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Triangular Difference Mean Graphs

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Abstract: In this paper, we define a new labeling namely triangular difference mean labeling and investigate triangular difference mean behaviours of some standard graphs. A triangular difference mean labeling of a graph $G = (p, q)$ is an injection $f : V \longrightarrow Z^+$, where Z^+ is a set of positive integers such that for each edge $e = uv$, the edge labels are defined as

$$f^*(e) = \left\lceil \frac{|f(u) - f(v)|}{2} \right\rceil$$

such that the values of the edges are the first q triangular numbers. A graph that admits a triangular difference mean labeling is called a triangular difference mean graph.

Key Words: Mean labeling, triangular difference mean labeling, Smarandachely k-triangular labeling, triangular difference mean graph.

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§1. Introduction

By a graph, we mean a finite, simple and undirected one. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. Terms and notations not defined here are used in the sense of Harary [2] and for number theory we follow Burton[1]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. There are several types of graph labeling and an excellent survey on graph labeling can be found in [3]. The notion of triangular mean labeling was due to Seenivasan et al. [7]. Let $G = (V, E)$ be a graph with p vertices and q edges. Consider an injection $f : V(G) \longrightarrow \{0, 1, 2, \cdots, T_q\}$, where T_q is the q^{th} triangular number. Define $f^* : E(G) \longrightarrow \{1, 3, \cdots, T_q\}$ such that $f^*(e) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ for all edges $e = uv$. If $f^*(E(G))$ is a sequence of consecutive triangular numbers $T_1, T_2, \cdots, T_q$, then the function f is said to be triangular mean labeling. Generally, If there are only k consecutive triangular numbers $T_i, T_{i+1}, \cdots, T_{i+k-1}$ with $k \leq q$ in $f^*(E(G))$, such a f is called a Smarandachely k-triangular labeling. A graph that admits a triangular mean labeling or Smarandachely k-triangular labeling is called a triangular mean graph or a Smarandachely

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k-triangular mean graph.

Murugan et al.[4] introduced skolem difference mean labeling and some standard results on skolem difference mean labeling were proved in [5] and [6]. A graph $G = (V, E)$ with p vertices and q edges is said to have skolem difference mean labeling if it is possible to label the vertices $x \in V$ with distinct elements $f(x)$ from $\{1, 2, 3, \cdots, p + q\}$ in such a way that for each edge $e = uv$, let $f^*(e) = \left\lceil \frac{|f(u)-f(v)|}{2} \right\rceil$ and the resulting labels of the edges are distinct and are $1, 2, 3, \cdots, q$. A graph that admits a skolem difference mean labeling is called a skolem difference mean graph.

Motivated by the concepts in [7] and [4], we define a new labeling namely triangular difference mean labeling. A triangular difference mean labeling of a graph $G = (p, q)$ is an injection $f : V \longrightarrow Z^+$, where Z^+ is a set of positive integers such that for each edge $e = uv$, the edge labels are defined as $f^*(e) = \left\lceil \frac{|f(u)-f(v)|}{2} \right\rceil$ such that the values of the edges are the first q triangular numbers. A graph that admits a triangular difference mean labeling is called a triangular difference mean graph. We use the following definitions in the subsequent sequel.

Definition 1.1 *A vertex of degree one is called a pendant vertex and a pendant edge is an edge incident with a pendant vertex. The corona $G_1 \odot G_2$ of the graphs G_1 and G_2 is obtained by taking one copy of G_1 (with p vertices) and p copies of G_2 and then join the i^{th} vertex of G_1 to every vertex of the i^{th} copy of G_2.*

Definition 1.2 *The bistar $B_{m,n}$ is a graph obtained from K_2 by joining m pendant edges to one end of K_2 and n pendant edges to the other end of K_2.*

Definition 1.3 *The graph $C_n@P_m$ is obtained by identifying one pendant vertex of the path P_m to a vertex of the cycle C_n.*

Definition 1.4 *A triangular number is a number obtained by adding all positive integers less than or equal to a given positive integer n. If the n^{th} triangular number is denoted by T_n, then $T_n = \frac{1}{2}n(n + 1)$.*

§2. Triangular Difference Mean Graphs

In this section, we establish that path $P_n (n \geq 1)$, $K_{1,n} (n \geq 1)$, $P_n \odot K_1 (n \geq 2)$, $B_{m,n} (m \geq 1,\ n \geq 1)$, $T(n, m)$, $S(\underbrace{n, n, \cdots, n}_{\text{m times}})$, $C_n (n > 3)$ and $C_n@P_m (n \geq 4, m \geq 2)$ admit triangular difference mean labeling . Further, we prove that C_3 is not a triangular difference mean graph.

Theorem 2.1 *Any path $P_n (n \geq 1)$ is a triangular difference mean graph.*

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices of the path P_n. Then $E(P_n) = \{e_i = v_i v_{i+1} : 1 \leq i \leq n - 1\}$. Define $f : V(P_n) \longrightarrow Z^+$ as follows:

$f(v_1) = 1$ and $f(v_i) = 2(T_1 + T_2 + \cdots + T_{i-1}) + 1$ for $2 \leq i \leq n$.

For the vertex label f, the induced edge label f^* is as follows:

$f^*(e_i) = T_i$ for $1 \leq i \leq n-1$. Hence P_n is a triangular difference mean graph. □

The triangular difference mean labeling of P_5 is given in Figure 1.

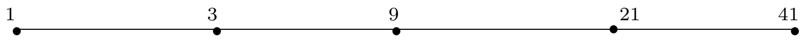

Figure 1

Theorem 2.2 *The star graph $K_{1,n}(n \geq 1)$ admits triangular difference mean labeling.*

Proof Let v be the apex vertex and $v_1, v_2, \cdots, v_n$ be the pendant vertices of the star $K_{1,n}$. Then $E(K_{1,n}) = \{vv_i : 1 \leq i \leq n\}$. Define $f : V(K_{1,n}) \longrightarrow Z^+$ as follows:

$f(v) = 1, f(v_i) = 2T_i + 1$ for $1 \leq i \leq n$.

For the vertex label f, the induced edge label f^* is as follows:

$f^*(vv_i) = T_i$ for $1 \leq i \leq n$.

Then the induced edge labels are the triangular numbers $T_1, T_2, \cdots, T_n$. Hence $K_{1,n}$is a triangular difference mean graph. □

The triangular difference mean labeling of $K_{1,8}$ is shown in Figure 2.

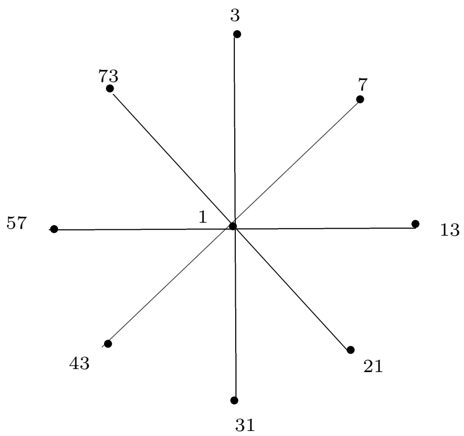

Figure 2

Theorem 2.3 *The comb graph $P_n \odot K_1(n \geq 2)$ admits triangular difference mean labeling.*

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices of the path P_n and $u_1, u_2, \cdots, u_n$ be the pendant vertices adjacent to $v_1, v_2, \cdots, v_n$ respectively. Then $E(P_n \odot K_1) = \{e_i = v_iv_{i+1}, e'_j = u_jv_j : 1 \leq i \leq n-1, 1 \leq j \leq n\}$. Define $f : V(P_n \odot K_1) \longrightarrow Z^+$ as follows:

$f(v_1) = 1, f(v_i) = 2(T_1 + T_2 + \cdots + T_{i-1}) + 1$ for $2 \leq i \leq n, f(u_1) = 2T_n$;
$f(u_i) = 2(T_1 + T_2 + \cdots + T_{i-1}) + 2T_{n+i-1} + 1$ for $2 \leq i \leq n$.

For the vertex label f, the induced edge label f^* is as follows:

$f^*(e_i) = T_i$ for $1 \leq i \leq n-1$, $f^*(e'_j) = T_{n+j-1}$ for $1 \leq j \leq n$.

Then the edge labels are the triangular numbers: $T_1, T_2, \cdots, T_{2n-1}$. Hence $P_n \odot K_1$ is a triangular difference mean graph. □

The triangular difference mean labeling of $P_5 \odot K_1$ is shown in Figure 3.

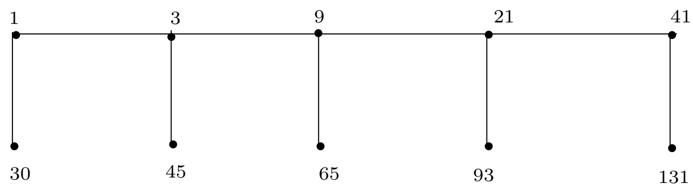

Figure 3

Theorem 2.4 *The bistar $B_{m,n}(m \geq 1,\ n \geq 1)$ is a triangular difference mean graph.*

Proof Let $V(B_{m,n}) = \{u, v, u_i, v_j : 1 \leq i \leq m,\ 1 \leq j \leq n\}$ and $E(B_{m,n}) = \{uv, uu_i, vv_j : 1 \leq i \leq m,\ 1 \leq j \leq n\}$. Define $f : V(B_{m,n}) \longrightarrow Z^+$ as follows:

$f(u) = 1, f(v) = 3, f(u_i) = 2T_{i+1} + 1$ for $1 \leq i \leq m$;
$f(v_j) = 2T_{m+j+1} + 3$ for $1 \leq j \leq n$.

For the vertex label f, the induced edge label f^* is as follows:

$f^*(uv) = T_1,\ f^*(uu_i) = T_{i+1}$ for $1 \leq i \leq m$;
$f^*(vv_j) = T_{m+j+1}$ for $1 \leq j \leq n$.

The induced edge labels are the first $m + n + 1$ triangular numbers and hence $B_{m,n}$ is a triangular difference mean graph. □

The triangular difference mean labeling of $B_{4,5}$ is shown in Figure 4.

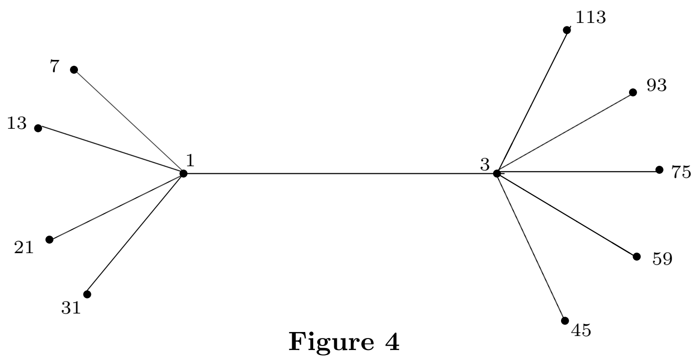

Figure 4

Theorem 2.5 *A graph obtained by joining the roots of different stars to a new vertex, is a triangular difference mean graph.*

Proof Let $K_{1,n_1}, K_{1,n_2}, \cdots, K_{1,n_k}$ be k stars. Let G be a graph obtained by joining the central vertices of the stars to a new vertex u.

Assign 1 to u; $2T_1+1, 2T_2+1, \cdots, 2T_k+1$ to the central vertices of the stars; $2T_{k+1}+2T_1+1, 2T_{k+2}+2T_1+1, \cdots, 2T_{k+n_1}+2T_1+1$ to the pendant vertices of the first star; $2T_{k+n_1+1}+2T_2+1, 2T_{k+n_1+2}+2T_2+1, \cdots, 2T_{k+n_1+n_2}+2T_2+1$ to the pendant vertices of the second star and so on, finally assign the numbers $2T_{k+n_1+n_2+\cdots+n_{k-1}+1}+2T_k+1, 2T_{k+n_1+n_2+\cdots+n_{k-1}+2}+2T_k+1, \cdots, 2T_{k+n_1+n_2+\cdots,+n_{k-1}+n_k}+2T_k+1$ to the pendant vertices of the last star. Then, the edge labels are the triangular numbers $T_1, T_2, \cdots, T_{k+n_1+n_2+\cdots+n_{k-1}+n_k}$ and also the vertex labels are all different. □

The triangular difference mean labeling of the tree given in Theorem 2.5 with $k = 3, n_1 = 4, n_2 = 5$ and $n_3 = 4$ is shown in Figure 5.

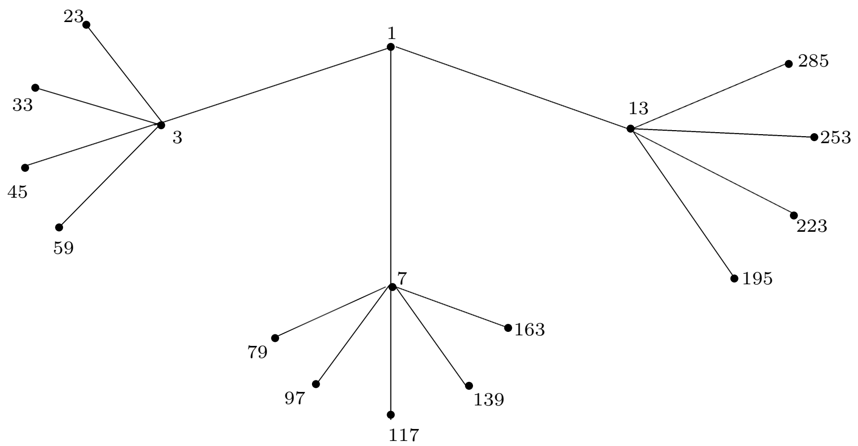

Figure 5

Theorem 2.6 *A tree $T(n,m)$, obtained by identifying a central vertex of a star with a pendant vertex of a path , is a triangular difference mean graph.*

Proof Let $v_0, v_1, v_2, \cdots, v_n$ be the vertices of the path $P_n$having path length $n (n \geq 1)$ and $u, u_1, u_2, \cdots, u_m$ be the vertices of the star $K_{1,m}$. Let $T(n,m)$ be a tree obtained by identifying v_0 with u.

Define $f : V(T(n,m)) \longrightarrow Z^+$ as follows:

$f(v_0) = 1$, $f(u_i) = 2T_i + 1$ for $1 \leq i \leq m$,

$f(v_j) = 2(T_{m+1} + T_{m+2} + \ldots + T_{m+j}) + 1$ for $1 \leq j \leq n$.

For a vertex label f, the induced edge label f^* is as follows:

$f^*(v_0u_i) = T_i$ for $1 \leq i \leq m$;

$f^*(v_{j-1}v_j) = T_{m+j}$ for $1 \leq j \leq n$.

Then the induced edge labels are the first $m + n$ triangular numbers. Hence the tree $T(n,m)$ admits a triangular difference mean labeling. □

The triangular difference mean labeling of a tree $T(3,7)$ is shown in Figure 6.

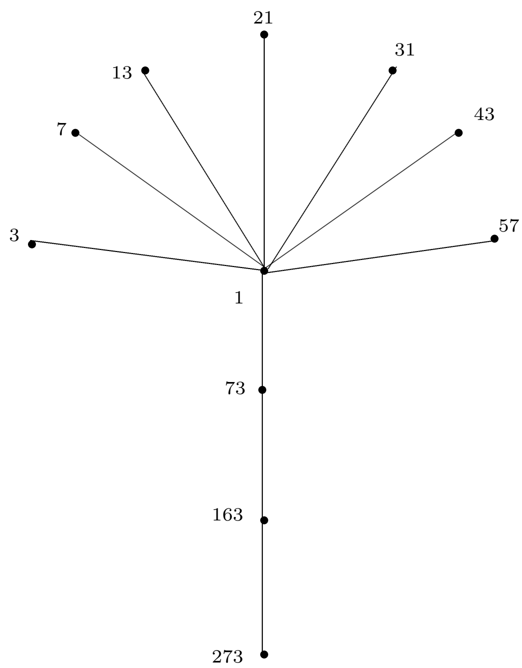

Figure 6

Theorem 2.7 *The caterpillar* $S(\underbrace{n,n,\cdots,n}_{mtimes})$ *is a triangular difference mean graph.*

Proof Let $v_1, v_2, \cdots, v_m$ be the vertices of the path P_m and $v_j^i (1 \leq i \leq n,\ 1 \leq j \leq m)$ be the pendant vertices incident with $v_j (1 \leq j \leq m)$.
Then $V(S(\underbrace{n,n,\cdots,n}_{\text{m times}})) = \{v_j, v_i^j :\ 1 \leq i \leq n,\ 1 \leq j \leq m\}$ and $E(S(\underbrace{n,n,\cdots,n}_{\text{m times}})) = \{v_t v_{t+1},\ v_j v_j^i :\ 1 \leq t \leq m-1,\ 1 \leq i \leq n,\ 1 \leq j \leq m\}$.

Define $f : V(S(\underbrace{n,n,\cdots,n}_{\text{m times}})) \longrightarrow Z^+$ as follows:

$f(v_1) = 1,\ f(v_j) = 2(T_1 + T_2 + \cdots + T_{j-1}) + 1$ for $2 \leq j \leq m$;
$f(v_j^i) = f(v_j) + 2T_{m+(j-1)n+i-1}$ for $1 \leq j \leq m$ and $1 \leq i \leq n$.

For each vertex label f, the induced edge label f^* is as follows:

$f^*(v_j v_{j+1}) = T_j$ for $1 \leq j \leq m-1$;
$f^*(v_j v_j^i) = T_{m+(j-1)n+i-1}$ for $1 \leq j \leq m$ and $1 \leq i \leq n$.

Then the edge labels are the triangular numbers $T_1, T_2, \cdots, T_{m-1}, T_m, \cdots, T_{m+n-1}$ and also the vertex labels are different. Hence $S((\underbrace{n,n,\cdots,n}_{\text{m times}}))$ is a triangular difference mean graph. □

Theorem 2.8 *Every cycle* $C_n (n > 3)$ *is a triangular difference mean graph.*

Proof We prove this theorem in two cases.

Case 1. $n = 4m + 1$.

Let $S = \left\lceil \frac{1}{2}\sum\limits_{i=1}^{n} T_i \right\rceil$. Select some of the $T_i^{'s}$ namely $T_{l_1}, T_{l_2}, \cdots, T_{l_k}$ from $T_1, T_2, \cdots, T_n$ such that $\sum\limits_{i=1}^{k} T_{l_i} = S$, where $k < n$ and assume $T_{l_1} > T_{l_2} > \cdots > T_{l_k}$. Then the remaining $T_i^{'s}$ namely, $T_{l_{k+1}}, T_{l_{k+2}}, \cdots, T_{l_n}$ are such that $T_{l_{k+1}} > T_{l_{k+2}} > \cdots, > T_{l_n}$ and $\sum\limits_{i=k+1}^{n} T_{l_i} = S - 1$. Let $v_1, v_2, \cdots, v_{k-1}, v_k, v_{k+1}, \cdots, v_n$ be the vertices of C_n. Label the first $k+1$ vertices $v_1, v_2, \ldots, v_{k+1}$ as follows:

$f(v_1) = 1,\ f(v_2) = 2T_{l_1},\ f(v_3) = 2T_{l_1} + 2T_{l_2} - 1;$

$f(v_4) = 2T_{l_1} + 2T_{l_2} + 2T_{l_3} - 1, \cdots, f(v_{k+1}) = 2T_{l_1} + 2T_{l_2} + \cdots + 2T_{l_k} - 1$ and then,

$f(v_{k+2}) = 2T_{l_1} + 2T_{l_2} + \cdots + 2T_{l_k} - 2T_{l_{k+1}} - 1;$

$f(v_{k+3}) = 2T_{l_1} + 2T_{l_2} + \cdots + 2T_{l_k} - 2T_{l_{k+1}} - 2T_{l_{k+2}} - 1, \cdots;$

$f(v_n) = 2T_{l_1} + 2T_{l_2} + \cdots + 2T_{l_k} - 2T_{l_{k+1}} - 2T_{l_{k+2}} - \cdots - 2T_{l_{n-1}} - 1.$

Hence, the edge labels are the triangular numbers $\{T_{l_1}, T_{l_2}, \cdots, T_{l_{k-1}}, T_{l_k}, T_{l_{k+1}}, \cdots, T_{l_n}\} = \{T_1, T_2, \cdots, T_n\}$ and also the vertex labels are all different.

Case 2. $n \neq 4m + 1, m \geqslant 1$.

Let $S = \left\lceil \frac{1}{2}\sum\limits_{i=1}^{n} T_i \right\rceil$. Select some of the T'^{s}_{i} namely $T_{l_1}, T_{l_2}, \cdots, T_{l_k}$ from $T_1, T_2, \cdots, T_n$ such that $\sum\limits_{i=1}^{k} T_{l_i} = S$, where $k < n$ and assume $T_{l_1} > T_{l_2} > \cdots > T_{l_k}$. Then the remaining T'^{s}_{i} namely, $T_{l_{k+1}}, T_{l_{k+2}}, \cdots, T_{l_n}$ are such that $T_{l_{k+1}} > T_{l_{k+2}} > \cdots > T_{l_n}$ and $\sum\limits_{i=k+1}^{n} T_{l_i} = S$. Let $v_1, v_2, \cdots, v_{k-1}, v_k, v_{k+1}, \cdots, v_n$ be the vertices of C_n. We label the vertices $v_1, v_2, \cdots, v_n$ as follows:

$f(v_1) = 1,\ f(v_2) = 2T_{l_1} + 1,\ f(v_3) = 2T_{l_1} + 2T_{l_2} + 1;$

$f(v_4) = 2T_{l_1} + 2T_{l_2} + 2T_{l_3} + 1, \cdots;$

$f(v_{k+1}) = 2T_{l_1} + 2T_{l_2} + \cdots + 2T_{l_k} + 1;$

$f(v_{k+2}) = 2T_{l_1} + 2T_{l_2} + \cdots + 2T_{l_k} - 2T_{l_{k+1}} + 1;$

$f(v_{k+3}) = 2T_{l_1} + 2T_{l_2} + \cdots, T_{l_k} - 2T_{l_{k+1}} - 2T_{l_{k+2}} + 1, \cdots;$

$f(v_n) = 2T_{l_1} + 2T_{l_2} + \cdots + 2T_{l_k} - 2T_{l_{k+1}} - 2T_{l_{k+2}} - \cdots - 2T_{l_{n-1}} + 1.$

Thus, the edge labels are the triangular numbers $\{T_{l_1}, T_{l_2}, \cdots, T_{l_{k-1}}, T_{l_k}, T_{l_{k+1}}, \cdots, T_{l_n}\}$ and also the vertex labels are all different. □

The triangular difference mean labeling of C_6 is shown in Figure 8.

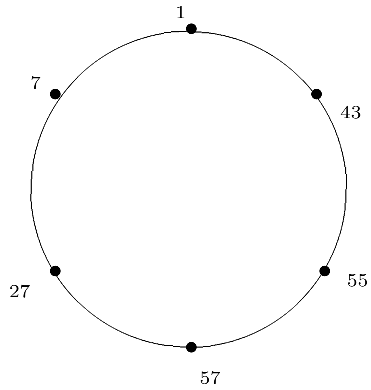

Figure 8

Theorem 2.9 *The graph $C_n@P_m (n \geq 4, m \geq 2)$ is a triangular difference mean graph.*

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices of the cycle C_n and $u_1, u_2, \cdots, u_m$ be the vertices of the path P_m. The graph $C_n@P_m$ is obtained by identifying the vertexu_1 with the vertex v_1. We label the vertices of C_n as in Theorem 2.9 and assign the number $2T_{n+1}+2T_{n+2}+\cdots+2T_{n+j-1}+1$ to vertex u_j of the path P_m for $2 \leq j \leq m$. Then the induced edge labels are the first $m+n-1$ triangular numbers. Hence, $C_n@P_m$ is a triangular difference mean graph. □

The triangular difference mean labeling of $C_4@P_3$ is shown in Figure 9.

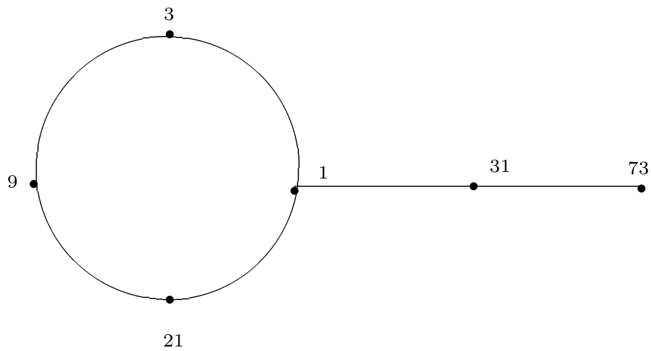

Figure 9

Theorem 2.10 *The cycle C_3 is not a triangular difference mean graph.*

Proof Suppose C_3 is a triangular difference mean graph with triangular difference mean labeling f. Let the vertices of C_3 be u, v, w. Let $f(u) = x$. Then to get 1 as an edge label we must have $f(v) \in \{x+1, x+2, x-1, x-2\}$. To get $T_2, f(w) \in \{x+5, x+6, x-5, x-6\}$ or $f(w) \in \{x-6, x-7, x+4, x+5\}$. Then we get either $\{1, 3, 2\}$ or $\{1, 3, 4\}$ as the set of induced edge labels. Therefore, $T_3 = 6$ can not be an edge label of C_3. Hence C_3 is not a triangular difference mean graph. □

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