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Second Status Connectivity Indices and its Coindices of Composite Graphs

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Abstract: In this paper, we obtain the exact formulae for the second status connectivity indices and its coindices of some composite graphs such as Cartesian product, join and composition of two connected graphs.

Key Words: Wiener index, status connectivity index, composite graph.

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§1. Introduction

A *topological index* is a mathematical measure which correlates to the chemical structures of any simple finite graph. They are invariant under the graph isomorphism. They play an important role in the study of $QSAR/QSPR$. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, nanoscience, biological and other properties of chemical compounds. Wiener index is the first distance-based topological index that were defined by Wiener [5]. For more details, see [9,10,11,12].

The *status* [2] of a vertex $v \in V(G)$ is defined as the sum of its distance from every other vertex in $V(G)$ and is denoted by $\sigma_G(v)$, that is, $\sigma_G(v) = \sum\limits_{u \in V(G)} d_G(u,v)$, where $d_G(u,v)$ is the distance between u and v in G. The status of vertex v is also called as *transmisson* of v [2].

The *Wiener index* $W(G)$ of a connected graph G is defined as the sum of the distances between all pairs of vertices of G, that is,

$$W(G) = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u,v) = \frac{1}{2} \sum_{u \in V(G)} \sigma_G(v).$$

The *first Zagreb index* is defined as

$$M_1(G) = \sum_{u \in V(G)} (d_G(u))^2 = \sum_{uv \in E(G)} (d_G(u) + d_G(v))$$

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and the *second Zagreb index* is defined as

$$M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v).$$

The Zagreb indices are found to have applications in QSPR and QSAR studies as well, see [7]. The *first* and *second Zagreb coindices* were first introduced by Ashrafi et al. [8]. They are defined as follows:

$$\overline{M}_1(G) = \sum_{uv \notin E(G)} (d_G(u) + d_G(v))$$

and the *second Zagreb index* is defined as

$$\overline{M}_2(G) = \sum_{uv \notin E(G)} d_G(u) d_G(v).$$

Motivated by the invariants like Zagreb indices, Ramane et al.[1] proposed the *first status connectivity index* $S_1(G)$ and *first status connectivity coindex* $\overline{S}_1(G)$ of a connected graph G as

$$S_1(G) = \sum_{uv \in E(G)} \Big(\sigma_G(u) + \sigma_G(v)\Big) \text{ and } \overline{S}_1(G) = \sum_{uv \notin E(G)} \Big(\sigma_G(u) + \sigma_G(v)\Big).$$

Similarly, the *second status connectivity index* $S_2(G)$ and *second status connectivity coindex* $\overline{S}_2(G)$ of a connected graph G as

$$S_2(G) = \sum_{uv \in E(G)} \sigma_G(u) \sigma_G(v) \text{ and } \overline{S}_2(G) = \sum_{uv \notin E(G)} \sigma_G(u) \sigma_G(v).$$

The bounds for the status connectivity indices are determined in [1]. Also they are discussed the linear regression analysis of the distance-based indices with the boiling points of benzenoid hydrocarbons and the linear model based on the status index is better than the models corresponding to the other distance based indices. In this sequence, here we obtain the exact formulae for second status connectivity indices and its coindices of some composite graphs such as Cartesian product, join, composition of two connected graphs.

§2. Main Results

In this section, we obtain the second status connectivity indices and its coindices of Cartesian product, join and composition of two graphs.

Lemma 2.1 *Let G be a connected graph on n vertices. Then*

$$\overline{S}_2(G) = 2(W(G)) - \frac{1}{2} \sum_{u \in V(G)} (\sigma_G(u))^2 - S_2(G).$$

Proof By the definition of $\overline{S}_2$, we obtain:

$$\begin{aligned}
\overline{S}_2(G) &= \sum_{uv\notin E(G)} \sigma_G(u)\sigma_G(v) \\
&= \sum_{\{u,v\}\subseteq V(G)} \sigma_G(u)\sigma_G(v) - \sum_{uv\in E(G)} \sigma_G(u)\sigma_G(v) \\
&= \frac{1}{2}\bigg(\Big(\sum_{u\in V(G)} \sigma_G(u)\Big)^2 - \sum_{u\in V(G)} (\sigma_G(u))^2\bigg) - S_2(G) \\
&= 2(W(G)) - \frac{1}{2}\sum_{u\in V(G)} (\sigma_G(u))^2 - S_2(G).
\end{aligned}$$

□

Let C_n and P_n denote the cycle and path on n vertices, respectively. It is known that [1]

$$S_1(P_n) = \frac{1}{3}n(n-1)(2n-1) \quad \text{and} \quad W(P_n) = \frac{n(n^2-1)}{6}$$

and

$$S_1(C_n) = \begin{cases} \frac{n^3}{2}, & \text{if } n \text{ is even,} \\ \frac{n(n^2-1)}{2}, & \text{otherwise;} \end{cases} \quad \text{and} \quad W(C_n) = \begin{cases} \frac{n^3}{8}, & \text{if } n \text{ is even,} \\ \frac{n(n^2-1)}{8}, & \text{otherwise.} \end{cases}$$

We therefore have that

Lemma 2.2 *For cycle C_n and path P_n, we get that*

(1) *For* $n \geq 3$, $S_2(C_n) = \begin{cases} \frac{n^5}{16} & \text{if } n \text{ is even} \\ \frac{n(n^2-1)^2}{16} & \text{if } n \text{ is odd;} \end{cases}$

(2) $S_2(P_n) = \frac{n^2(n-1)}{4}$.

2.1 Cartesian Product

The *Cartesian product*, $G\,\square\,H$, of the graphs G and H has the vertex set $V(G\,\square\,H) = V(G)\times V(H)$ and $(u,x)(v,y)$ is an edge of $G\,\square\,H$ if $u = v$ and $xy \in E(H)$ or, $uv \in E(G)$ and $x = y$. To each vertex $u \in V(G)$, there is an isomorphic copy of H in $G\,\square\,H$ and to each vertex $v \in V(H)$, there is an isomorphic copy of G in $G\,\square\,H$.

Theorem 2.3 *Let G and H be two connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then*

$$\begin{aligned}
S_2(G\square H) &= n_2^3 S_2(G) + n_1^3 S_2(H) + 2n_1n_2(S_1(G)W(H) + S_1(H)W(G)) \\
&\quad + n_2^2 m_2 \sum_{u_i\in V(G)} (\sigma_G(u_i))^2 + n_1^2 m_1 \sum_{v_s\in V(H)} (\sigma_H(v_s))^2.
\end{aligned}$$

Proof From the structure of $G\square H$, the distance between two vertices (u_i, v_r) and (u_k, v_s) of $G\square H$ is $d_G(u_i,u_k) + d_H(v_r,v_s)$. Moreover, the degree of a vertex (u_i,v_r) in $V(G\square H)$ is $d_G(u_i) + d_H(v_r)$. By the definition of $\sigma(u)$ for the graph $G\Box H$ and a vertex $(u_i, v_r) \in V(G\Box H)$, we have

$$\begin{aligned}
\sigma_{G\Box H}((u_i, v_r)) &= \sum_{(u_k,v_s)\in V(G\Box H)} d_{G\Box H}((u_i, v_r), (u_k, v_s)) \\
&= \sum_{u_k\in V(G)} \sum_{v_s\in V(H)} \Big(d_G(u_i, u_k) + d_G(v_r, v_s)\Big) \\
&= n_2\sigma_G(u_i) + n_1\sigma_H(v_r). \qquad (2.1)
\end{aligned}$$

Hence by the definitions of S_2 and $G\Box H$, we have

$$\begin{aligned}
S_2(G\Box H) &= \sum_{(u_i,v_s)(u_k,v_s)\in E(G\Box H)} \sigma_{G\Box H}((u_i, v_s))\sigma_{G\Box H}((u_k, v_s)) \\
&\quad + \sum_{(u_i,v_s)(u_k,v_s)\in E(G\Box H)} \sigma_{G\Box H}((u_i, v_r))\sigma_{G\Box H}((u_i, v_s)) \\
&= A_1 + A_2, \qquad (2.2)
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \sum_{(u_i,v_s)(u_k,v_s)\in E(G\Box H)} \sigma_{G\Box H}((u_i, v_s))\sigma_{G\Box H}((u_k, v_s)) \\
&= \sum_{u_iu_k\in E(G)} \sum_{v_s\in V(H)} \Big(n_2\sigma_G(u_i) + n_1\sigma_H(v_s)\Big)\Big(n_2\sigma_G(u_k) + n_1\sigma_H(v_s)\Big), \text{ by } (2.1) \\
&= \sum_{u_iu_k\in E(G)} \sum_{v_s\in V(H)} \Big(n_2^2\sigma_G(u_i)\sigma_G(u_k) + n_1n_2\sigma_G(u_i)\sigma_H(v_s) \\
&\quad + n_1n_2\sigma_H(v_s)\sigma_G(u_k) + n_1^2(\sigma_H(v_s))^2\Big) \\
&= n_2^3 \sum_{u_iu_k\in E(G)} \sigma_G(u_i)\sigma_G(u_k) + n_1n_2 \sum_{v_s\in V(H)} \sigma_H(v_s) \sum_{u_iu_k\in E(G)} (\sigma_G(u_i) + \sigma_G(u_k)) \\
&\quad + n_1^2m_1 \sum_{v_s\in V(H)} (\sigma_H(v_s))^2 \\
&= n_2^3S_2(G) + 2n_1n_2S_1(G)W(H) + n_1^2m_1 \sum_{v_s\in V(H)} (\sigma_H(v_s))^2.
\end{aligned}$$

and a similar argument of A_1, we obtain

$$\begin{aligned}
A_2 &= \sum_{(u_i,v_s)(u_k,v_s)\in E(G\Box H)} \sigma_{G\Box H}((u_i, v_r))\sigma_{G\Box H}((u_i, v_s)) \\
&= n_1^3S_2(H) + 2n_1n_2S_1(H)W(G) + n_2^2m_2 \sum_{u_i\in V(G)} (\sigma_G(u_i))^2.
\end{aligned}$$

From (2.2) and A_1, A_2, we obtain:

$$\begin{aligned}
S_2(G\Box H) &= n_2^3S_2(G) + n_1^3S_2(H) + 2n_1n_2(S_1(G)W(H) + S_1(H)W(G)) \\
&\quad + n_2^2m_2 \sum_{u_i\in V(G)} (\sigma_G(u_i))^2 + n_1^2m_1 \sum_{v_s\in V(H)} (\sigma_H(v_s))^2. \qquad \Box
\end{aligned}$$

Remark 2.4 For each vertex (u_i, v_r) in $G\Box H$,

$$\begin{aligned}
&\sum_{(u_i,v_r)\in V(G\Box H)} \Big(\sigma_{G\Box H}((u_i,v_r))\Big)^2 \\
&= \sum_{u_i\in V(G)}\sum_{v_r\in V(H)} \Big(n_2\sigma_G(u_i)+n_1\sigma_H(v_r)\Big)^2, \text{ by } (2.1) \\
&= \sum_{u_i\in V(G)}\sum_{v_r\in V(H)} \Big(n_2^2(\sigma_G(u_i))^2+n_1^2(\sigma_H(v_r))^2+2n_1n_2\sigma_G(u_i)\sigma_H(v_r)\Big) \\
&= n_2^3\sum_{u_i\in V(G)}(\sigma_G(u_i))^2+n_1^3\sum_{v_r\in V(H)}(\sigma_H(v_r))^2+8n_1n_2W(G)W(H).
\end{aligned}$$

By Theorem 2.3, Lemma 2.1, Remark 2.4 and this fact that [3], $W(G\Box H) = n_2^2W(G)+n_1^2W(H)$, the following theorem is straightforward.

Theorem 2.5 *Let G and H be two connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then*

$$\begin{aligned}
\overline{S}_2(G\Box H) &= 2[n_2^2W(G)+n_1^2W(H)]^2-n_2^3S_2(G)-n_1^3S_2(H) \\
&\quad -2n_1n_2[S_1(G)W(H)+S_1(H)W(G)+2W(G)W(H)] \\
&\quad -\frac{n_2^2(n_2+2m_2)}{2}\sum_{u_i\in V(G)}(\sigma_G(u_i))^2-\frac{n_1^2(n_1+2m_1)}{2}\sum_{v_r\in V(H)}(\sigma_H(v_r))^2.
\end{aligned}$$

2.2 Join

The *join* $G+H$ of two graphs G and H is the union $G\cup H$ together with all the edges joining $V(G)$ and $V(H)$. From the structure of $G+H$, the distance between two vertices u and v of $G+H$ is

$$d_{G+H}(u,v)=\begin{cases} 0, \text{ if } u=v, \\ 1, \text{ if } uv\in E(G) \text{ or } uv\in E(H) \text{ or } (u\in V(G) \text{ and } v\in V(H)), \\ 2, \text{ otherwise.}\end{cases}$$

Moreover, the degree of a vertex v in $V(G+H)$ is

$$d_{G+H}(v)=\begin{cases} d_G(v)+|V(H)|, if \ \ v\in V(G), \\ d_H(v)+|V(G)|, if \ \ v\in V(H).\end{cases}$$

Theorem 2.6 *Let G and H be two connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then*

$$\begin{aligned}
S_2(G+H) &= M_2(G)+M_2(H)-(2n_1+n_2-2)M_1(G) \\
&\quad -(2n_2+n_1-2)M_1(H) \\
&\quad +(2n_1+n_2-2)[(2n_1+n_2-2)m_1-2n_1m_2] \\
&\quad -(2n_2+n_1-2)[(2n_2+n_1-2)m_2-2n_2m_1 \\
&\quad +n_1n_2(2n_1+n_2-2)]+4m_1m_2.
\end{aligned}$$

Proof Let u be a vertex in $V(G)$. Then from the structure of $G+H$, we obtain:

$$\begin{aligned}\sigma_{G+H}(u) &= \sum_{v\in V(G+H)} d_{G+H}((u,v)) \\ &= \sum_{v\in V(G)\ u\neq v, uv\notin E(G)} 2 + \sum_{v\in V(G),\ u\neq v, uv\in E(G)} 1 + \sum_{v\in V(H)} 1 \\ &= 2n_1+n_2-2-d_G(u).\end{aligned}$$

Similarly, if v is a vertex of H, then $\sigma_{G+H}(v) = 2n_2+n_1-2-d_G(v)$.

The edge set of $G+H$ can be partitioned into three subsets, namely,

$E_1 = \{uv \in E(G+H)|uv \in E(G)\}$,

$E_2 = \{uv \in E(G+H)|uv \in E(H)\}$ and

$E_3 = \{uv \in E(G+H)|u \in V(G),\ v \in V(H)\}$.

The contribution of the edges in E_1 is given by

$$\begin{aligned}S_2(G+H) &= \sum_{uv\in E_1} \sigma_{G+H}(u)\sigma_{G+H}(v) \\ &= \sum_{uv\in E(G)} \Big(2n_1+n_2-2-d_G(u)\Big)\Big(2n_1+n_2-2-d_G(v)\Big) \\ &= \sum_{uv\in E(G)} \big[(2n_1+n_2-2)^2 - (2n_1+n_2-2)d_G(v) \\ &\quad -(2n_1+n_2-2)d_G(u) + d_G(u)d_G(v)] \\ &= (2n_1+n_2-2)^2 m_1 - (2n_1+n_2-2)M_1(G) + M_2(G). \qquad (2.3)\end{aligned}$$

Similarly, the contribution of the edges in E_2 is given by

$$\begin{aligned}S_2(G+H) &= \sum_{uv\in E_2} \sigma_{G+H}(u)\sigma_{G+H}(v) \\ &= (2n_2+n_1-2)^2 m_2 - (2n_2+n_1-2)M_1(H) + M_2(H). \qquad (2.4)\end{aligned}$$

The contribution of the edges in E_3 is given by

$$\begin{aligned}S_2(G+H) &= \sum_{uv\in E_3} \sigma_{G+H}(u)\sigma_{G+H}(v) \\ &= \sum_{u\in V(G)}\sum_{v\in V(H)} \Big(2n_1+n_2-2-d_G(u)\Big)\Big(2n_2+n_1-2-d_H(v)\Big) \\ &= \sum_{u\in V(G)}\sum_{v\in V(H)} \Big[(2n_1+n_2-2)(2n_2+n_1-2) - (2n_1+n_2-2)d_H(v) \\ &\quad -(2n_2+n_1-2)d_G(u) + d_G(u)d_H(v)\Big] \\ &= (2n_1+n_2-2)(2n_2+n_1-2)n_1n_2 - 2n_1m_2(2n_1+n_2-2) \\ &\quad -2n_2m_1(2n_2+n_1-2) + 4m_1m_2. \qquad (2.5)\end{aligned}$$

The total contribution of the edges in $G+H$ and its $S_2(G+H)$ is given by

$$\begin{aligned}S_2(G+H) &= M_2(G)+M_2(H)-(2n_1+n_2-2)M_1(G)\\&-(2n_2+n_1-2)M_1(H)\\&+(2n_1+n_2-2)[(2n_1+n_2-2)m_1-2n_1m_2]\\&-(2n_2+n_1-2)[(2n_2+n_1-2)m_2\\&-2n_2m_1+n_1n_2(2n_1+n_2-2)]+4m_1m_2. \qquad \square\end{aligned}$$

Remark 2.7 For each vertex v in $G+H$,

$$\begin{aligned}\sum_{v\in V(G+H)}(\sigma_{G+H}(v))^2 &= \sum_{v\in V(G)}(\sigma_{G+H}(v))^2+\sum_{v\in V(H)}(\sigma_{G+H}(v))^2\\&= \sum_{v\in V(G)}(2n_1+n_2-2-d_G(u))^2+\sum_{v\in V(H)}(2n_2+n_1-2-d_G(v))^2\\&= \sum_{v\in V(G)}\Big((2n_1+n_2-2)^2+(d_G(v))^2-2(2n_1+n_2-2)d_G(v)\Big)\\&+\sum_{v\in V(H)}\Big((2n_2+n_1-2)^2+(d_H(v))^2-2(2n_2+n_1-2)d_H(v)\Big)\\&= (2n_1+n_2-2)^2n_1+M_1(G)-4m_1(2n_1+n_2-2)\\&+(2n_2+n_1-2)^2n_2+M_1(H)-4m_2(2n_2+n_1-2).\end{aligned}$$

According to [3], we know that

$$\begin{aligned}W(G+H) &= |V(G)|\,(|V(G)|-1)+|V(H)|\,(|V(H)|-1)\\&\qquad +|V(G)|\,|V(H)|-|E(G)|-|E(H)|\,.\end{aligned}$$

By this formula, Theorem 2.6, Lemma 2.1 and Remark 2.7, we obtain the following theorem.

Theorem 2.8 *Let G and H be two connected graphs with n_1,n_2 vertices and m_1,m_2 edges, respectively. Then*

$$\begin{aligned}\overline{S}_2(G+H) &= \frac{M_1(G)}{2}\Big(4n_1+2n_2-5\Big)+\frac{M_1(H)}{2}\Big(4n_2+2n_1-5\Big)\\&-M_2(G)-M_2(H)+2\Big(n_1(n_1-1)+n_2(n_2-1)+n_1n_2-m_1-m_2\Big)\\&-(2n_1+n_2-2)\Big((2n_1+n_2-2)(\frac{n_1}{2}+m_1)-2(m_1+n_1m_2)\Big)\\&-(2n_2+n_1-2)\Big((2n_2+n_1-2)(\frac{n_2}{2}-m_2)-2(m_2-n_2m_1)\\&-n_1n_2(2n_1+n_2-2)\Big)-4m_1m_2.\end{aligned}$$

2.3 Composition

The *composition* of two graphs G and H is denoted by $G[H]$. The vertex set of $G[H]$ is $V(G)\times V(H)$ and any two vertices (u_i,v_r) and (u_k,v_s) are adjacent if and only if $u_iu_k\in E(G)$ or $u_i=u_k$ and $v_rv_s\in E(H)$.

Theorem 2.9 *Let G and H be two connected graphs with n_1,n_2 vertices and m_1,m_2 edges, respectively.*

Then

$$\begin{aligned} S_2(G[H]) &= n_2^4 S_2(G) + 2n_2^2(n_2(n_2-1)-m_2)S_1(G) + 8n_2m_2(n_2-1)W(G) \\ &\quad -2n_2W(G)M_1(H) - 2n_1(n_2-1)M_1(H) + n_1M_2(H) \\ &\quad +n_2^2m_2 \sum_{u_i \in V(G)} (\sigma_G(u_i))^2 + 4(n_2-1)^2(n_1m_2+m_1n_2^2) \\ &\quad +4m_1m_2(m_2-2n_2(n_2-1)). \end{aligned}$$

Proof For the composition of two graphs, the degree of a vertex (u,v) of $G[H]$ is given by $d_{G[H]}((u,v)) = n_2d_G(u) + d_H(v)$. Moreover, the distance between two vertices (u_i,v_r) and (u_k,v_s) of $G[H]$ is

$$d_{G[H]}((u_i,v_r),(u_k,v_s)) = \begin{cases} d_G(u_i,u_k) & u_i \neq u_k \\ 2 & u_i = u_k,\ v_rv_s \notin E(H) \\ 1 & u_i = u_k,\ v_rv_s \in E(H). \end{cases}$$

Let (u_i,v_r) be a vertex of $G[H]$. Then

$$\begin{aligned} \sigma_{G[H]}((u_i,v_r)) &= \sum_{(u_k,v_s)\in V(G[H])} d_{G[H]}((u_i,v_r),(u_k,v_s)) \\ &= \sum_{(u_k,v_s)\in V(G[H]),\ u_i\neq u_k} d_G(u_i,u_k) + \sum_{(u_i,v_s)\in V(G[H])} d_{G[H]}((u_i,v_r),(u_i,v_s)) \\ &= n_2\sigma_G(u_i) + d_H(v_r) + 2(n_2-1-d_H(v_r)) \\ &= n_2\sigma_G(u_i) + 2(n_2-1) - d_H(v_r). \end{aligned} \tag{2.6}$$

From the structure of $G[H]$ and definition of S_2, we have

$$\begin{aligned} S_2(G[H]) &= \sum_{u_i\in V(G)} \sum_{v_rv_s\in E(H)} \sigma_{G[H]}((u_i,v_r))\sigma_{G[H]}((u_i,v_s)) \\ &\quad + \sum_{u_iu_k\in E(G)} \sum_{v_r\in V(H)} \sum_{v_s\in V(H)} \sigma_{G[H]}((u_i,v_r))\sigma_{G[H]}((u_i,v_s)) \\ &= A_1 + A_2, \end{aligned} \tag{2.7}$$

where,

$$\begin{aligned} A_1 &= \sum_{u_i\in V(G)} \sum_{v_rv_s\in E(H)} \sigma_{G[H]}((u_i,v_r))\sigma_{G[H]}((u_i,v_s)) \\ &= \sum_{u_i\in V(G)} \sum_{v_rv_s\in E(H)} \Big(n_2\sigma_G(u_i) + 2(n_2-1) - d_H(v_r)\Big)\Big(n_2\sigma_G(u_i) + 2(n_2-1) - d_H(v_s)\Big) \\ &= \sum_{u_i\in V(G)} \sum_{v_rv_s\in E(H)} \Big[n_2^2(\sigma_G(u_i))^2 + 2(n_2-1)n_2\sigma_G(u_i) - n_2\sigma_G(u_i)d_H(v_s) + 2(n_2-1)n_2\sigma_G(u_i) \\ &\quad +4(n_2-1)^2 - 2(n_2-1)d_H(v_s) - n_2\sigma_G(u_i)d_H(v_r) - 2(n_2-1)d_H(v_r) + d_H(v_r)d_H(v_s)\Big] \\ &= \sum_{u_i\in V(G)} \sum_{v_rv_s\in E(H)} \Big[n_2^2(\sigma_G(u_i))^2 + 4n_2(n_2-1)\sigma_G(u_i) + 4(n_2-1) - n_2\sigma_G(u_i)(d_H(v_r)+d_H(v_s)) \\ &\quad -2(n_2-1)(d_H(v_r)+d_H(v_s)) + d_H(v_r)d_H(v_s)\Big] \\ &= n_2^2m_2 \sum_{u_i\in V(G)} (\sigma_G(u_i))^2 + 8n_2(n_2-1)m_2W(G) + n_1M_2(H) \end{aligned}$$

$$-2n_2W(G)M_1(H) - 2(n_2-1)n_1M_1(H).$$

$$\begin{aligned}
A_2 &= \sum_{u_iu_k\in V(G)}\sum_{v_r\in V(H)}\sum_{v_s\in V(H)} \sigma_{G[H]}((u_i,v_r))\sigma_{G[H]}((u_k,v_s))\\
&= \sum_{u_iu_k\in V(G)}\sum_{v_r\in V(H)}\sum_{v_s\in V(H)} \Big(n_2\sigma_G(u_i)+2(n_2-1)-d_H(v_r)\Big)\Big(n_2\sigma_G(u_k)+2(n_2-1)-d_H(v_s)\Big)\\
&= \sum_{u_iu_k\in V(G)}\sum_{v_r\in V(H)}\sum_{v_s\in V(H)} \Big[n_2^2\sigma_G(u_i)\sigma_G(u_k)+2(n_2-1)n_2(\sigma_G(u_i)+\sigma_G(u_k))+4(n_2-1)^2\\
&\quad -n_2\sigma_G(u_i)d_H(v_s)-n_2d_H(v_r)\sigma_G(u_k)-2(n_2-1)(d_H(v_r)+d_H(v_r))+d_H(v_r)d_H(v_s)\Big]\\
&= n_2^4S_2(G)+2n_2^2(n_2(n_2-1)-m_2)S_1(G)-8n_2m_1m_2(n_2-1)+4m_1m_2^2+4(n_2-1)^2m_1n_2^2.
\end{aligned}$$

Hence

$$\begin{aligned}
S_2(G[H]) &= n_2^4S_2(G)+2n_2^2(n_2(n_2-1)-m_2)S_1(G)+8n_2m_2(n_2-1)W(G)\\
&\quad -2n_2W(G)M_1(H)-2n_1(n_2-1)M_1(H)+n_1M_2(H)\\
&\quad +n_2^2m_2\sum_{u_i\in V(G)}(\sigma_G(u_i))^2+4(n_2-1)^2(n_1m_2+m_1n_2^2)\\
&\quad +4m_1m_2(m_2-2n_2(n_2-1)). \qquad \square
\end{aligned}$$

Remark 2.10 Let (u_i,v_r) be a vertex of $G[H]$. Then

$$\begin{aligned}
\sum_{(u_i,v_r)\in V(G[H])}(\sigma_{G[H]}((u_i,v_r)))^2 &= \sum_{u_i\in V(G)}\sum_{v_r\in V(H)}(n_2\sigma_G(u_i)+2(n_2-1)-d_H(v_r))^2\\
&= \sum_{u_i\in V(G)}\sum_{v_r\in V(H)}\Big(n_2^2(\sigma_G(u_i))^2+4(n_2-1)^2+(d_H(v_r))^2\\
&\quad +4n_2(n_2-1)\sigma_G(u_i)-2n_2\sigma_G(u_i)d_H(v_r)-2(n_2-1)d_H(v_r)\Big)\\
&= n_2^3\sum_{u_i\in V(G)}(\sigma_G(u_i))^2+n_1M_1(H)\\
&\quad +4n_2(n_2(n_2-1)-m_2)\sum_{u_i\in V(G)}\sigma_G(u_i)\\
&\quad +4(n_2-1)(n_1n_2(n_2-1)-m_2).
\end{aligned}$$

Recall from [3] that

$$W(G[H]) = |V(H)|^2\,(W(G)+|V(G)|) - |V(G)|\,(|V(H)|+|E(H)|).$$

In the next theorem, we obtain a formula for $\overline{S}_1(G[H])$ according to $W(G[H])$, $S_2(G[H])$ and Remark 2.10.

Theorem 2.11 *Let G and H be two connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then*

$$\begin{aligned}
\overline{S}_2(G[H]) &= \Big(2n_2W(G)+2n_1(n_2-1)-\frac{n_1}{2}\Big)M_1(H)-n_1M_2(H)-n_2^2S_2(G)\\
&\quad -2n_2^2(n_2(n_2-1)-m_2)S_1(G)-\Big(8n_2m_2(n_1-1)+2n_2^2\Big)W(G)
\end{aligned}$$

$$-\frac{n_2^2}{2}(n_2-2m_2)\sum_{u_i\in V(G)}(\sigma_G(u_i))^2-2n_2(n_2(n_2-1)-m_2)\sum_{u_i\in V(G)}\sigma_G(u_i)$$
$$+n_1n_2(2n_2-1)-n_1m_2-2(n_2-1)^2(n_1n_2+2n_1m_2+2m_1n_2^2)$$
$$+2m_2(n_2-1)(4m_1n_2+1)-4m_1m_2^2.$$

References

[1] H. S. Ramane, A.S. Yalnaik, Status connectivity indices of graphs and its applications to the boiling point of benzenoid hydrocarbons, *J. Appl. Math. Comput.*, (2016), DOI 10.1007/s12190-016-1052-5.

[2] F. Harary, Status and contrastatus, *Sociometry*, 22(1959), 23-43.

[3] Y.N. Yeh, I. Gutman, On the sum of all distances in composite graphs, *Discrete Math.*, 135 (1994), 359-365.

[4] A. R. Ashrafi, A. Hamzeh, S. Hossein-Zadeh, Calculation of some topological indices of splice and link graphs, *J. Appl. Math. & Informatics*, 29(2011), 327 - 335.

[5] H. Wiener, Structural determination of the paraffin boiling points, *J. Amer. Chem. Soc.*, 69 (1947), 17-20.

[6] T.Doslić, Splices, links and their degree weighted Wiener polynomials, *Graph Theory Notes N.Y.*, 48 (2005), 47-55.

[7] J. Devillers, A.T. Balaban, Eds., *Topological Indices and Related Descriptors in QSAR and QSPR*, Gordon and Breach, Amsterdam, The Netherlands, 1999.

[8] A.R. Ashrafi, T. Doslić, A. Hamzeh, The Zagreb coindices of graph operations, *Discrete Appl. Math.*, 158(2010), 1571-1578.

[9] A.A. Dobrynin, R. Entringer, I. Gutman, Wiener index of trees: theory and applications, *Acta Appl. Math.*, 66 (2001), 211-249.

[10] K. Pattabiraman, P. Paulraja, On some topological indices of the tensor product of graphs, *Discrete Appl. Math.*, 160 (2012), 267-279.

[11] K. Pattabiraman, P. Paulraja, Wiener and vertex PI indices of the strong product of graphs, *Discuss. Math. Graph Thoery*, 32 (2012), 749-769.

[12] K. Pattabiraman, P. Paulraja, Wiener index of the tensor product of a path and a cycle, *Discuss. Math. Graph Thoery*, 31 (2011), 737-751.