underline{b}\right)\left(e^{\sqrt{\lambda}t}-e^{-\sqrt{\lambda}t}\right)=\overline{y}\left(t,1\right)$$

and

$$\begin{aligned}\underline{y}\left(t,1\right)-\underline{y}\left(t,\alpha\right)&=\left(1-\alpha\right)\left(\frac{1}{4}\left(\overline{a}-\underline{a}\right)\left(e^{\sqrt{\lambda}t}+e^{-\sqrt{\lambda}t}\right)\right.\\&\quad\left.\frac{1}{4\sqrt{\lambda}}\left(\overline{b}-\underline{b}\right)\left(e^{\sqrt{\lambda}t}-e^{-\sqrt{\lambda}t}\right)\right)\\&=\overline{y}\left(t,\alpha\right)-\overline{y}\left(t,1\right),\end{aligned}$$

the (1,1) solution of the initial value problem (1)-(2) is a symmetric triangular fuzzy number for any $t>0$.

For (1,2) solution, since

$$
\begin{aligned}
\underline{y}(t,1) &= e^{\sqrt{\lambda}t}\left(\frac{1}{4\sqrt{\lambda}}\left(\overline{b}+\underline{b}\right)+\frac{1}{4}\left(\overline{a}+\underline{a}\right)\right) \\
&\quad +e^{-\sqrt{\lambda}t}\left(-\frac{1}{4\sqrt{\lambda}}\left(\overline{b}+\underline{b}\right)+\frac{1}{4}\left(\overline{a}+\underline{a}\right)\right) \\
&= \overline{y}(t,1)
\end{aligned}
$$

and

$$
\begin{aligned}
\underline{y}(t,1)-\underline{y}(t,\alpha) &= (1-\alpha)\left(\frac{\sin\left(\sqrt{\lambda}t\right)}{2\sqrt{\lambda}}\left(\overline{b}-\underline{b}\right)+\frac{\cos\left(\sqrt{\lambda}t\right)}{2}\left(\overline{a}-\underline{a}\right)\right) \\
&= \overline{y}(t,\alpha)-\overline{y}(t,1),
\end{aligned}
$$

the (1,2) solution of the initial value problem (1)-(2) is a symmetric triangular fuzzy number for any $t>0$.

For the cases of (1,2) and (2,2) solutions, the proof is similar.   □

Example 3.6 Consider the solutions of the fuzzy initial value problem

$$
y''(t)=y(t),\quad y(0)=[1]^{\alpha},\quad y'(0)=[0]^{\alpha} \tag{19}
$$

by fuzzy Laplace transform, where $[1]^{\alpha}=[\alpha,2-\alpha]$, $[0]^{\alpha}=[-1+\alpha,1-\alpha]$.

Its (1,1) solution is

$$
\underline{y}(t,\alpha)=\frac{1}{2}\left(\alpha\left(e^{t}+e^{-t}\right)+(-1+\alpha)\left(e^{t}-e^{-t}\right)\right),
$$

$$
\overline{y}(t,\alpha)=\frac{1}{2}\left((2-\alpha)\left(e^{t}+e^{-t}\right)+(1-\alpha)\left(e^{t}-e^{-t}\right)\right),
$$

$$
[y(t)]^{\alpha}=\left[\underline{y}(t,\alpha),\overline{y}(t,\alpha)\right].
$$

Its (1,2) solution is

$$
\underline{y}(t,\alpha)=\frac{1}{2}\left(e^{t}+e^{-t}\right)+(\alpha-1)\left(\sin(t)+\cos(t)\right),
$$

$$
\overline{y}(t,\alpha)=\frac{1}{2}\left(e^{t}+e^{-t}\right)+(1-\alpha)\left(\sin(t)+\cos(t)\right),
$$

$$
[y(t)]^{\alpha}=\left[\underline{y}(t,\alpha),\overline{y}(t,\alpha)\right].
$$

Its (2,1) solution is

$$
\underline{y}(t,\alpha)=\frac{1}{2}\left(e^{t}+e^{-t}\right)+(1-\alpha)\left(\sin(t)-\cos(t)\right),
$$

$$
\overline{y}(t,\alpha)=\frac{1}{2}\left(e^{t}+e^{-t}\right)+(\alpha-1)\left(\sin(t)-\cos(t)\right),
$$

$$[y(t)]^\alpha = \left[\underline{y}(t,\alpha), \overline{y}(t,\alpha)\right].$$

and its (2,2) solution is

$$\underline{y}(t,\alpha) = \frac{1}{2}\left(\alpha\left(e^t + e^{-t}\right) + (1-\alpha)\left(e^t - e^{-t}\right)\right),$$

$$\overline{y}(t,\alpha) = \frac{1}{2}\left((2-\alpha)\left(e^t + e^{-t}\right) + (-1+\alpha)\left(e^t - e^{-t}\right)\right).$$

$$[y(t)]^\alpha = \left[\underline{y}(t,\alpha), \overline{y}(t,\alpha)\right].$$

According to Theorem 3.1 and 3.2, (1,1) solution is a valid $\alpha-$level set, and (1,2) solution is a valid $\alpha-$level set since the function $f(t) > 0$ for $t \in \left(0, \frac{\pi}{2}\right)$ in Figure 1. By Theorem 3.3, (2,1) solution is a valid $\alpha-$level set for $t \in [0, 0.785398]$ since the function $g(t) \leq 0$ in Figure 2. That is, (1,2) solution is not solution of the problem. Also by Theorem 3.4, (2,2) solution is a valid $\alpha-$level set since the function $h(t) > 0$ in Figure 3. All of the solutions are symmetric triangular fuzzy numbers for any $t > 0$. We can see that the graphics of solutions in Figure 4-Figure 7. Also, we can see that in Figure 4, (1,1) solution is fuzzier as time goes by and (2,1) is not a valid fuzzy function for $t \geq 0.785398$ in Figure 6.
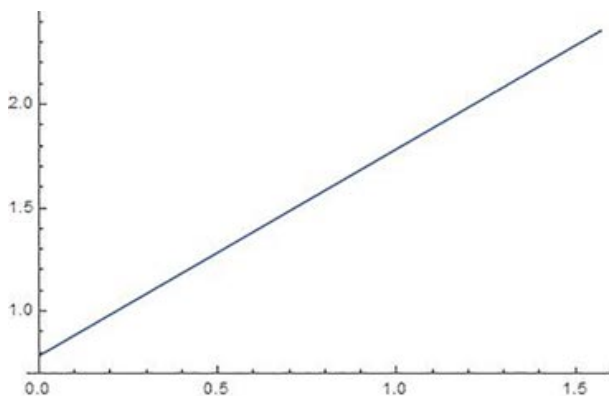
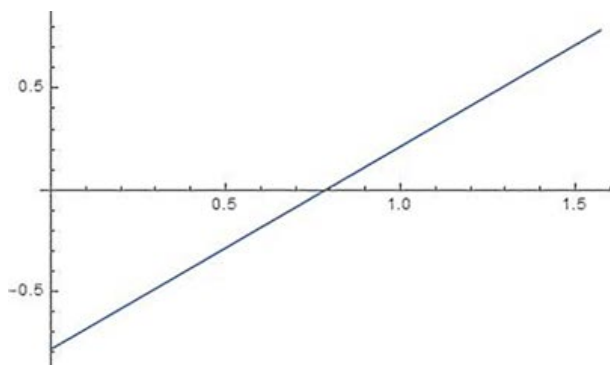**Figure 1** Graphic of the function $f(t) = t - \tan^{-1}(-1)$

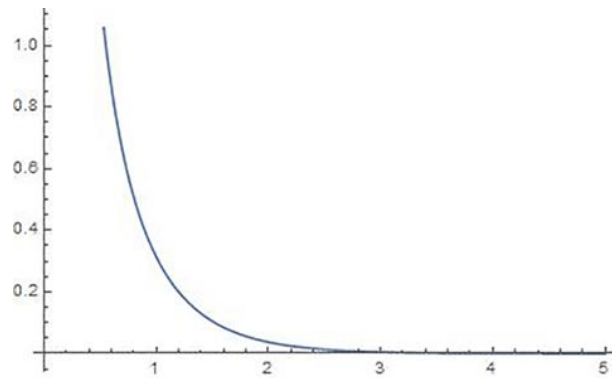**Figure 2** Graphic of the function $g(t) = t - \tan^{-1}(1)$

Figure 3 Graphic of the function $h(t) = \frac{e^t + e^{-t}}{e^t - e^{-t}} - 1$
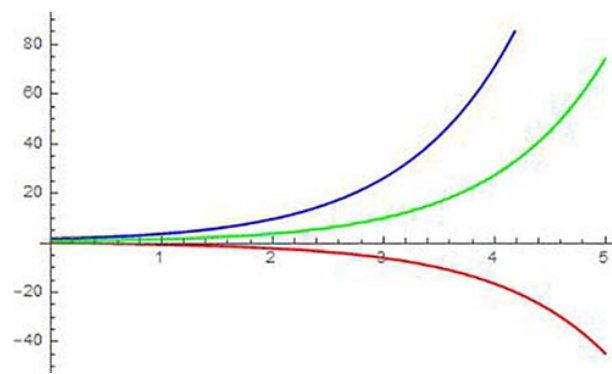

Figure 4 Graphic of (1,1) solution for $\alpha = 0.2$
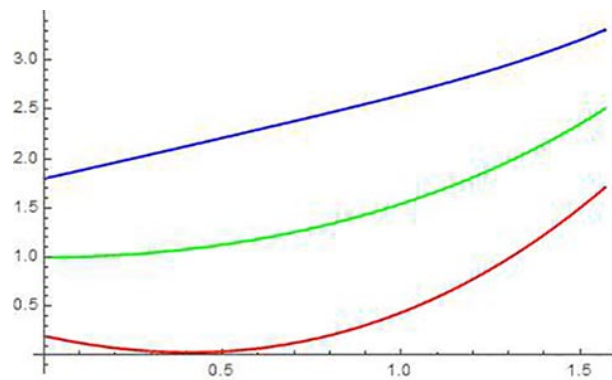

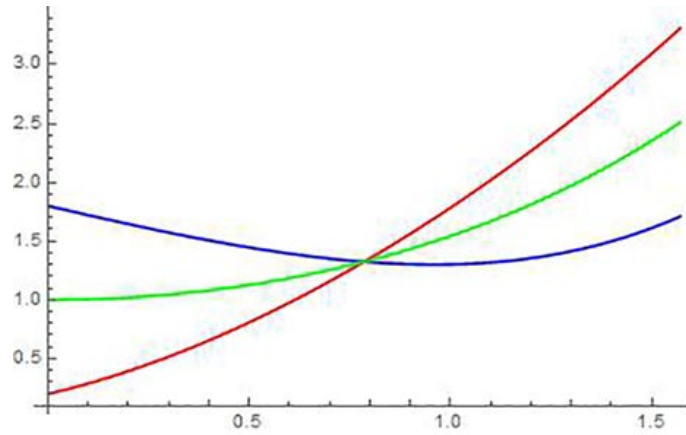**Figure 5** Graphic of (1,2) solution for $\alpha = 0.2$

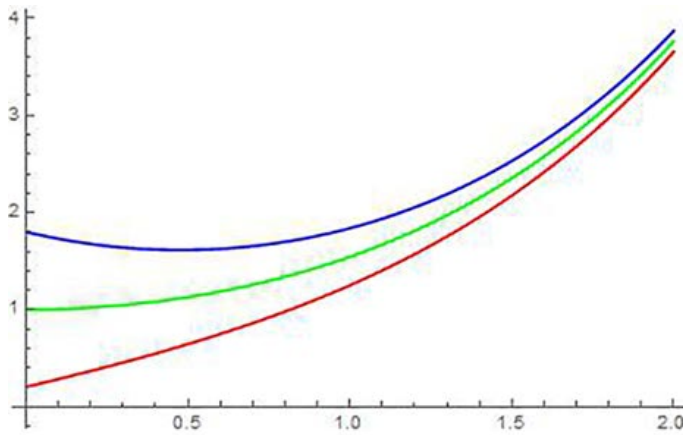**Figure 6** Graphic of (2,1) solution for $\alpha = 0.2$

Figure 7 Graphic of (2,2) solution for $\alpha = 0.2$

$$\text{Blue} \to \overline{y}_\alpha(t)$$
$$\text{Red} \to \underline{y}_\alpha(t)$$
$$\text{Green} \to \overline{y}_1(t) = \underline{y}_1(t)$$

§4. Conclusions

In this paper, the solutions of second-order fuzzy initial value problem with positive coefficient are investigated by fuzzy Laplace transform. Generalized differentiability, Hukuhara difference and fuzzy arithmetic are used. Solutions are found by fuzzy Laplace transform using the generalized differentiability. It is shown that whether the solutions valid fuzzy functions or not. Studied problem is shown on an example. Graphics of found solutions are drawn. It is found that (1,1), (1,2) and (2,2) solutions are valid fuzzy level sets and (2,1) solution is a valid fuzzy level set for $t \in [0, 0.785398]$. But (1,1) solution has a drawback : it is fuzzier as time goes by.

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