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Mannheim Partner D-Curves in Minkowski 3-Space

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Abstract: In this paper, we give the definition, different types and characterizations of Mannheim partner D-curves in Minkowski 3-space E_1^3. We find the relations between the geodesic curvatures, the normal curvatures and the geodesic torsions of these associated curves. Furthermore, we show that the definition and the characterizations of Mannheim partner D-curves include those of Mannheim partner curves in some special cases in Minkowski 3-space E_1^3.

Key Words: Minkowski 3-space, Mannheim partner D-curves, Darboux frame.

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§1. Introduction

In the study of the fundamental theory and the characterizations of space curves, the related curves for which there exist corresponding relations between the curves are very interesting and an important problem. The most fascinating examples of such curves are associated curves, the curves for which at the corresponding points of them one of the Frenet vectors of a curve coincides with the one of the Frenet vectors of the other curve. The well known of the associated curves is Bertrand curve which is characterized as a kind of corresponding relation between the two curves. The relation is that the principal normal of a curve is the principal normal of another curve i.e, the Bertrand curve is a curve which shares the normal line with another curve. Over years many mathematicians have studied on Bertrand curves in different spaces and consider the properties of these curves [1-6].

Furthermore, Bertrand curves are not only the example of associated curves. Recently, a new definition of the associated curves was given by Liu and Wang [9,17]. They called these new curves as Mannheim partner curves: Let x and x_1 be two curves in the three dimensional Euclidean E^3. If there exists a corresponding relationship between the space curves x and x_1 such that, at the corresponding points of the curves, the principal normal lines of x coincides with the binormal lines of x_1, then x is called a Mannheim curve, and x_1 is called a Mannheim

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partner curve of x. The pair $\{x, x_1\}$is said to be a Mannheim pair. They showed that the curve $x_1(s_1)$ is the Mannheim partner curve of the curve $x(s)$ if and only if the curvature κ_1 and the torsion τ_1 of $x_1(s_1)$ satisfy the following equation

$$\dot{\tau} = \frac{d\tau}{ds_1} = \frac{\kappa_1}{\lambda}(1 + \lambda^2 \tau_1^2)$$

for some non-zero constant λ. They also study the Mannheim curves in Minkowski 3-space[9,16]. Some different characterizations of Mannheim partner curves are given by Orbay and others [12]. The differential geometry of the curves fully lying on a surface in Minkowski 3-spaceE_1^3 is given by Ugurlu, Kocayigit and Topal [8,14,15]. They have given the Darboux frame of the curves according to the Lorentzian characters of surfaces and the curves. Finally, in the Euclidean 3-space, Mannheim partner D -curves is defined by Kazaz, M. and others [7]

In this paper we consider the notion of the Mannheim partner curve for the curves lying on the surfaces. We call these new associated curves as Mannheim partner D-curves and by using the Darboux frame of the curves we give the definition, different types and the characterizations of these curves in Minkowski 3-spaceE_1^3.

§2. Preliminaries

The Minkowski 3-space E_1^3 is the real vector space IR^3 provided with the standard flat metric given by

$$\langle \ , \ \rangle = -dx_1^2 + dx_2^2 + dx_3^2$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. An arbitrary vector $\vec{v} = (v_1, v_2, v_3)$ in E_1^3 can have one of three Lorentzian causal characters; it can be spacelike if $\langle \vec{v}, \vec{v} \rangle > 0$ or $\vec{v} = 0$, timelike if $\langle \vec{v}, \vec{v} \rangle < 0$ and null (lightlike) if $\langle \vec{v}, \vec{v} \rangle = 0$ and $\vec{v} \neq 0$. Similarly, an arbitrary curve $\vec{\alpha} = \vec{\alpha}(s)$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null (lightlike)[11]. We say that a timelike vector is future pointing or past pointing if the first compound of the vector is positive or negative, respectively. For any vectors $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ in E_1^3, Lorentz vector product of $\vec{x}$ and $\vec{y}$ is defined by

$$\vec{x} \times \vec{y} = \begin{vmatrix} e_1 & -e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2y_3 - x_3y_2,\ x_1y_3 - x_3y_1,\ x_2y_1 - x_1y_2)$$

where

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases}$$

$$e_i = (\delta_{i1}, \delta_{i2}, \delta_{i3}) \text{ and } e_1 \times e_2 = -e_3, \quad e_2 \times e_3 = e_1, \quad e_3 \times e_1 = -e_2.$$

Denote by $\left\{\vec{T},\ \vec{N},\ \vec{B}\right\}$ the moving Frenet frame along the curve $\alpha(s)$ in the Minkowski space E_1^3. For an arbitrary spacelike curve $\alpha(s)$ in the space E_1^3, the following Frenet formulae are given,

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -\varepsilon k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix},$$

where $\left\langle \vec{T},\vec{T}\right\rangle = 1,\ \left\langle \vec{N},\vec{N}\right\rangle = \varepsilon = \pm 1,\ \left\langle \vec{B},\vec{B}\right\rangle = -\varepsilon,\ \left\langle \vec{T},\vec{N}\right\rangle = \left\langle \vec{T},\vec{B}\right\rangle = \left\langle \vec{N},\vec{B}\right\rangle = 0$ and k_1 and k_2 are curvature and torsion of the spacelike curve $\alpha(s)$ respectively. Here, ε determines the kind of spacelike curve $\alpha(s)$. If $\varepsilon = 1$, then $\alpha(s)$ is a spacelike curve with spacelike first principal normal $\vec{N}$ and timelike binormal $\vec{B}$. If $\varepsilon = -1$, then $\alpha(s)$ is a spacelike curve with timelike principal normal $\vec{N}$ and spacelike binormal $\vec{B}$. Furthermore, for a timelike curve $\alpha(s)$ in the space E_1^3, the following Frenet formulae are given in as follows,

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix}.$$

where $\left\langle \vec{T},\vec{T}\right\rangle = -1,\ \left\langle \vec{N},\vec{N}\right\rangle = \left\langle \vec{B},\vec{B}\right\rangle = 1,\ \left\langle \vec{T},\vec{N}\right\rangle = \left\langle \vec{T},\vec{B}\right\rangle = \left\langle \vec{N},\vec{B}\right\rangle = 0$ and k_1 and k_2 are curvature and torsion of the timelike curve $\alpha(s)$ respectively [14,15].

Definition 2.1([11]) (i) (Hyperbolic angle) *Let $\vec{x}$ and $\vec{y}$ be future pointing (or past pointing) timelike vectors in IR_1^3. Then there is a unique real number $\theta \geqslant 0$ such that $< \vec{x}, \vec{y} >= -|\vec{x}|\ |\vec{y}| \cosh\theta$. This number is called the hyperbolic angle between the vectors $\vec{x}$ and $\vec{y}$.*

(ii) (Central angle) *Let $\vec{x}$ and $\vec{y}$ be spacelike vectors in IR_1^3 that span a timelike vector subspace. Then there is a unique real number $\theta \geqslant 0$ such that $< \vec{x}, \vec{y} >= |\vec{x}|\ |\vec{y}| \cosh\theta$. This number is called the central angle between the vectors $\vec{x}$ and $\vec{y}$.*

(iii) (Spacelike angle) *Let $\vec{x}$ and $\vec{y}$ be spacelike vectors in IR_1^3 that span a spacelike vector subspace. Then there is a unique real number $\theta \geqslant 0$ such that $< \vec{x}, \vec{y} >= |\vec{x}|\ |\vec{y}| \cos\theta$. This number is called the spacelike angle between the vectors $\vec{x}$ and $\vec{y}$.*

(iv) (Lorentzian timelike angle) *Let $\vec{x}$ be a spacelike vector and $\vec{y}$ be a timelike vector in IR_1^3. Then there is a unique real number $\theta \geqslant 0$ such that $< \vec{x}, \vec{y} >= |\vec{x}|\ |\vec{y}| \sinh\theta$. This number is called the Lorentzian timelike angle between the vectors $\vec{x}$ and $\vec{y}$.*

Definition 2.2([11]) *A surface in the Minkowski 3-space IR_1^3 is called a timelike surface if the induced metric on the surface is a Lorentz metric and it is called a spacelike surface if the induced metric on the surface is a positive definite Riemannian metric, i.e., the normal vector on the spacelike (timelike) surface is a timelike (spacelike) vector, respectively.*

Lemma 2.1([11]) *In the Minkowski 3-space IR_1^3, the following properties are satisfied:*

(i) *Two timelike vectors are never orthogonal;*

(ii) *Two null vectors are orthogonal if and only if they are linearly dependent;*

(iii) *A timelike vector is never orthogonal to a null (lightlike) vector .*

§3. Darboux Frame of a Curve Lying on a Surface in Minkowski 3-space E_1^3

Let S be an oriented surface in three-dimensional Minkowski space E_1^3 and let consider a non-null curve $x(s)$ lying on S fully. Since the curve $x(s)$ is also in space, there exists Frenet frame $\left\{\vec{T},\vec{N},\vec{B}\right\}$ at each points of the curve where $\vec{T}$ is unit tangent vector, $\vec{N}$ is principal normal vector and $\vec{B}$ is binormal vector, respectively.

Since the curve $x(s)$ lies on the surface S there exists another frame of the curve $x(s)$which is called Darboux frame and denoted by $\left\{\vec{T},\vec{g},\vec{n}\right\}$. In this frame $\vec{T}$ is the unit tangent of the curve, $\vec{n}$ is the unit normal of the surface S and $\vec{g}$ is a unit vector given by $\vec{g}=\vec{n}\times\vec{T}$. Since the unit tangent $\vec{T}$ is common in both Frenet frame and Darboux frame, the vectors $\vec{N}$, $\vec{B}$, $\vec{g}$ and $\vec{n}$ lie on the same plane. Then, if the surface S is an oriented timelike surface, the relations between these frames can be given as follows:

If the curve $x(s)$ is timelike, If the curve $x(s)$ is spacelike

$$\begin{bmatrix}\vec{T}\\ \vec{g}\\ \vec{n}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\varphi & \sin\varphi\\ 0 & -\sin\varphi & \cos\varphi\end{bmatrix}\begin{bmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{bmatrix},\quad \begin{bmatrix}\vec{T}\\ \vec{g}\\ \vec{n}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & \cosh\varphi & \sinh\varphi\\ 0 & \sinh\varphi & \cosh\varphi\end{bmatrix}\begin{bmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{bmatrix}.$$

If the surface S is an oriented spacelike surface, then the curve $x(s)$lying on S is a spacelike curve. So, the relations between the frames can be given as follows

$$\begin{bmatrix}\vec{T}\\ \vec{g}\\ \vec{n}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & \cosh\varphi & \sinh\varphi\\ 0 & \sinh\varphi & \cosh\varphi\end{bmatrix}\begin{bmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{bmatrix}.$$

In all cases, φ is the angle between the vectors $\vec{g}$ and $\vec{N}$.

According to the Lorentzian causal characters of the surface S and the curve $x(s)$ lying on S, the derivative formulae of the Darboux frame can be changed as follows:

(i) If the surface S is a timelike surface, then the curve $x(s)$ lying on S can be a spacelike or a timelike curve. Thus, the derivative formulae of the Darboux frame of $x(s)$ is given by

$$\begin{bmatrix}\dot{\vec{T}}\\ \dot{\vec{g}}\\ \dot{\vec{n}}\end{bmatrix}=\begin{bmatrix}0 & k_g & -\varepsilon k_n\\ k_g & 0 & \varepsilon\tau_g\\ k_n & \tau_g & 0\end{bmatrix}\begin{bmatrix}\vec{T}\\ \vec{g}\\ \vec{n}\end{bmatrix},\ \left\langle\vec{T},\vec{T}\right\rangle=\varepsilon=\pm1,\quad \langle\vec{g},\vec{g}\rangle=-\varepsilon,\quad \langle\vec{n},\vec{n}\rangle=1. \qquad (1)$$

(ii) If the surface S is a spacelike surface, then the curve $x(s)$ lying on S is a spacelike curve. Thus, the derivative formulae of the Darboux frame of $x(s)$ is given by

$$\begin{bmatrix} \dot{\vec{T}} \\ \dot{\vec{g}} \\ \dot{\vec{n}} \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_g \\ k_n & \tau_g & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{g} \\ \vec{n} \end{bmatrix}, \left\langle \vec{T}, \vec{T} \right\rangle = 1, \quad \langle \vec{g}, \vec{g} \rangle = 1, \quad \langle \vec{n}, \vec{n} \rangle = -1. \tag{2}$$

In these formulae k_g, k_n and τ_g are called the geodesic curvature, the normal curvature and the geodesic torsion, respectively. Here and in the following, we use "dot" to denote the derivative with respect to the arc length parameter of a curve.

The relations between geodesic curvature, normal curvature, geodesic torsion and κ, τ are given as follows (See [9,14,15]):

- if both S and $x(s)$ are timelike or spacelike,

$$k_g = \kappa \cos\varphi, k_n = \kappa \sin\varphi, \tau_g = \tau + \frac{d\varphi}{ds}; \tag{3}$$

- if S is timelike and $x(s)$ is spacelike

$$k_g = \kappa \cosh\varphi, k_n = \kappa \sinh\varphi, \tau_g = \tau + \frac{d\varphi}{ds}. \tag{4}$$

Furthermore, the geodesic curvature k_g and geodesic torsion τ_g of the curve $x(s)$ can be calculated as follows:

$$k_g = -\left\langle \frac{dx}{ds}, \frac{d^2x}{ds^2} \times n \right\rangle, \tau_g = -\varepsilon \left\langle \frac{dx}{ds}, n \times \frac{dn}{ds} \right\rangle, \tag{5}$$

$$k_g = -\left\langle \frac{dx}{ds}, \frac{d^2x}{ds^2} \times n \right\rangle, \tau_g = \left\langle \frac{dx}{ds}, n \times \frac{dn}{ds} \right\rangle. \tag{6}$$

where $\varepsilon = \left\langle \vec{T}, \vec{T} \right\rangle = \pm 1$.

In the differential geometry of surfaces, for a curve $x(s)$lying on a surface S the followings are well-known

- $x(s)$is a geodesic curve $\Leftrightarrow k_g = 0$,
- $x(s)$is an asymptotic line $\Leftrightarrow k_n = 0$,
- $x(s)$is a principal line $\Leftrightarrow \tau_g = 0$ [10].

Along every point of the surface passes a geodesic in every direction. A geodesic is uniquely determined by an initial point and tangent at that point. All straight lines on a surface are geodesics. Along all curved geodesics the principal normal coincides with the surface normal. Along asymptotic lines osculating planes and tangent planes coincide, along geodesics they are normal. Through a point of a nondevelopable surface pass two asymptotic lines which can be real or imaginary [13].

§4. Mannheim Partner D-Curves in Minkowski 3-Space E_1^3

In this section, by considering the Darboux frame, we define Mannheim partner D-curves and give the characterizations of these curves in Minkowski 3-space E_1^3.

Definition 4.1 *Let S and S_1 be oriented surfaces in three-dimensional Minkowski space E_1^3 and let consider the curves $x(s)$ and $x_1(s_1)$ parametrized by the arc-lenght lying fully on S and S_1, respectively. Denote the Darboux frames of $x(s)$ and $x_1(s_1)$ by $\{T, g, n\}$ and $\{T_1, g_1, n_1\}$, respectively. If there exists a corresponding relationship between the curves x and x_1 such that, at the corresponding points of the curves, the Darboux frame element g of x coincides with the Darboux frame element n_1 of x_1, then x is called a Mannheim D-curve, and x_1 is a Mannheim partner D-curve of x. Then, the pair $\{x, x_1\}$is said to be a Mannheim D-pair. If there exist such curves lying on the oriented surfaces S and S_1, respectively, we call the pair $\{S, S_1\}$ as Mannheim pair surfaces.*

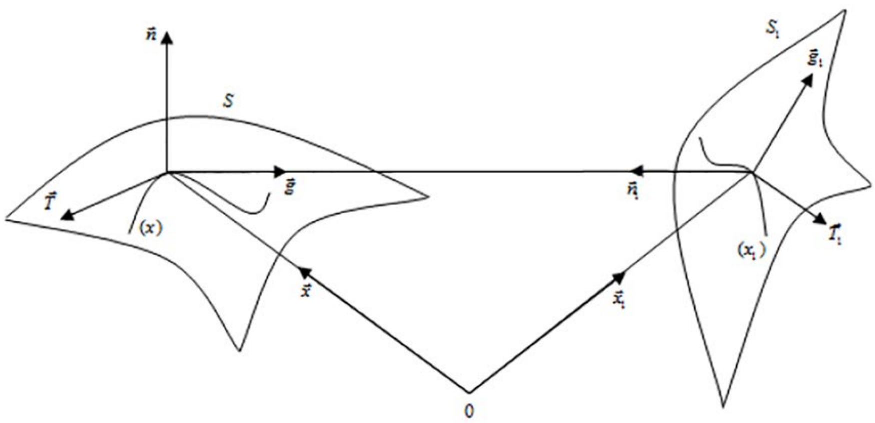

Figure 1 Mannheim partner D-curves

By considering the Lorentzian casual characters of the surfaces and the curves, from Definition 4.1, it is easily seen that there are five different types of the Mannheim D-curves in Minkowski 3-space. Let the pair $\{x, x_1\}$ be a Mannheim D-pair. Then according to the character of the surface S we have the followings:

Case 1. The oriented surface S is spacelike.

If both the surface S and the curve $x(s)$lying on S are spacelike then, there are two cases; first one is that the surface S_1 is a timelike surface and the curve $x_1(s_1)$ fully lying on S_1 is spacelike. In this case we say that the pair $\{x, x_1\}$ is a Mannheim D-pair of the type 1. The second one is that both the surface S_1 and the curve $x_1(s_1)$ fully lying on S_1 are timelike. In this case we say that the pair $\{x, x_1\}$ is a Mannheim D-pair of the type 2.

Case 2. The oriented surface S is timelike.

If the curve $x(s)$lying on S is a timelike curve then there are two cases; one is that both the surface S_1 and the curve $x_1(s_1)$ fully lying on S_1 are timelike. In this case we say that the pair $\{x, x_1\}$ is a Mannheim D-pair of the type 3. The other case is that the curve $x_1(s_1)$ fully lying on S_1 is a spacelike curve. Then the pair $\{x, x_1\}$ is a Mannheim D-pair of the type 4. If the curve $x(s)$lying on S is a spacelike curve then both the surface S_1 and the curve $x_1(s_1)$ fully lying on S_1 are spacelike. Then we say that the pair $\{x, x_1\}$ is a Mannheim D-pair of the type 5.

Theorem 4.1 *Let S be an oriented surface and $x(s)$ be a Mannheim D-curve in E_1^3 with arc length parameter s fully lying on S. If S_1 is another oriented surface and $x_1(s_1)$ is a curve with arc length parameter s_1 fully lying on S_1, then $x_1(s_1)$ is Mannheim partner D-curve of $x(s)$ if and only if the normal curvature k_n of $x(s)$ and the geodesic curvature k_{g_1}, the normal curvature k_{n_1} and the geodesic torsion τ_{g_1} of $x_1(s_1)$ satisfy the following equations.*

(i) if the pair $\{x, x_1\}$ is a Mannheim D-pair of the type 1 or 3, then

$$\dot{\tau}_{g_1} = \frac{1}{\lambda}\left[\left(\frac{(1+\lambda k_{n_1})^2 - \lambda^2\tau_{g_1}^2}{(1+\lambda k_{n_1})}\right)\left(k_n\frac{1+\lambda k_{n_1}}{\cosh\theta} - k_{g_1}\right) + \frac{\lambda^2\tau_{g_1}\dot{k}_{n_1}}{1+\lambda k_{n_1}}\right],$$

(ii) if the pair $\{x, x_1\}$ is a Mannheim D-pair of the type 2 or 4, then

$$\dot{\tau}_{g_1} = \frac{1}{\lambda}\left[\left(\frac{(1+\lambda k_{n_1})^2 - \lambda^2\tau_{g_1}^2}{(1+\lambda k_{n_1})}\right)\left(k_n\frac{1+\lambda k_{n_1}}{\sinh\theta} - k_{g_1}\right) + \frac{\lambda^2\tau_{g_1}\dot{k}_{n_1}}{1+\lambda k_{n_1}}\right],$$

(iii) if the pair $\{x, x_1\}$ is a Mannheim D-pair of the type 5, we have

$$\dot{\tau}_{g_1} = \frac{1}{\lambda}\left[\left(\frac{(1+\lambda k_{n_1})^2 + \lambda^2\tau_{g_1}^2}{(1+\lambda k_{n_1})}\right)\left(-k_n\frac{1+\lambda k_{n_1}}{\cos\theta} - k_{g_1}\right) + \frac{\lambda^2\tau_{g_1}\dot{k}_{n_1}}{1+\lambda k_{n_1}}\right],$$

for some nonzero constants λ, where θ is the angle between the tangent vectors T and T_1 at the corresponding points of the curves x and x_1.

Proof (i) Suppose that the pair $\{x, x_1\}$ is a Mannheim D-pair of the type 1. Denote the Darboux frames of $x(s)$ and $x_1(s_1)$by $\{T, g, n\}$ and $\{T_1, g_1, n_1\}$, respectively. Then by the definition we can assume that

$$x(s) = x_1(s_1) + \lambda(s_1)n_1(s_1) \tag{7}$$

for some function $\lambda(s_1)$. By taking derivative of (7) with respect to s_1 and applying the Darboux formulas (1) we have

$$T\frac{ds}{ds_1} = (1+\lambda k_{n_1})T_1 + \dot{\lambda}n_1 + \lambda\tau_{g_1}g_1. \tag{8}$$

Since the direction of n_1 coincides with the direction of g, we get

$$\dot{\lambda}(s_1) = 0.$$

This means that λ is a nonzero constant. Thus, the equality (8) can be written as follows

$$T\cdot\frac{ds}{ds_1}=(1+\lambda k_{n_1})T_1+\lambda\tau_{g_1}g_1. \tag{9}$$

On the other hand we have

$$T=\cosh\theta\, T_1+\sinh\theta\, g_1, \tag{10}$$

where θ is the angle between the tangent vectors T and T_1 at the corresponding points of x and x_1. By differentiating this last equation with respect to s_1, we get

$$(k_g g+k_n n)\frac{ds}{ds_1}=(\dot\theta+k_{g_1})\sinh\theta T_1+(\dot\theta+k_{g_1})\cosh\theta g_1+(-k_{n_1}\cosh\theta+\tau_{g_1}\sinh\theta)n_1. \tag{11}$$

From this equation and the fact that

$$n=\sinh\theta\, T_1+\cosh\theta\, g_1, \tag{12}$$

we get

$$\begin{aligned}(k_g g+k_n\sinh\theta T_1+k_n\cosh\theta g_1)\frac{ds}{ds_1} &= (\dot\theta+k_{g_1})\sinh\theta T_1+(\dot\theta+k_{g_1})\cosh\theta g_1\\ &\quad +(-k_{n_1}\cosh\theta+\tau_{g_1}\sinh\theta)n_1.\end{aligned} \tag{13}$$

Since the direction of n_1 is coincident with g we have

$$\dot\theta=k_n\frac{ds}{ds_1}-k_{g_1}. \tag{14}$$

From (9) and (10) and notice that T_1 is orthogonal to g_1 we obtain

$$\frac{ds}{ds_1}=\frac{1+\lambda k_{n_1}}{\cosh\theta}=\frac{\lambda\tau_{g_1}}{\sinh\theta}. \tag{15}$$

Equality (15) gives us

$$\tanh\theta=\frac{\lambda\tau_{g_1}}{1+\lambda k_{n_1}}. \tag{16}$$

By taking the derivative of this equation and applying (15) we get

$$\dot\tau_{g_1}=\frac{1}{\lambda}\left[\left(\frac{(1+\lambda k_{n_1})^2-\lambda^2\tau_{g_1}^2}{(1+\lambda k_{n_1})}\right)\left(k_n\frac{1+\lambda k_{n_1}}{\cosh\theta}-k_{g_1}\right)+\frac{\lambda^2\tau_{g_1}\dot k_{n_1}}{1+\lambda k_{n_1}}\right], \tag{17}$$

that is desired.

Conversely, assume that the equation (17) holds for some nonzero constants λ. Then by using (14) and (15), (16) gives us

$$k_n\left(\frac{ds}{ds_1}\right)^3=\lambda\dot\tau_{g_1}(1+\lambda k_{n_1})-\lambda^2\tau_{g_1}\dot k_{n_1}+\left((1+\lambda k_{n_1})^2-\lambda^2\tau_{g_1}^2\right)k_{g_1}. \tag{18}$$

Define a curve

$$x(s) = x_1(s_1) + \lambda n_1(s_1), \tag{19}$$

where λ is non-zero constant. We will prove that x is a Mannheim D-curve and x_1 is the Mannheim partner D-curve of x. By taking the derivative of (19) with respect to s_1 twice, we get

$$\frac{ds}{ds_1}T = (1 + \lambda k_{n_1})T_1 + \lambda \tau_{g_1} g_1 \tag{20}$$

and

$$\begin{aligned}(k_g g + k_n n)\left(\frac{ds}{ds_1}\right)^2 + T\frac{d^2 s}{ds_1^2} &= (\lambda \dot{k}_{n_1} + \lambda \tau_{g_1} k_{g_1})T_1 + \left((1+\lambda k_{n_1})k_{g_1} + \lambda \dot{\tau}_{g_1}\right) g_1 \\ &\quad + \left(-(1+\lambda k_{n_1})k_{n_1} + \lambda \tau_{g_1}^2\right) n_1,\end{aligned} \tag{21}$$

respectively. Taking the cross product of (20) with (21) we have

$$\begin{aligned}&[k_g n + k_n g]\left(\frac{ds}{ds_1}\right)^3 \\ &= \left(\lambda \tau_{g_1} k_{n_1}(1+\lambda k_{n_1}) - \lambda^2 \tau_{g_1}^3\right) T_1 - \left[-(1+\lambda k_{n_1})^2 k_{n_1} + \lambda \tau_{g_1}^2 (1+\lambda k_{n_1})\right] g_1 \\ &\quad + \left[-k_{g_1}(1+\lambda k_{n_1})^2 - \lambda \dot{\tau}_{g_1}(1+\lambda k_{n_1}) + \lambda^2 \tau_{g_1} \dot{k}_{n_1} + \lambda^2 \tau_{g_1}^2 k_{g_1}\right] n_1.\end{aligned} \tag{22}$$

By substituting (18) in (22) we get

$$\begin{aligned}[k_g n + k_n g]\left(\frac{ds}{ds_1}\right)^3 &= \left(\lambda \tau_{g_1} k_{n_1}(1+\lambda k_{n_1}) - \lambda^2 \tau_{g_1}^3\right) T_1 \\ &\quad - \left(-k_{n_1}(1+\lambda k_{n_1})^2 + \lambda \tau_{g_1}^2 (1+\lambda k_{n_1})\right) g_1 - k_n \left(\frac{ds}{ds_1}\right)^3 n_1.\end{aligned} \tag{23}$$

Taking the cross product of (20) with (23) we have

$$\begin{aligned}[k_g g + k_n n]\left(\frac{ds}{ds_1}\right)^4 &= k_n \left(\frac{ds}{ds_1}\right)^3 \lambda \tau_{g_1} T_1 + k_n \left(\frac{ds}{ds_1}\right)^3 (1+\lambda k_{n_1}) g_1 \\ &\quad + \left((1+\lambda k_{n_1})^2 - \lambda^2 \tau_{g_1}^2\right)\left(\lambda \tau_{g_1}^2 - k_{n_1}(1+\lambda k_{n_1})\right) n_1.\end{aligned} \tag{24}$$

From (23) and (24) we have

$$\begin{aligned}&\left(k_g^2 - k_n^2\right)\left(\frac{ds}{ds_1}\right)^4 \vec{n} \\ &= \left[k_g \frac{ds}{ds_1}\left(\lambda \tau_{g_1} k_{n_1}(1+\lambda k_{n_1}) - \lambda^2 \tau_{g_1}^3\right) - \lambda \tau_{g_1} k_n^2 \left(\frac{ds}{ds_1}\right)^3\right] \vec{T}_1 \\ &\quad - \left[k_g \frac{ds}{ds_1}\left(-k_{n_1}(1+\lambda k_{n_1})^2 + \lambda \tau_{g_1}^2 (1+\lambda k_{n_1})\right) + (1+\lambda k_{n_1}) k_n^2 \left(\frac{ds}{ds_1}\right)^3\right] \vec{g}_1 \\ &\quad - \left[k_n k_g \left(\frac{ds}{ds_1}\right)^4 + k_n \left((1+\lambda k_{n_1})^2 - \lambda^2 \tau_{g_1}^2\right)\left(\lambda \tau_{g_1}^2 - k_{n_1}(1+\lambda k_{n_1})\right)\right] \vec{n}_1.\end{aligned} \tag{25}$$

Furthermore, from (20) and (23) we get

$$\begin{cases} \left(\lambda^2\tau_{g_1}^2 - (1+\lambda k_{n_1})^2\right) = \left(\frac{ds}{ds_1}\right)^2 , \\ k_g\left(\frac{ds}{ds_1}\right)^2 - \lambda\tau_{g_1}^2 + k_{n_1}(1+\lambda k_{n_1}) = 0 \end{cases} \tag{26}$$

respectively. Substituting (26) in (25) we obtain

$$\begin{aligned} &\left(k_g^2 - k_n^2\right)\left(\frac{ds}{ds_1}\right)^4 \vec{n} \\ &= \left[k_g\frac{ds}{ds_1}\left(\lambda\tau_{g_1}k_{n_1}(1+\lambda k_{n_1}) - \lambda^2\tau_{g_1}^3\right) - \lambda\tau_{g_1}k_n^2\left(\frac{ds}{ds_1}\right)^3\right]\vec{T}_1 \\ &\quad - \left[k_g\frac{ds}{ds_1}\left(-k_{n_1}(1+\lambda k_{n_1})^2 + \lambda\tau_{g_1}^2(1+\lambda k_{n_1})\right) + (1+\lambda k_{n_1})k_n^2\left(\frac{ds}{ds_1}\right)^3\right]\vec{g}_1. \end{aligned} \tag{27}$$

Equality (20) and (27) shows that the vectors $\vec{T}$ and $\vec{n}$ lie on the plane $sp\left\{\vec{T}_1, \vec{g}_1\right\}$. So, at the corresponding points of the curves, the Darboux frame element $\vec{g}$ of x coincides with the Darboux frame element $\vec{n}_1$ of x_1, i.e, the curves x and x_1 are Mannheim D-pair curves of the type 1. $\square$

Let now give the characterizations of Mannheim partner D-curves in some special cases. Let the pair $\{x, x_1\}$ be a Mannheim D-pair of the type 1 or 3 in Minkowski 3-space E_1^3. Assume that $x(s)$ be an asymptotic Mannheim D-curve. Then, from (16) we have the following special cases:

(i) Consider that $x_1(s_1)$ is a geodesic curve. Then $x_1(s_1)$ is Mannheim partner D-curve of $x(s)$if and only if the following equation holds,

$$\dot{\tau}_{g_1} = -\frac{\lambda\tau_{g_1}\dot{k}_{n_1}}{1-\lambda k_{n_1}}.$$

(ii) Assume that $x_1(s_1)$ is an asymptotic line. Then $x_1(s_1)$ is Mannheim partner D-curve of $x(s)$if and only if the geodesic curvature $k_{g_1}$and the geodesic torsion τ_{g_1} of $x_1(s_1)$ satisfy the following equation,

$$\lambda\dot{\tau}_{g_1} = (1+\lambda^2\tau_{g_1}^2)k_{g_1}.$$

In this case, the Frenet frame of the curve $x_1(s_1)$ coincides with its Darboux frame. Thus, we have $k_{g_1} = \kappa_1$ and $\tau_{g_1} = \tau_1$. So, in Minkowski 3-space the Mannheim partner D-curves become the Mannheim partner curves, i.e., if both $x(s)$ and $x_1(s_1)$ are asymptotic lines then, the definition and the characterizations of the Mannheim partner D-curves involve those of the Mannheim partner curves in Minkowski 3-space.

(iii) If $x_1(s_1)$ is a principal line then $x_1(s_1)$ is Mannheim partner D-curve of $x(s)$ if and only if the geodesic curvature $k_{g_1} = 0$, that is $x_1(s_1)$ is also a geodesic curve or $k_{n_1} = -1/\lambda = const.$

The proofs of the statement (ii) and (iii) of Theorem 4. 1 and the particular cases given above can be given by the same way of the proof of statement (i).

Theorem 4.2 *Let the pair* $\{x, x_1\}$ *be a Mannheim D-pair in Minkowski 3-space* E_1^3*. Then the relation between geodesic curvature* k_g*, geodesic torsion* τ_g *of* $x(s)$*and the normal curvature* k_{n_1}*, the geodesic torsion* τ_{g_1} *of* $x_1(s_1)$ *is given as follows:*

(i) *if the pair* $\{x, x_1\}$ *is a Mannheim D-pair of the type 1, 3, 4 or 5 then*

$$k_g - k_{n_1} = \lambda(-k_g k_{n_1} + \tau_g \tau_{g_1}),$$

(ii) *if the pair* $\{x, x_1\}$ *is a Mannheim D-pair of the type 2, then*

$$k_g + k_{n_1} = \lambda(-k_g k_{n_1} + \tau_g \tau_{g_1}).$$

Proof (i) Let $x(s)$ be a Mannheim D-curve and $x_1(s_1)$ be a Mannheim partner D-curve of $x(s)$ in Minkowski 3-space E_1^3 and the pair $\{x, x_1\}$ be of the type 1. Then by definition we can write

$$x_1(s_1) = x(s) - \lambda(s)g(s) \tag{28}$$

for some constants λ. By differentiating (28) with respect to s_1 we have

$$T_1 = (1 + \lambda k_g)\frac{ds}{ds_1}T - \lambda\tau_g\frac{ds}{ds_1}n. \tag{29}$$

By definition we have

$$T_1 = \cosh\theta T - \sinh\theta n. \tag{30}$$

From (29) and (30) we obtain

$$\cosh\theta = (1 + \lambda k_g)\frac{ds}{ds_1}, \quad \sinh\theta = \lambda\tau_g\frac{ds}{ds_1}. \tag{31}$$

Using (13) and (31) it is easily seen that

$$k_g - k_{n_1} = \lambda(-k_g k_{n_1} + \tau_g \tau_{g_1}) \tag{32}$$

This completes the proof. □

Let the pair $\{x, x_1\}$be a Mannheim D-pair of the type 1 in Minkowski 3-space E_1^3. Then, we obtain the following special cases by Theorem 4.2.

(i) If x_1 is an asymptotic line, then

$$k_g = \lambda\tau_g\tau_{g_1}$$

(ii) If x_1 is a principal line, then

$$k_g - k_{n_1} = -\lambda k_g k_{n_1}$$

(iii) If x is a geodesic curve, then

$$k_{n_1} = -\lambda \tau_g \tau_{g_1}$$

(iv) If x is a principal line then

$$k_g - k_{n_1} = -\lambda k_g k_{n_1}$$

The proof of the cases that the pair $\{x, x_1\}$be a Mannheim D-pair of the type 2, 3, 4 or 5 can be given by a similar procedure used in the proof of the case that the pair $\{x, x_1\}$ is of the type 1.

Theorem 4.3 *Let $\{x, x_1\}$ be Mannheim D-pair of the type 1. Then the following relations hold:*

(i) $k_{g_1} = k_n \frac{ds}{ds_1} - \frac{d\theta}{ds_1}$;
(ii) $\tau_g \frac{ds}{ds_1} = -k_{n_1} \sinh\theta + \tau_{g_1} \cosh\theta$;
(iii) $k_g \frac{ds}{ds_1} = -k_{n_1} \cosh\theta + \tau_{g_1} \sinh\theta$;
(iv) $\tau_{g_1} = (-k_g \sinh\theta + \tau_g \cosh\theta) \frac{ds}{ds_1}$.

Proof (i) Since the pair $\{x, x_1\}$ is of the type 1, we have $\langle T, T_1 \rangle = \cosh\theta$. By differentiating this equality with respect to s_1 we have

$$\left\langle (k_g g + k_n n) \frac{ds}{ds_1}, T_1 \right\rangle + \langle T, k_{g_1} g_1 - k_{n_1} n_1 \rangle = \sinh\theta \frac{d\theta}{ds_1}.$$

Using the fact that the direction of n_1 coincides with the direction of g and

$$\begin{cases} T_1 = \cosh\theta T - \sinh\theta n, \\ g_1 = -\sinh\theta T + \cosh\theta n, \end{cases} \tag{33}$$

we easily get that

$$k_{g_1} = k_n \frac{ds}{ds_1} - \frac{d\theta}{ds_1}.$$

(ii) By definition we get $\langle n, n_1 \rangle = 0$. Differentiating this equality with respect to s_1 we have

$$\left\langle (k_n T + \tau_g g) \frac{ds}{ds_1}, n_1 \right\rangle + \langle n, k_{n_1} T_1 + \tau_{g_1} g_1 \rangle = 0.$$

By (29) we obtain

$$\tau_g \frac{ds}{ds_1} = -k_{n_1} \sinh\theta + \tau_{g_1} \cosh\theta.$$

(iii) By differentiating the equation $\langle T, n_1 \rangle = 0$ with respect to s_1 we get

$$\left\langle (k_g g + k_n n) \frac{ds}{ds_1}, n_1 \right\rangle + \langle T, k_{n_1} T_1 + \tau_{g_1} g_1 \rangle = 0.$$

From (29) it follows that

$$k_g \frac{ds}{ds_1} = -k_{n_1} \cosh\theta + \tau_{g_1} \sinh\theta.$$

(iv) By differentiating the equation $\langle g, g_1 \rangle = 0$ with respect to s_1 we obtain

$$\left\langle (-k_g T + \tau_g n)\frac{ds}{ds_1}, g_1 \right\rangle + \langle g, k_{g_1} T_1 + \tau_{g_1} n_1 \rangle = 0.$$

By considering (29) we get

$$\tau_{g_1} = (-k_g \sinh\theta + \tau_g \cosh\theta)\frac{ds}{ds_1}. \quad (34)$$

This completes the proof. □

The statements of Theorem 4.3 for the pairs $\{x, x_1\}$ of the type 2, 3, 4, and 5 are given in Tables 1 and 2, and the proofs can be easily done by the same way of the case the pairs $\{x, x_1\}$ is of the type 1.

For the pair $\{x, x_1\}$ of the type 2	For the pair $\{x, x_1\}$ of the type 3
$(i)\ k_{g_1} = k_n \frac{ds}{ds_1} + \frac{d\theta}{ds_1}$ $(ii)\ \tau_g \frac{ds}{ds_1} = k_{n_1} \cosh\theta - \tau_{g_1} \sinh\theta$ $(iii)\ k_g \frac{ds}{ds_1} = k_{n_1} \sinh\theta - \tau_{g_1} \cosh\theta$ $(iv)\ \tau_{g_1} = (-k_g \cosh\theta + \tau_g \sinh\theta)\frac{ds}{ds_1}$	$(i)\ k_{g_1} = k_n \frac{ds}{ds_1} + \frac{d\theta}{ds_1}$ $(ii)\ \tau_g \frac{ds}{ds_1} = k_{n_1} \sinh\theta - \tau_{g_1} \cosh\theta$ $(iii)\ k_g \frac{ds}{ds_1} = k_{n_1} \cosh\theta - \tau_{g_1} \sinh\theta$ $(iv)\ \tau_{g_1} = (k_g \sinh\theta - \tau_g \cosh\theta)\frac{ds}{ds_1}$

Table 1

For the pair $\{x, x_1\}$ of the type 4	For the pair $\{x, x_1\}$ of the type 5
$(i)\ k_{g_1} = k_n \frac{ds}{ds_1} + \frac{d\theta}{ds_1}$ $(ii)\ \tau_g \frac{ds}{ds_1} = -k_{n_1} \cosh\theta + \tau_{g_1} \sinh\theta$ $(iii)\ k_g \frac{ds}{ds_1} = -k_{n_1} \sinh\theta + \tau_{g_1} \cosh\theta$ $(iv)\ \tau_{g_1} = (k_g \cosh\theta - \tau_g \sinh\theta)\frac{ds}{ds_1}$	$(i)\ k_{g_1} = -k_n \frac{ds}{ds_1} - \frac{d\theta}{ds_1}$ $(ii)\ \tau_g \frac{ds}{ds_1} = -k_{n_1} \cos\theta + \tau_{g_1} \sin\theta$ $(iii)\ k_g \frac{ds}{ds_1} = k_{n_1} \sin\theta + \tau_{g_1} \cos\theta$ $(iv)\ \tau_{g_1} = (k_g \cos\theta + \tau_g \sin\theta)\frac{ds}{ds_1}$

Table 2

Let now x be a Mannheim D-curve and x_1 be a Mannheim partner D-curve of x and the pair $\{x, x_1\}$ be of the type 1. From (5) and by using the fact that n_1 is coincident with g we have

$$\begin{aligned} k_{g_1} &= -\langle \dot{x}_1, \ddot{x}_1 \times n_1 \rangle = -\langle \dot{x}_1, \ddot{x}_1 \times g \rangle \\ &= k_n \left[-(1+\lambda k_g)^2 + \lambda^2 \tau_g^2 \right] \left(\frac{ds}{ds_1} \right)^3 + \left[\lambda \dot{\tau}_g (1+\lambda k_g) - \lambda^2 \tau_g \dot{k}_g \right] \left(\frac{ds}{ds_1} \right)^2. \end{aligned}$$

Then the relations between the geodesic curvature k_{g_1} of $x_1(s_1)$ and the geodesic curvature k_g, the normal curvature $k_n$and the geodesic torsion τ_g of $x(s)$ are given as follows:

If $k_g = 0$ then from (33) the geodesic curvature k_{g_1} of $x_1(s_1)$ is

$$k_{g_1} = \left(\frac{ds}{ds_1}\right)^3 (-1 + \lambda^2 \tau_g^2) k_n + \left(\frac{ds}{ds_1}\right)^2 \lambda \dot{\tau}_g. \tag{34}$$

If $k_n = 0$ then the relation between k_g, τ_g and k_{g_1} is

$$k_{g_1} = \lambda \left(\frac{ds}{ds_1}\right)^2 \left(\dot{\tau}_g (1 + \lambda k_g) - \lambda \tau_g \dot{k}_g\right). \tag{35}$$

If $\tau_g = 0$ then, for the geodesic curvature k_{g_1}, we have

$$k_{g_1} = -\left(\frac{ds}{ds_1}\right)^3 (1 + \lambda k_g)^2 k_n. \tag{36}$$

From (34),(35) and (36) we give the following corollary.

Corollary 4.1 *Let x be a Mannheim D-curve and x_1 be a Mannheim partner D-curve of x and the pair the pair $\{x, x_1\}$ be of the type 1. Then the relations between the geodesic curvature k_{g_1} of $x_1(s_1)$ and the geodesic curvature, k_g, the normal curvature k_n and the geodesic torsion τ_g of $x(s)$ are given as follows*

(*i*) *If x is a geodesic curve, then the geodesic curvature k_{g_1} of $x_1(s_1)$is*

$$k_{g_1} = \left(\frac{ds}{ds_1}\right)^3 (-1 + \lambda^2 \tau_g^2) k_n + \left(\frac{ds}{ds_1}\right)^2 \lambda \dot{\tau}_g$$

(*ii*) *If x is an asymptotic line, then the equation of k_{g_1} is*

$$k_{g_1} = \lambda \left(\frac{ds}{ds_1}\right)^2 \left(\dot{\tau}_g (1 + \lambda k_g) - \lambda \tau_g \dot{k}_g\right)$$

(*iii*) *If x is a principal line, then the geodesic curvature k_{g_1} of $x_1(s_1)$ is*

$$k_{g_1} = -\left(\frac{ds}{ds_1}\right)^3 (1 + \lambda k_g)^2 k_n.$$

If the pair $\{x, x_1\}$is of the type 2, 3, 4 or 5 then the geodesic curvature of the curve $x_1(s_1)$ is given in Tables 3 and 4 following.

If the pair $\{x, x_1\}$is of the type 2	If the pair $\{x, x_1\}$ is of the type 3
$k_{g_1} = k_n \left[(1+\lambda k_g)^2 + \lambda^2 \tau_g^2\right] \left(\frac{ds}{ds_1}\right)^3 + \left[-\lambda \dot{\tau}_g (1+\lambda k_g) + \lambda^2 \tau_g \dot{k}_g\right] \left(\frac{ds}{ds_1}\right)^2$	$k_{g_1} = -k_n \left[(1-\lambda k_g)^2 + \lambda^2 \tau_g^2\right] \left(\frac{ds}{ds_1}\right)^3 + \left[-\lambda \dot{\tau}_g (1-\lambda k_g) - \lambda^2 \tau_g \dot{k}_g\right] \left(\frac{ds}{ds_1}\right)^2$

Table 3

If the pair $\{x, x_1\}$ is of the type 4	If the pair $\{x, x_1\}$is of the type 5
$k_{g_1} = k_n \left[(1-\lambda k_g)^2 - \lambda^2 \tau_g^2\right] \left(\frac{ds}{ds_1}\right)^3 + \left[\lambda \dot{\tau}_g (1-\lambda k_g) + \lambda^2 \tau_g \dot{k}_g\right] \left(\frac{ds}{ds_1}\right)^2$	$k_{g_1} = -k_n \left[(1-\lambda k_g)^2 + \lambda^2 \tau_g^2\right] \left(\frac{ds}{ds_1}\right)^3 + \left[-\lambda \dot{\tau}_g (1-\lambda k_g) - \lambda^2 \tau_g \dot{k}_g\right] \left(\frac{ds}{ds_1}\right)^2$

Table 4

and the statements in Corollary 4.1 are obtained by the same way.

Similarly, if the pair $\{x, x_1\}$ is of the type 1, from(6) and by using the fact that g is coincident with n_1, the relation between the geodesic torsion $\tau_{g_1}$of $x_1(s_1)$ and the geodesic torsion $\tau_g$of $x(s)$ is given by

$$\tau_{g_1} = \tau_g \left(\frac{ds}{ds_1}\right)^2. \tag{37}$$

Furthermore, by using (15), from (37) we have

$$\tau_g \tau_{g_1} = \frac{\sinh^2\theta}{\lambda^2}. \tag{38}$$

Then, from (37) and (38) we can give the following corollary.

Corollary 4.2 *Let x be a Mannheim D-curve and x_1 be a Mannheim partner D-curve of x and $\{x, x_1\}$ be of the type 1. Then the relation between the geodesic torsion τ_{g_1} of $x_1(s_1)$ and the geodesic torsion τ_g of $x(s)$ is given by one of the followings:*

(i) $\tau_{g_1} = \tau_g \left(\frac{ds}{ds_1}\right)^2$;

(ii) $\tau_g \tau_{g_1} = \frac{\sinh^2\theta}{\lambda^2}$

and so, the Mannheim partner D-curve x_1 is a principal line when the Mannheim D-curve x is a principal line.

Similarly, from (15) and (37) we get

(iii) $\frac{\tau_g}{\tau_{g_1}} = \frac{\cosh^2\theta}{(1+\lambda k_{n_1})^2}$.

Then, if $x_1(s_1)$ is an asymptotic curve, i.e.,$k_{n_1} = 0$, we have

$$\tau_g = \cosh^2\theta \tau_{g_1}. \tag{39}$$

From (39) we have the following corollary.

Corollary 4.3 *Let x be a Mannheim D-curve and x_1 be a Mannheim partner D-curve of x and $\{x, x_1\}$ be of the type 1. If $x_1(s_1)$ is an asymptotic curve then the relation between the geodesic torsion τ_g of $x(s)$ and the geodesic torsion $\tau_{g_1}$of $x_1(s_1)$ is given as follows:*

(iv) $\tau_g = \cosh^2\theta \ \tau_{g_1}$,

where θ is the angle between the tangent vectors T and T_1 at the corresponding points of x and x_1.

When the pair $\{x, x_1\}$ is of the type 2, 3, 4 or 5, then the relations which give the geodesic torsion τ_{g_1} of $x_1(s_1)$ are given in Tables 5 and 6 following.

For the pair $\{x, x_1\}$ of the type 2	For the pair $\{x, x_1\}$ of the type 3
$(i)\ \tau_{g_1} = \tau_g \left(\frac{ds}{ds_1}\right)^2$ $(ii)\ \tau_g \tau_{g_1} = \frac{\cosh^2\theta}{\lambda^2}$ $(iii)\ \frac{\tau_g}{\tau_{g_1}} = \frac{\sinh^2\theta}{(1+\lambda k_{n_1})^2}$ $(iv)\ \tau_g = \sinh^2\theta\tau_{g_1}$, if $x_1(s_1)$ is an asymptotic curve.	$(i)\ \tau_{g_1} = -\tau_g \left(\frac{ds}{ds_1}\right)^2$ $(ii)\ \tau_g \tau_{g_1} = -\frac{\sinh^2\theta}{\lambda^2}$ $(iii)\ \frac{\tau_g}{\tau_{g_1}} = -\frac{\cosh^2\theta}{(1+\lambda k_{n_1})^2}$ $(iv)\ \tau_g = -\cosh^2\theta\tau_{g_1}$, if $x_1(s_1)$ is an asymptotic curve.

Table 5

For the pair $\{x, x_1\}$ of the type 4	For the pair $\{x, x_1\}$ of the type 5
$(i)\ \tau_{g_1} = -\tau_g \left(\frac{ds}{ds_1}\right)^2$ $(ii)\ \tau_g \tau_{g_1} = -\frac{\cosh^2\theta}{\lambda^2}$ $(iii)\ \frac{\tau_g}{\tau_{g_1}} = -\frac{\sinh^2\theta}{(1+\lambda k_{n_1})^2}$ $(iv)\ \tau_g = -\sinh^2\theta\tau_{g_1}$, if $x_1(s_1)$ is an asymptotic curve.	$(i)\ \tau_{g_1} = \tau_g \left(\frac{ds}{ds_1}\right)^2$ $(ii)\ \tau_g \tau_{g_1} = \frac{\sin^2\theta}{\lambda^2}$ $(iii)\ \frac{\tau_g}{\tau_{g_1}} = \frac{\cos^2\theta}{(1+\lambda k_{n_1})^2}$ $(iv)\ \tau_g = \cos^2\theta\tau_{g_1}$, if $x_1(s_1)$ is an asymptotic curve.

Table 6

§5. Conclusions

In this paper, in Minkowski 3-space E_1^3, the definition and characterizations of Mannheim partner D-curves are given which is a new study of associated curves lying on surfaces. It is shown that in Minkowski 3-space E_1^3, the definition and the characterizations of Mannheim partner D-curves include those of Mannheim partner curves in some special cases. Furthermore, the relations between the geodesic curvature, the normal curvature and the geodesic torsion of these curves are given.

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