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Position Vectors of the Curves in Affine 3-Space According to Special Affine Frames

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Abstract: In literature, there are three affine frames commonly used for space curves, which are called equi-affine frame, Winternitz frame and intrinsic affine frame, respectively. In this study, we examined the position vectors of the space curves in affine 3-space for each of these three frames separately, in terms of lying in the planes $\{T, N\}$, $\{T, B\}$ and $\{N, B\}$ which are known as osculating, rectifying and normal planes, respectively and we obtained the position vectors and we gave some conclusions.

Key Words: Position vector, equi-affine frame, Winternitz frame, intrinsic affine frame.

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§1. Introduction

Affine differential geometry is the study of differential invariants with respect to the group of affine transformation. The group of affine motions special linear transformation namely the group of equi-affine or unimodular transformations consist of volume preserving ($\det(a_{jk}) = 1$) linear transformations together with translation such that

$$x_j^* = \sum_{k=1}^{3} a_{jk} x_k + c_j \qquad j = 1, 2, 3$$

This transformations group denoted by $ASL(3, IR) := SL(3, IR) \times IR^3$ and comprising diffeomorphisms of IR^3 that preserve some important invariants such curvatures that in curve theory as well. An equi-affine group is also called an Euclidean group [3].

Salkowski and Schells gave the equi-affine frame [4], Kreyszig and Pendl gave the characterization of spherical curves in both Euclidean and affine 3-spaces [3]. Su classified the curves in affine 3-space by using equi-affine frame [6]. Winternitz dwelled on the insufficiency of equi-affine frame for curves class of C^5 and defined the new frame known as Winternitz frame [5,1]. Davis obtained new affine frame by defining intrinsic affine binormal and in this study, we called that frame as intrinsic affine frame [2].

A set of points that corresponds to a vector of vector space constructed on a field is called an *affine space* associated with vector space. We denote A_3 as affine 3-space associated with IR^3. Let

$$\alpha : J \longrightarrow A_3$$

represent a curve in A_3, where $t \in J = (t_1, t_2) \subset IR$ is fixed and open interval. Regularity of a curve

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in A_3 is defined as $\left| \begin{array}{ccc} \dot{\alpha} & \ddot{\alpha} & \dddot{\alpha} \end{array} \right| \neq 0$ on J, where $\dot{\alpha} = d\alpha/dt$, etc. Then, we may associate α with the invariant parameter

$$s = \sigma(t) = \int_{t_1}^{t} \left| \begin{array}{ccc} \dot{\alpha} & \ddot{\alpha} & \dddot{\alpha} \end{array} \right|^{1/6} dt$$

which is called the *affine arc length* of $\alpha(s)$. The coordinates of a curve are given by three linearly independent solutions of the equations

$$\alpha^{(iv)}(s) + k(s)\,\alpha''(s) + \tau_\alpha(s)\,\alpha'(s) = 0 \tag{1}$$

under the condition

$$\left| \begin{array}{ccc} \alpha'(s) & \alpha''(s) & \alpha'''(s) \end{array} \right| = 1 \tag{2}$$

where $k(s)$ and $\tau_\alpha(s)$ are differentiable functions of s.

§2. Position Vectors of the Curves in Affine 3-Space According to Equi-Affine Frame

Let $\alpha(s)$ be a regular curve with affine arc lengh parameter s. The vectors $\alpha'(s)$, $\alpha''(s)$ and $\alpha'''(s)$ are called tangent, affine normal and affine binormal vectors respectively, and the planes $sp\{\alpha'(s), \alpha''(s)\}$, $sp\{\alpha'(s), \alpha'''(s)\}$ and $sp\{\alpha''(s), \alpha'''(s)\}$ are called osculating, rectifying and normal planes of the curve $\alpha(s)$. Thus, the frame

$$\left\{ \begin{array}{rcl} T'(s) & = & N(s), \\ N'(s) & = & B(s), \\ B'(s) & = & -\tau_\alpha(s)T(s) - k(s)N(s) \end{array} \right. \tag{3}$$

is called equi-affine frame, where $k(s)$ and $\tau_\alpha(s)$ are called equi-affine curvature and equi-affine torsion, which are given as follow

$$k(s) = \left| \begin{array}{ccc} \alpha'(s) & \alpha'''(s) & \alpha^{(iv)}(s) \end{array} \right| \tag{4}$$

$$\tau_\alpha(s) = -\left| \begin{array}{ccc} \alpha''(s) & \alpha'''(s) & \alpha^{(iv)}(s) \end{array} \right|. \tag{5}$$

Let $f(s)$, $g(s)$ and $h(s)$ be differentiable functions then we can write

$$\alpha(s) = f(s)\,T(s) + g(s)\,N(s) + h(s)\,B(s) \tag{6}$$

and by differentiating equation (6) with respect to s and by using equations (3), we obtain

$$0 = \left\{ f'(s) - h(s)\,k_2(s) - 1 \right\} T(s) + \left\{ f(s) + g'(s) \right\} N(s) + \left\{ \begin{array}{c} h'(s) + g(s) \\ -h(s)\,k_1(s) \end{array} \right\} B(s)$$

Therefore, for $\alpha''(s) = N(s)$ and $B(s) = \alpha'''(s)$, we obtain the following theorem.

Theorem 2.1 *Let $\alpha(s)$ be a unit speed curve in A_3, with equi-affine curvature $k(s)$ and with equi-affine torsion $\tau_\alpha(s)$, then $\alpha(s)$ has the position vector in (6) according to equi-affine frame for some* *differentiable functions $f(s)$, $g(s)$ and $h(s)$ satisfy the equations*

$$\begin{cases} f'(s) - h(s)\,\tau_\alpha(s) &= 1, \\ f(s) + g'(s) &= 0, \\ h'(s) + g(s) - h(s)\,k(s) &= 0. \end{cases}$$

Assume that the position vector of $\alpha(s)$ always lies in the plane $sp\{N(s), B(s)\}$. Position vector of the curve $\alpha(s)$ satisfies the equation

$$\alpha(s) = g(s)\,N(s) + h(s)\,B(s) \tag{7}$$

for some differentiable functions $g(s)$ and $h(s)$. Differentiating equation (7) with respect to s, we obtain

$$0 = \{-h(s)\,\tau_\alpha(s) - 1\}\,T(s) + \{g'(s) - h(s)\,k(s)\}\,N + \{h'(s) + g(s)\}\,B(s)$$

It follows that

$$\begin{cases} h(s)\,\tau_\alpha(s) &= -1, \\ g'(s) - h(s)\,k(s) &= 0, \\ h'(s) + g(s) &= 0 \end{cases} \tag{8}$$

and $h(s) = \frac{-1}{\tau_\alpha(s)}$, $g'(s) = -h''(s)$. Therefore, from the second equation we get

$$h''(s) + h(s)\,k(s) = 0 \tag{9}$$

and also

$$\left(\frac{1}{\tau_\alpha(s)}\right)'' + \frac{k(s)}{\tau_\alpha(s)} = 0, \tag{10}$$

and we find

$$\alpha(s) = \left(\frac{1}{\tau_\alpha(s)}\right)' N(s) - \frac{1}{\tau_\alpha(s)} B(s).$$

By considering $\alpha''(s) = N(s)$ and $\alpha'''(s) = B(s)$, we have the following theorem.

Theorem 2.2 *Let $\alpha(s)$ be a unit speed curve in A_3, with nonzero equi-affine curvatures satisfying*

$$\left(\frac{1}{\tau_\alpha(s)}\right)'' + \frac{k(s)}{\tau_\alpha(s)} = 0,$$

then, $\alpha(s)$ is a curve whose position vector according to equi-affine frame always lies in the sp$\{N(s), B(s)\}$ if and only if $\alpha(s)$ is the solution of the differential equation of

$$\frac{1}{\tau_\alpha(s)}\alpha'''(s) - \left(\frac{1}{\tau_\alpha(s)}\right)' \alpha''(s) + \alpha(s) = 0.$$

In the case of $k(s) = 0$, from the first and the second equation of (8) $g(s) = c_0$, $h(s) = \frac{-1}{\tau_\alpha(s)}$ and from the third equation of (8), we get $\tau_\alpha(s) = \frac{1}{c_0 s - c_1}$. Thus, from (7), the position vector of $\alpha(s)$ satisfies the following differential equation

$$(c_0 s - c_1)\alpha'''(s) - c_0\alpha''(s) + \alpha(s) = 0.$$

In the case of $k(s)$ nonzero constant, from the second and the third equation of (8)

$$\begin{cases} g(s) &= c_2\sqrt{k}\sin(\sqrt{k}s) - c_1\sqrt{k}\cos(\sqrt{k}s) \\ h(s) &= c_1\sin(\sqrt{k}s) + c_2\cos(\sqrt{k}s) \end{cases}$$

and from the first equation of (8)

$$\tau_\alpha(s) = \frac{-1}{c_1\sin(\sqrt{k}s) + c_2\cos(\sqrt{k}s)}.$$

From (7), the position vector of $\alpha(s)$ satisfies the following differential equation

$$\left(c_2\sqrt{k}\sin(\sqrt{k}s) - c_1\sqrt{k}\cos(\sqrt{k}s)\right)\alpha''(s) + \left(c_1\sin(\sqrt{k}s) + c_2\cos(\sqrt{k}s)\right)\alpha'''(s) = \alpha(s)$$

It is clear that $\tau_\alpha(s)$ cannot be zero from the first equation of (8).

In the case of $\tau_\alpha(s)$ nonzero constant, from the first and the third equation of (8) $g(s) = 0$, $h(s) = \frac{-1}{\tau_\alpha}$ and from the second equation of (8), we obtain $k(s) = 0$. From (7), the position vector of $\alpha(s)$ satisfies the following differential equation

$$\alpha'''(s) + \tau_\alpha\alpha(s) = 0$$

$$\alpha(s) = c_1 e^{\frac{\sqrt[3]{-\tau_\alpha}bs}{2}} + c_2 e^{\frac{-\sqrt[3]{-\tau_\alpha}as}{2}} + c_3 e^{\sqrt[3]{-\tau_\alpha}s}.$$

In the case of $\tau_\alpha(s)$ and $k(s)$ nonzero constants, from the first and the third equation of (8) $g(s) = 0$, $h(s) = \frac{-1}{\tau_\alpha}$ and from the second equation of (8), we obtain $k(s) = 0$. By using (7), the position vector of $\alpha(s)$ satisfies the following differential equation

$$\alpha'''(s) + \tau_\alpha\alpha(s) = 0$$

$$\alpha(s) = c_1 e^{\frac{\sqrt[3]{-\tau_\alpha}bs}{2}} + c_2 e^{\frac{-\sqrt[3]{-\tau_\alpha}as}{2}} + c_3 e^{\sqrt[3]{-\tau_\alpha}s}$$

where a and b are scalars that can be complex in general. Hence, we know the following theorem.

Theorem 2.3 *Let $\alpha(s)$ be a unit speed curve in A_3 , with the equi-affine curvature $k(s)$ and with the intrinsic affine torsion $\widetilde{\tau}_\alpha(s)$ whose position vector lies in $sp\{N(s), B(s)\}$ then the followings are true,*

(1) *If $k(s) = 0$ and $\tau_\alpha(s) = \frac{1}{c_0 s - c_1}$ then position vector of $\alpha(s)$ satisfies the equation*

$$(c_0 s - c_1)\alpha'''(s) - c_0\alpha''(s) + \alpha(s) = 0;$$

(2) *If $k(s) > 0$ constant and $\tau_\alpha(s) = \frac{-1}{\omega}$ then position vector of $\alpha(s)$ satisfies the equation*

$$\omega\alpha'''(s) - \omega'\alpha''(s) - \alpha(s) = 0,$$

where $\omega = c_1\sin(\sqrt{k}s) + c_2\cos(\sqrt{k}s)$;

(3) *There is no curve whose $\tau_\alpha(s) = 0$ in A_3;*

(4) *If $\tau_\alpha(s) < 0$ constant then $k(s) = 0$ and position vector of $\alpha(s)$ is*

$$\alpha(s) = c_1 e^{\rho_1 s} + c_2 e^{\rho_2 s} + c_3 e^{\rho_3 s}.$$

where $\rho_1 = \frac{\sqrt[3]{-\tau_\alpha}b}{2}$, $\rho_2 = \frac{-\sqrt[3]{-\tau_\alpha}a}{2}$ and $\rho_3 = \sqrt[3]{-\tau_\alpha}$. Here, a and b are scalars that can be complex in general.

We assume that the position vector of $\alpha(s)$ always lies in the plane $\mathrm{sp}\{T(s), B(s)\}$. Position vector of the curve $\alpha(s)$ satisfies equation

$$\alpha(s) = f(s)\,T(s) + h(s)\,B(s) \tag{11}$$

for some differentiable functions $f(s)$ and $h(s)$. Differentiating equation (11) with respect to s, we obtain

$$0 = \left\{f'(s)\,T(s) - h(s)\,\tau_\alpha(s) - 1\right\}T + \{f(s) - h(s)\,k(s)\}\,N + h'(s)\,B(s).$$

It follows that

$$\begin{cases} f'(s) - h(s)\,\tau_\alpha(s) & = & 1, \\ f(s) - h(s)\,k(s) & = & 0, \\ h'(s) = 0 \end{cases}$$

and it is clear that$h(s) = c_0$ and then,

$$\begin{cases} f'(s) - c_0\tau_\alpha(s) & = & 1, \\ f(s) - c_0 k(s) & = & 0, \\ k'(s) - \tau_\alpha(s) & = & \frac{1}{c_0}. \end{cases}$$

Therefore, we obtained

$$\alpha(s) = c_0 k_1(s) T(s) + c_0 B(s).$$

By considering $\alpha'(s) = T(s)$ and $\alpha'''(s) = B(s)$, we can give the following theorem.

Theorem 2.4 *Let $\alpha(s)$ be a unit speed curve in A_3 , with nonzero affine curvatures satisfying*

$$k'(s) - \tau_\alpha(s) = \frac{1}{c_0},$$

then, α is a curve whose position vector according to equi-affine frame always lies in the sp$\{T(s), B(s)\}$ if and only if α is the solution of the differential equation of

$$c_0 k(s)\alpha'(s) + c_0\alpha'''(s) - \alpha(s) = 0.$$

In the case of $k(s) = 0$, from the second and the third equation of (12)

$$h(s) = c_0, \quad f(s) = 0, \; \widetilde{\tau}_\alpha(s) \neq 0$$

and from the first equation of (12) we get

$$\tau_\alpha(s) = \frac{-1}{c_0}.$$

Thus, from (11), the position vector of $\alpha(s)$ satisfies the following differential equation

$$c_0\alpha'''(s) - \alpha(s) = 0$$

$$\alpha(s) = c_1 e^{\frac{-as}{2(c_0)^{1/3}}} + c_2 e^{\frac{bs}{2(c_0)^{1/3}}} + c_3 e^{\frac{s}{(c_0)^{1/3}}}.$$

In the case of $k(s)$ nonzero constant, from the second and the third equation of (12) $h(s) = c_0$, $f(s) = c_0 k$ and from the first equation of (12) $\tau_\alpha(s) = \frac{-1}{c_0}$. From (11), the position vector of $\alpha(s)$

satisfies the following differential equation

$$c_0\alpha'''(s) + c_0k\alpha'(s) - \alpha(s) = 0$$

$$\begin{aligned}\alpha(s) &= c_1 e^{\frac{\left\{-(c_0)^{1/3}2^{-1/3}a\lambda^2+k\left\{-2\lambda(c_0)^{2/3}+c_0k^3 2^{1/3}b\right\}\right\}s}{6(c_0)^{2/3}\lambda}} \\ &\quad + c_2 e^{\frac{-\left\{-(c_0)^{1/3}2^{-1/3}b\lambda^2+k\left\{2\lambda(c_0)^{2/3}+c_0k^3 2^{1/3}a\right\}\right\}s}{6(c_0)^{2/3}\lambda}} \\ &\quad + c_3 e^{\frac{\left\{(c_0)^{1/3}2^{-1/3}\lambda^2+c_0k^2 2^{1/3}-(c_0)^{2/3}k\lambda\right\}s}{3(c_0)^{2/3}\lambda}}\end{aligned}$$

where

$$\lambda = (-2k^3c_0 + 3\sqrt{-12c_0k^3+81} + 27)^{1/3}$$

and a, b are scalars that can be complex in general.

In the case of $\tau_\alpha(s) = 0$, from the second and the third equation of (12) $h(s) = c_0$, $f(s) = c_0k(s)$ and from the first equation of (12), we obtain

$$k(s) = \frac{1}{c_0}s + c_1.$$

From (11), the position vector of $\alpha(s)$ satisfies the following differential equation

$$c_0\alpha'''(s) + c_0k(s)\alpha'(s) - \alpha(s) = 0.$$

In the case of $\tau_\alpha(s)$ nonzero constant, from the second and the third equation of (12) $h(s) = c_0$, $f(s) = c_0k(s)$ and from the first equation of (11), we obtain $k(s) = \frac{1+c_0\tau_\alpha}{c_0}s + c_1$. From (12), the position vector of $\alpha(s)$ satisfies the following differential equation

$$c_0\alpha'''(s)c_0 + k(s)\alpha'(s) - \alpha(s) = 0.$$

In the case of $\tau_\alpha(s)$ and $k(s)$ nonzero constants, from the second and the third equation of (12) $h(s) = c_0$, $f(s) = c_0k$ and from the first equation of (12) we obtain $\tau_\alpha = \frac{1}{c_0}$. By using (11), the position vector of $\alpha(s)$ satisfies the following differential equation

$$c_0\alpha'''(s) + c_0k\alpha'(s) - \alpha(s) = 0.$$

Hence, we obtain the following theorem.

Theorem 2.5 *Let $\alpha(s)$ be a unit speed curve in A_3 , with the equi-affine curvature $k(s)$ and with the intrinsic affine torsion $\widetilde{\tau}_\alpha(s)$ whose position vector lies in $sp\{N(s), B(s)\}$ then, the followings are true.*

(1) *If $k(s) = 0$ and $\tau_\alpha(s) = \frac{-1}{c_0}$ then position vector of $\alpha(s)$ satisfies the equation*

$$\alpha(s) = c_1e^{\phi_1 s} + c_2e^{\phi_2 s} + c_3e^{\phi_3 s}$$

where $\phi_1 = \frac{-a}{2(c_0)^{1/3}}$, $\phi_2 = \frac{b}{2(c_0)^{1/3}}$ and $\phi_3 = \frac{1}{(c_0)^{1/3}}$;

(2) *If $\tau_\alpha(s)$ and $k(s)$ are nonzero constants then position vector of $\alpha(s)$ satisfies the equation*

$$\alpha(s) = c_1e^{\varphi_1 s} + c_2e^{\varphi_2 s} + c_3e^{\varphi_3 s}$$

where

$$\varphi_1 = \frac{\left\{-(c_0)^{1/3}2^{-1/3}a\lambda^2 + k\left\{-2\lambda(c_0)^{2/3} + c_0k^3 2^{1/3}b\right\}\right\}}{6(c_0)^{2/3}\lambda}$$

$$\varphi_2 = \frac{-\left\{-(c_0)^{1/3}2^{-1/3}b\lambda^2 + k\left\{2\lambda(c_0)^{2/3} + c_0k^3 2^{1/3}a\right\}\right\}}{6(c_0)^{2/3}\lambda}$$

$$\varphi_3 = \frac{\left\{(c_0)^{1/3}2^{-1/3}\lambda^2 + c_0k^2 2^{1/3} - (c_0)^{2/3}k\lambda\right\}}{3(c_0)^{2/3}\lambda},$$

$$\lambda = (-2k^3c_0 + 3\sqrt{-12c_0k^3 + 81} + 27)^{1/3}$$

and $c_1, c_2, c_3 \in IR^3$ *such that* $\begin{vmatrix} c_1 & c_2 & c_3 \end{vmatrix} = 1$ *and a, b are scalars that can be complex in general;*

(3) *If* $\tau_\alpha(s) = 0$*, and* $k(s) = \frac{1}{c_0}s + c_1$ *or* $\tau_\alpha(s)$ *nonzero constant and* $k(s) = \frac{1+c_0\tau_\alpha}{c_0}s + c_1$ *then position vector of* $\alpha(s)$ *satisfies the equation*

$$c_0\alpha'''(s)c_0 + k(s)\alpha'(s) - \alpha(s) = 0.$$

Now, assume that the position vector of $\alpha(s)$ always lies in the plane sp$\{T(s), N(s)\}$. Position vector of the curve α satisfies equation

$$\alpha(s) = f(s)T(s) + g(s)N(s) \qquad (13)$$

for some differentiable functions $f(s)$ and $g(s)$. Differentiating equation (13) with respect to s, we obtain

$$0 = \left\{f'(s) - 1\right\}T(s) + \left\{g'(s) + f(s)\right\}N(s) + g(s)B(s).$$

It follows that

$$\begin{cases} f'(s) & = 1, \\ g'(s) + f(s) & = 0, \\ g(s) & = 0. \end{cases} \qquad (14)$$

There is no $f(s)$ and $g(s)$ satisfying equations (14). Thus, we get the following theorem.

Theorem 2.6 *There is no curve in* A_3 *whose position vector always lies in the sp*$\{T(s), N(s)\}$ *according to equi-affine frame.*

§3. Position Vectors of the Curves in Affine 3-Space According to Winternitz Frame

Let $\alpha(s)$ be regular C^5-curve with affine arclenght parameter s. A. Winternitz in [5] defined a new equi-affine frame by taking

$$T(s) = \alpha'(s), N(s) = \alpha''(s), \quad B(s) = \alpha'''(s) + \frac{k(s)}{4}\alpha'(s)$$

which are called tangent, affine normal, binormal vectors and

$$k_1(s) = \frac{-k(s)}{4}, \quad k_2(s) = \frac{k'(s)}{4} - \tau_\alpha(s)$$

which are called the first and the secod affine curvatures (also we called them the first and the second Winternitz affine curvatures). Here, $k(s)$ and $\tau_\alpha(s)$ are called equi-affine curvature and equi-affine torsion given in (4) and (5). Winternitz frame (also called equi-affine frame for C^5-curves) is defined with the equations

$$\begin{cases} T'(s) &= N(s) \\ N'(s) &= k_1(s)T(s)+B(s) \\ B'(s) &= k_2(s)T(s)+3k_1(s)N(s). \end{cases} \tag{15}$$

Let $f(s)$, $g(s)$ and $h(s)$ be differentiable functions, then we can write

$$\alpha(s) = f(s)\,T(s) + g(s)\,N(s) + h(s)\,B(s). \tag{16}$$

Differentiating equation (16) with respect to s and by using equations (15), we obtain

$$0 = \left\{ \begin{array}{c} f'(s) + g(s)\,k_1(s) \\ +h(s)\,k_2(s) - 1 \end{array} \right\} T(s) + \left\{ \begin{array}{c} g'(s) + f(s) \\ +3h(s)\,k_1(s) \end{array} \right\} N(s) + \left\{ h'(s) + g(s) \right\} B(s)$$

Therefore, for $\alpha''(s) = N(s)$ and $B(s) = \alpha'''(s) + \frac{k(s)}{4}\alpha'(s)$, we get the following theorem.

Theorem 3.1 *Let $\alpha(s)$ be a unit speed curve in A_3, with Winternitz curvatures $k_1(s)$ and $k_2(s)$, then $\alpha(s)$ has the position vector in (17) according to Winternitz frame for some differentiable functions $f(s)$, $g(s)$ and $h(s)$ satisfies the equations*

$$\begin{cases} f'(s) + g(s)\,k_1(s) + h(s)\,k_2(s) &= 1 \\ g'(s) + f(s) + 3h(s)\,k_1(s) &= 0 \\ h'(s) + g(s) &= 0. \end{cases}$$

Assume that the position vector of $\alpha(s)$ always lies in the plane $\mathrm{sp}\{N(s), B(s)\}$. Position vector of the curve $\alpha(s)$ satisfies the equation

$$\alpha(s) = g(s)\,N(s) + h(s)\,B(s) \tag{17}$$

for some differentiable functions $g(s)$ and $h(s)$. Differentiating equation (17) with respect to s, we obtain

$$0 = \{g(s)\,k_1(s) + h(s)\,k_2(s) - 1\}\,T(s) + \left\{g'(s) + 3h(s)\,k_1(s)\right\} N(s) + \left\{h'(s) + g(s)\right\} B(s).$$

Thus, we have the following equations

$$\begin{cases} g(s)\,k_1(s) + h(s)\,k_2(s) &= 1 \\ g'(s) + 3h(s)\,k_1(s) &= 0 \\ h'(s) + g(s) &= 0. \end{cases} \tag{18}$$

From the first and the third equation of (18)

$$h'(s)\,k_1(s) - h(s)\,k_2(s) + 1 = 0$$

then solution is

$$h(s) = \varphi \left\{ -\int \frac{ds}{\varphi k_1(s)} + c_0 \right\}, \tag{19}$$

where $\varphi = e^{\int \frac{k_2(s)}{k_1(s)} ds}$ and from the second equation for $g'(s) = -h''(s)$ we get

$$h''(s) - 3h(s)\, k_1(s) = 0. \tag{20}$$

Then by using (19), (20) it turns to

$$\varphi \left\{ \int \frac{ds}{\varphi k_1(s)} - c_0 \right\} \left\{ 3k_1(s) - \frac{(k_2(s))^2}{(k_1(s))^2} - \left\{ \frac{k_2(s)}{k_1(s)} \right\}' \right\} - \frac{k_2(s)}{(k_1(s))^2} + \frac{k_1'(s)}{(k_1(s))^2} = 0 \tag{21}$$

and we find

$$\alpha(s) = \left\{ \varphi' \int \frac{ds}{\varphi k_1(s)} - c_0 \varphi' + \frac{\varphi}{\varphi k_1(s)} \right\} N(s) - \varphi \left\{ \int \frac{ds}{\varphi k_1(s)} - c_0 \right\} B(s).$$

By considering $\alpha''(s) = N(s)$ and $B(s) = \alpha'''(s) + \frac{k(s)}{4} \alpha'(s)$, we get the following theorem.

Theorem 3.2 *Let $\alpha(s)$ be a unit speed curve in A_3 , with nonzero Winternitz curvatures satisfing (21), then $\alpha(s)$ is a curve whose position vector according to Winternitz affine frame always lies in $sp\{N(s), B(s)\}$ if and only if $\alpha(s)$ is the solution of the differential equation of*

$$\left\{ \begin{array}{c} \varphi \left\{ \int \frac{ds}{\varphi k_1(s)} - c_0 \right\} \alpha'''(s) - \left\{ \varphi' \int \frac{ds}{\varphi k_1(s)} - c_0 \varphi' + \frac{\varphi}{\varphi k_1(s)} \right\} \alpha''(s) \\ -\varphi \left\{ \int \frac{ds}{\varphi k_1(s)} - c_0 \right\} k_1(s) \alpha'(s) + \alpha(s) \end{array} \right\} = 0.$$

In the case of $k_1(s) = 0$, from (18), we obtain $g(s) = c_0$, $h(s) = -c_0 s + c_1$ and $k_2(s) = \frac{1}{-c_0 s + c_1}$. From (17), position vector of $\alpha(s)$ satisfies

$$(c_0 s - c_1)\, \alpha'''(s) - c_0 \alpha''(s) + \alpha(s) = 0.$$

In the case of $k_1(s) \neq 0$ constant, from the second and the third equation of (18), we get

$$h''(s) - 3k_1 h(s) = 0$$

and the solution is

$$h(s) = c_1 e^{s\sqrt{3k_1}} + c_2 e^{-s\sqrt{3k_1}}.$$

Also, from the first equation, we get the second curvature is

$$k_2(s) = \frac{1 + c_1 \sqrt{3k_1} e^{s\sqrt{3k_1}} - c_2 \sqrt{3k_1} e^{-s\sqrt{3k_1}}}{\left(c_1 e^{s\sqrt{3k_1}} + c_2 e^{-s\sqrt{3k_1}} \right)}$$

and so $g(s)$ is

$$g(s) = \sqrt{3k_1} \left\{ c_2 e^{-s\sqrt{3k_1}} - c_1 e^{s\sqrt{3k_1}} \right\}.$$

From (17), position vector of $\alpha(s)$ satisfies

$$\left\{ \begin{array}{c} \left(c_1 e^{s\sqrt{3k_1}} + c_2 e^{-s\sqrt{3k_1}} \right) \alpha'''(s) + \sqrt{3k_1} \left\{ c_2 e^{-s\sqrt{3k_1}} - c_1 e^{s\sqrt{3k_1}} \right\} \alpha''(s) \\ -k_1 \left(c_1 e^{s\sqrt{3k_1}} + c_2 e^{-s\sqrt{3k_1}} \right) \alpha'(s) - \alpha(s) \end{array} \right\} = 0.$$

In the case of $k_2(s)=0$, from the first and the third equation of (18), we obtain $g(s)=\frac{1}{k_1(s)}$, $h(s)=-\int\frac{ds}{k_1(s)}+c_0$ and from the second equation of (18), we obtain the relation of the curvatures

$$k_1''(s)k_1(s)-3k_1'(s)+3k_1(s)^3=0.$$

Thus, from (17), position vector of $\alpha(s)$ satisfies

$$\left(-\int\frac{ds}{k_1(s)}+c_0\right)\alpha'''(s)+\frac{1}{k_1(s)}\alpha''(s)-\left(-\int\frac{ds}{k_1(s)}+c_0\right)k_1(s)\alpha'(s)-\alpha(s)=0.$$

In the case of $k_2(s)$ nonzero constant, from the first and the third equation of (18), we obtain

$$h(s)=\left\{\frac{1}{k_2}+c_0e^{k_2\int\frac{ds}{k_1(s)}}\right\},\quad g(s)=-\frac{c_0k_2}{k_1(s)}e^{k_2\int\frac{ds}{k_1(s)}}$$

and from the second equation of (18), we obtain the relation between the curvatures as follows

$$\left\{\frac{3k_1(s)^3-(k_2)^2}{k_1(s)^2}\right\}c_0e^{k_2\int\frac{ds}{k_1(s)}}+\frac{3k_1(s)}{k_2}=0.$$

From (17), position vector of $\alpha(s)$ satisfies

$$\left\{\begin{array}{c}\left\{\frac{1}{k_2}+c_0e^{k_2\int\frac{ds}{k_1(s)}}\right\}\alpha'''(s)-\frac{c_0k_2}{k_1(s)}e^{k_2\int\frac{ds}{k_1(s)}}\alpha''(s)\\ -\left\{\frac{1}{k_2}+c_0e^{k_2\int\frac{ds}{k_1(s)}}\right\}k_1(s)\alpha'(s)-\alpha(s)\end{array}\right\}=0.$$

In the case of $k_1(s)$ and $k_2(s)$ nonzero constants, from the first and the third equation of (18), we get

$$h(s)=\frac{1}{k_2}+c_0e^{\frac{k_2}{k_1}s},\quad g(s)=-c_0\frac{k_2}{k_1}e^{\frac{k_2}{k_1}s}$$

and also from the second equation of (18), we get the relation between the curvatures as follows

$$\left\{3(k_1)^2-(k_2)^2\right\}c_0k_2e^{\frac{k_2}{k_1}s}+3(k_1)^3=0.$$

From (17), position vector of $\alpha(s)$ satisfies

$$\left(\frac{1}{k_2}+c_0e^{\frac{k_2}{k_1}s}\right)\alpha'''(s)\ -c_0\frac{k_2}{k_1}e^{\frac{k_2}{k_1}s}\alpha''(s)-\left(\frac{1}{k_2}+c_0e^{\frac{k_2}{k_1}s}\right)k_1\alpha'(s)-\alpha(s)=0.$$

Therefore, we get the following theorem.

Theorem 3.3 *Let $\alpha(s)$ be a unit speed curve in A_3 , with the Winternitz curvatures $k_1(s)$ and $k_2(s)$, whose position vector lies in $sp\{N(s),B(s)\}$ then, the followings are true.*

(1) *If $k_1(s)=0$ and $k_2(s)=\frac{1}{-c_0s+c_1}$ then position vector of $\alpha(s)$ satisfies the equation*

$$(c_0s-c_1)\alpha'''(s)-c_0\alpha''(s)+\alpha(s)=0;$$

(2) *If $k_1(s)>0$ is constant and $k_2(s)=\frac{1+\widetilde{\varphi}'}{\widetilde{\varphi}}$ then position vector of $\alpha(s)$ satisfies the equation*

$$\widetilde{\varphi}v\alpha'''(s)-\widetilde{\varphi}'\alpha''(s)-k_1\widetilde{\varphi}\alpha'(s)-\alpha(s)=0$$

where $\widetilde{\varphi}=c_1e^{s\sqrt{3k_1}}+c_2e^{-s\sqrt{3k_1}}$;

(3) *If $k_2(s)=0$ and $k_1(s)$ satisfy $k_1''(s)k_1(s)-3k_1'(s)+3k_1(s)^3=0$ then position vector of $\alpha(s)$*

satisfies the equation

$$\psi\alpha'''(s)+\frac{1}{k_1(s)}\alpha''(s)-\psi k_1(s)\alpha'(s)-\alpha(s)=0$$

where $\psi=-\int\frac{ds}{k_1(s)}+c_0$;

(4) *If* $k_2(s)$ *is nonzero constant,* $k_1(s)$ *and* k_2 *satisfy*

$$\left\{\frac{3k_1(s)^3-(k_2)^2}{k_1(s)^2}\right\}c_0e^{k_2\int\frac{ds}{k_1(s)}}+\frac{3k_1(s)}{k_2}=0$$

then, the position vector of $\alpha(s)$ *satisfies the equation*

$$\left\{\frac{1}{k_2}+c_0\phi\right\}\alpha'''(s)-\frac{c_0\phi k_2}{k_1(s)}\alpha''(s)-\left\{\frac{1}{k_2}+c_0\phi\right\}k_1(s)\alpha'(s)-\alpha(s)=0$$

where $\phi=e^{k_2\int\frac{ds}{k_1(s)}}$;

(5) *If* $k_1(s)$, $k_2(s)$ *nonzero constants and satisfy the equation*

$$\left\{3(k_1)^2-(k_2)^2\right\}c_0k_2e^{\frac{k_2}{k_1}s}+3(k_1)^3=0$$

then, the position vector of $\alpha(s)$ *satisfies the equation*

$$\left(\frac{1}{k_2}+c_0\upsilon\right)\alpha'''(s)\ -c_0\frac{k_2}{k_1}\upsilon\alpha''(s)-\left(\frac{1}{k_2}+c_0\upsilon\right)k_1\alpha'(s)-\alpha(s)=0$$

where $\upsilon=e^{\frac{k_2}{k_1}s}$.

We assume that the position vector of $\alpha(s)$ always lies in the plane $\mathrm{sp}\{T(s),B(s)\}$. Position vector of the curve $\alpha(s)$ satisfies equation

$$\alpha(s)=f(s)T(s)+h(s)B(s) \tag{22}$$

for some differentiable functions $f(s)$ and $h(s)$. Differentiating equation (22) with respect to s, we obtain

$$0=\left\{f'(s)+h(s)k_2(s)-1\right\}T(s)+\{f(s)+3h(s)k_1(s)\}N(s)+h'(s)B(s),$$

it follows that

$$\begin{cases} f'(s)+h(s)k_2(s)=1\\ f(s)+3h(s)k_1(s)=0\\ h'(s)=0\end{cases} \tag{23}$$

for $h(s)=c_0$. From the first and second equations of (23), we get

$$k_2(s)-3k_1'(s)=\frac{1}{c_0}$$

and

$$f(s)=-3c_0k_1(s).$$

Therefore, we obtained

$$\alpha(s)=-3c_0k_1(s)T(s)+c_0B(s).$$

By considering $\alpha'(s)=T(s)$ and $B(s)=\alpha'''(s)+\frac{k(s)}{4}\alpha'(s)$, we get the following theorem.

Theorem 3.4 *Let $\alpha(s)$ be a unit speed curve in A_3 with nonzero Winternitz curvatures satisfing $k_2(s) - 3k_1'(s) = \frac{1}{c_0}$, then, $\alpha(s)$ is a curve whose position vector according to Winternitz affine frame always lies in $sp\{T(s), B(s)\}$ if and only if $\alpha(s)$ is the solution of the differential equation of*

$$c_0\alpha'''(s) - 4c_0k_1(s)\alpha'(s) - \alpha(s) = 0$$

In the case of $k_1(s) = 0$, from (23), we find $h(s) = c_0$, $f(s) = 0$ and $k_2(s) \neq 0$. From (22), we get

$$c_0\alpha'''(s) - \alpha(s) = 0$$

and the solution is

$$\alpha(s) = c_1 e^{\frac{-as}{2(c_0)^{1/3}}} + c_2 e^{\frac{bs}{2(c_0)^{1/3}}} + c_3 e^{\frac{s}{(c_0)^{1/3}}}.$$

In the case of $k_1(s)$ nonzero constant, from the second and the third equation of (23), we obtained $h(s) = c_0$, $f(s) = -3c_0k_1$ and also $k_2(s) = \frac{1}{c_0}$. From (22), we get $\alpha(s)$ satisfies the equation

$$c_0\alpha'''(s) - 4c_0k_1\alpha'(s) - \alpha(s) = 0.$$

In the case of $k_2(s) = 0$ constant, from the first and the third equation of (23) $f(s) = s + c_1$, $h(s) = c_0$ and from the second equation of (23), we obtain $k_1(s) = \frac{s+c_1}{-3c_0}$. From (22), we get that $\alpha(s)$ satisfies the equation

$$3c_0\alpha'''(s) + 4(s + c_1)\alpha'(s) - 3\alpha(s) = 0.$$

In the case of $k_2(s)$ nonzero constant, from the first and the third equation of (23) $h(s) = c_0$, $f(s) = (1 - c_0k_2)s + c_1$ and from the second equation of (23), we obtain $k_1(s) = \frac{(c_0k_2-1)s-c_1}{3c_0}$. From (22), we get that $\alpha(s)$ satisfies the equation

$$3c_0\alpha'''(s) + 4\left((1 - c_0k_2)s + c_1\right)\alpha'(s) - 3\alpha(s) = 0.$$

In the case of $k_1(s)$ and $k_2(s)$ nonzero constants, from the first and the third equation of (23) $h(s) = c_0$, $f(s) = -3c_0k_1$ and also from the second equation of (23), we obtain $k_2 = \frac{-1}{c_0}$. From (22), we get

$$c_0\alpha'''(s) - 4c_0k_1\alpha'(s) - \alpha(s) = 0$$

and the solution is

$$\begin{aligned}\alpha(s) \quad = \quad & c_1 e^{\frac{\left\{4k_1(c_0)^{2/3}b12^{1/3} - a\varphi^2\right\}s}{12\varphi}} + c_2 e^{\frac{-\left\{4k_1(c_0)^{2/3}12^{1/3}a - \varphi^2 b\right\}s}{12\varphi}} \\ & + c_3 e^{\frac{12^{1/3}\left\{4k_1(c_0)^{2/3}12^{1/3} + \varphi^2\right\}s}{6\varphi(c_0)^{1/3}}},\end{aligned}$$

where $\varphi = \left\{9 + \sqrt{-768(c_0)^2(k_1)^3 + 81}\right\}^{1/3}$. Hence, we can give the following theorem.

Theorem 3.5 *Let $\alpha(s)$ be a unit speed curve in A_3 with the Winternitz curvatures $k_1(s)$ and $k_2(s)$, whose position vector lies in $sp\{N(s), B(s)\}$, then, the followings are true.*

(1) *If $k_1(s) = 0$ then position vector of $\alpha(s)$ is*

$$\alpha(s) = c_1 e^{r_1 s} + c_2 e^{r_2 s} + c_3 e^{r_3 s}$$

for some $k_2(s)$, where $r_1 = \frac{-a}{2(c_0)^{1/3}}$, $r_2 = \frac{-b}{2(c_0)^{1/3}}$ and $r_3 = \frac{1}{(c_0)^{1/3}}$;

(2) *If $k_1(s)$ and $k_2(s)$ are nonzero constants then, position vector of $\alpha(s)$ is*

$$\alpha(s) = c_1 e^{\psi_1 s} + c_2 e^{\psi_2 s} + c_3 e^{\psi_3 s},$$

where

$$\begin{aligned} \psi_1 &= \frac{\left\{4k_1(c_0)^{2/3} b 12^{1/3} - a\varphi^2\right\}}{12\varphi} \\ \psi_2 &= \frac{-\left\{4k_1(c_0)^{2/3} 12^{1/3} a - \varphi^2 b\right\}}{12\varphi} \\ \psi_3 &= \frac{12^{1/3}\left\{4k_1(c_0)^{2/3} 12^{1/3} + \varphi^2\right\}}{6\varphi(c_0)^{1/3}} \end{aligned}$$

$$\varphi = \left\{9 + \sqrt{-768(c_0)^2(k_1)^3 + 81}\right\}^{1/3}$$

and a, b are scalars that can be complex in general;

(3) *If $k_2(s) = 0$ and $k_1(s) = \frac{s+c_1}{-3c_0}$ then, position vector of $\alpha(s)$ satisfies the equation*

$$3c_0\alpha'''(s) + 4(s + c_1)\alpha'(s) - 3\alpha(s) = 0;$$

(4) *If $k_2(s)$ is nonzero constant and $k_1(s) = \frac{(c_0k_2-1)s-c_1}{3c_0}$ then, position vector of $\alpha(s)$ satisfies the equation*

$$3c_0\alpha'''(s) + 4\left((1 - c_0k_2)s + c_1\right)\alpha'(s) - 3\alpha(s) = 0.$$

Now, assume that the position vector of $\alpha(s)$ always lies in the plane $\mathrm{sp}\{T(s), N(s)\}$. Position vector of the curve $\alpha(s)$ satisfies equations

$$\alpha(s) = f(s)T(s) + g(s)N(s) \tag{24}$$

and

$$0 = \left\{f'(s) + g(s)k_1(s) - 1\right\}T(s) + \left\{g'(s) + f(s)\right\}N(s) + g(s)B(s)$$

for some differentiable functions $f(s)$ and $g(s)$. Differentiating equation (24) with respect to s, we obtain

$$\begin{cases} f'(s) + g(s)k_1(s) &= 1 \\ g'(s) + f(s) &= 0 \\ g(s) &= 0. \end{cases} \tag{25}$$

There is no $f(s)$ and $g(s)$ satisfying equations (25). Thus, we get the following theorem.

Theorem 3.6 *There is no curve in A_3 whose position vector always lies in sp$\{T(s), N(s)\}$ according to Winternitz affine frame.*

§4. Position Vectors of the Curves in Affine 3-Space According to Intrinsic Equi-Affine Frame

In [2], D. Davis obtained a new affine frame by taking $T(s) := \alpha'(s)$, $N(s) := \alpha''(s)$, $B(s) := k(s)\alpha'(s) + \alpha'''(s)$ (which is called intrinsic affine binormal) and $\widetilde{\tau}_\alpha(s) := k(s) - \tau'_\alpha(s)$ (which is called intrinsic affine torsion) with the equations

$$\begin{cases} T'(s) &= N(s) \\ N'(s) &= -k(s)T(s)+B(s) \\ B'(s) &= -\widetilde{\tau}_\alpha(s)T(s). \end{cases} \tag{26}$$

We called $\{T(s),N(s),B(s)\}$ is intrinsic affine frame. Here, $k(s)$ and $\tau_\alpha(s)$ are called equi-affine curvature and equi-affine torsion given in (4) and (5).

Let $f(s)$, $g(s)$ and $h(s)$ be differentiable functions then, we can write

$$\alpha(s) = f(s)\,T(s) + g(s)\,N(s) + h(s)\,B(s). \tag{27}$$

Differentiating equation (27) with respect to s and by using equations (26), we obtain

$$0 = \left\{f'(s) - h(s)\,\widetilde{\tau}_\alpha(s) - 1\right\}T(s) + \left\{ f(s) + g'(s)\right\}N(s) + \left\{ \begin{array}{c} h'(s) + g(s) \\ -h(s)\,k(s) \end{array} \right\} B(s).$$

For $\alpha''(s) = N(s)$ and $B(s) = k(s)\alpha'(s) + \alpha'''(s)$, we can give the following theorem.

Theorem 4.1 *Let $\alpha(s)$ be a unit speed curve in A_3 with equi-affine curvature $k(s)$ and with intrinsic torsion $\widetilde{\tau}_\alpha(s)$, then, $\alpha(s)$ has the position vector in* (27) *according to intrinsic equi-affine frame for some differentiable functions $f(s)$, $g(s)$ and $h(s)$ satisfy the equations*

$$\begin{cases} f'(s) - h(s)\,\widetilde{\tau}_\alpha(s) &= 1, \\ f(s) + g'(s) &= 0, \\ h'(s) + g(s) - h(s)\,k(s) &= 0. \end{cases}$$

Assume that the position vector of $\alpha(s)$ always lies in the plane $\mathrm{sp}\{N(s),B(s)\}$ then, position vector of the curve $\alpha(s)$ satisfies the equation

$$\alpha(s) = g(s)\,N(s) + h(s)\,B(s) \tag{28}$$

for some differentiable functions $g(s)$ and $h(s)$. Differentiating equation (28) with respect to s, we obtain

$$0 = \{-h(s)\,\widetilde{\tau}_\alpha(s) - g(s)\,k(s) - 1\}\,T(s) + g'(s)\,N(s) + \left\{h'(s) + g(s)\right\}B(s).$$

It follows that

$$\begin{cases} h(s)\,\widetilde{\tau}_\alpha(s) + g(s)\,k(s) &= -1, \\ g'(s) &= 0, \\ h'(s) + g(s) &= 0. \end{cases} \tag{29}$$

and we get $g(s) = c_0$ and $h(s) = -c_0 s + c_1$. From the second, the third and the first equation of (29), we obtain

$$(-c_0 s + c_1)\,\widetilde{\tau}_\alpha(s) + c_0 k(s) = -1.$$

In this case, we can write

$$\alpha(s) = c_0 N(s) + (-c_0 s + c_1)\,B(s).$$

By considering $\alpha''(s) = N(s)$ and $B(s) = k(s)\alpha'(s) + \alpha'''(s)$ we can give the following theorem.

Theorem 4.2 *Let $\alpha(s)$ be a unit speed curve in A_3 with nonzero equi-affine curvature $k(s)$ and with intrinsic torsion $\widetilde{\tau}_\alpha(s)$ satisfying*

$$(c_0 s - c_1)\,\widetilde{\tau}_\alpha(s) - c_0 k(s) = 1,$$

then, $\alpha(s)$ is a curve whose position vector according to intrinsic equi-affine frame always lies in $sp\{N(s), B(s)\}$ if and only if $\alpha(s)$ is the solution of the equation

$$(-c_0 s + c_1)\,\alpha'''(s) + c_0\alpha''(s) + (-c_0 s + c_1)\,k(s)\alpha'(s) - \alpha(s) = 0.$$

In the case of $k(s) = 0$, from the first and the second equation of (29), we obtained $g(s) = c_0$, $h(s) = \frac{-1}{\widetilde{\tau}_\alpha(s)}$ and $\widetilde{\tau}_\alpha(s) \neq 0$. From the third equation of (29), we get $\widetilde{\tau}_\alpha(s) = \frac{1}{c_0 s - c_1}$. Thus, from (28), the position vector of $\alpha(s)$ satisfies the following differential equation

$$\alpha'''(s) - c_0\widetilde{\tau}_\alpha(s)\alpha''(s) + \widetilde{\tau}_\alpha(s)\alpha(s) = 0.$$

In the case of $k(s)$ nonzero constant, from the second and the third equation of (29), we obtained $g(s) = c_0$, $h(s) = -c_0 s + c_1$. From the first equation of (29), we get $\widetilde{\tau}_\alpha(s) = \frac{1+c_0 k}{c_0 s - c_1}$. From (28), the position vector of $\alpha(s)$ satisfies the following differential equation

$$(-c_0 s + c_1)\,\alpha'''(s) + c_0\alpha''(s) + (-c_0 s + c_1)\,k\alpha'(s) - \alpha(s) = 0.$$

In the case of $\widetilde{\tau}_\alpha(s) = 0$, from the second and the third equation of (29), we obtained $g(s) = c_0$, $h(s) = -c_0 s + c_1$. From the first equation of (29), we obtained $k(s) = \frac{-1}{c_0}$. From (28), the position vector of $\alpha(s)$ satisfies the following differential equation

$$(-c_0 s + c_1)\,\alpha'''(s) + c_0\alpha''(s) + (-c_0 s + c_1)\,k(s)\alpha'(s) - \alpha(s) = 0.$$

In the case of $\widetilde{\tau}_\alpha(s)$ nonzero constant, from the second and the third equation of (29) $g(s) = c_0$, $h(s) = -c_0 s + c_1$ and from the first equation of (29), we obtain $k(s) = \frac{(c_0 s - c_1)\widetilde{\tau}_\alpha - 1}{c_0}$. From (28), the position vector of $\alpha(s)$ satisfies the following differential equation

$$(-c_0 s + c_1)\,\alpha'''(s) + c_0\alpha''(s) + (-c_0 s + c_1)\,k(s)\alpha'(s) - \alpha(s) = 0.$$

In the case of $\widetilde{\tau}_\alpha(s)$ and $k(s)$ nonzero constants, from the second and the third equation of (29), we obtained $g(s) = c_0$, $h(s) = -c_0 s + c_1$ and from the first equation of (29), we obtained $\widetilde{\tau}_\alpha = \frac{1+c_0 k}{c_0 s - c_1}$. By using (28), the position vector of $\alpha(s)$ satisfies the following differential equation,

$$(-c_0 s + c_1)\,\alpha'''(s) + c_0\alpha''(s) + (-c_0 s + c_1)\,k\alpha'(s) - \alpha(s) = 0.$$

Hence, we get the following theorem.

Theorem 4.3 *Let $\alpha(s)$ be a unit speed curve in A_3 with the equi-affine curvature $k(s)$ and with the intrinsic affine torsion $\widetilde{\tau}_\alpha(s)$ whose position vector lies in $sp\{N(s), B(s)\}$ then, the followings are true.*

(1) *If $k(s) = 0$ then position vector of $\alpha(s)$ satisfies the equation*

$$\alpha'''(s) - c_0\widetilde{\tau}_\alpha(s)\alpha''(s) + \widetilde{\tau}_\alpha(s)\alpha(s) = 0$$

for some $\widetilde{\tau}_\alpha(s)$*;*

(2) *If* $\widetilde{\tau}_\alpha(s) = 0$ *and* $k(s) = \frac{-1}{c_0}$ *or* $\widetilde{\tau}_\alpha(s)$ *is nonzero constant and*

$$k(s) = \frac{(c_0 s - c_1)\widetilde{\tau}_\alpha - 1}{c_0}$$

then, position vector of $\alpha(s)$ *satisfies the equation*

$$(-c_0 s + c_1)\,\alpha'''(s) + c_0\alpha''(s) + (-c_0 s + c_1)\,k(s)\alpha'(s) - \alpha(s) = 0;$$

(3) *If* $k(s)$ *is nonzero constant and* $\widetilde{\tau}_\alpha(s) = \frac{1+c_0 k}{c_0 s - c_1}$ *or* $\widetilde{\tau}_\alpha(s)$ *and* $k(s)$ *are nonzero constants then, position vector of* $\alpha(s)$ *satisfies the equation*

$$(-c_0 s + c_1)\,\alpha'''(s) + c_0\alpha''(s) + (-c_0 s + c_1)\,k\alpha'(s) - \alpha(s) = 0.$$

We assume that the position vector of α always lies in the plane $\mathrm{sp}\{T(s), B(s)\}$. Position vector of the curve α satisfies equation

$$\alpha(s) = f(s)\,T(s) + h(s)\,B(s) \tag{30}$$

for some differentiable functions $f(s)$ and $h(s)$. Differentiating equation (30), with respect to s, we obtain

$$0 = \left\{f'(s) - h(s)\,\tau_\alpha(s) - 1\right\}T(s) + f(s)\,N(s) + h'(s)\,B(s).$$

It follows that

$$\begin{cases} f'(s) - h(s)\,\tau_\alpha(s) &= 1, \\ f(s) &= 0, \\ h'(s) &= 0 \end{cases} \tag{31}$$

and we get $h(s) = c_0$ and $\tau_\alpha(s) = \frac{-1}{c_0}$. Thus, we can write

$$\alpha(s) = c_0 B(s).$$

If $\tau_\alpha(s) = 0$ then, there is no function $f(s)$ that satisfies the first and the second equation of (31). By considering $\alpha'(s) = T(s)$ and $B(s) = k(s)\alpha' + \alpha'''$, we can give the following theorem.

Theorem 4.4 *Let* $\alpha(s)$ *be a unit speed curve in* A_3 *with nonzero intrinsic affine torsion, then,* $\alpha(s)$ *is a curve whose position vector according to intrinsic equi-affine frame always lies in* $sp\{T(s), B(s)\}$ *if and only if* $\alpha(s)$ *is the solution of the equation*

$$c_0\alpha'''(s) + c_0 k(s)\alpha'(s) - \alpha(s) = 0$$

for some $k(s)$.

Now, assume that the position vector of $\alpha(s)$ always lies in the plane $\mathrm{sp}\{T(s), N(s)\}$. The position vector of the curve $\alpha(s)$ satisfies equation

$$\alpha(s) = f(s)\,T(s) + g(s)\,N(s) \tag{32}$$

for some differentiable functions $f(s)$ and $g(s)$. Differentiating equation (32) with respect to s, we obtain

$$0 = \left\{ f'(s)\, T(s) - g(s)\, \kappa_\alpha(s) - 1 \right\} T(s) + \left\{ g'(s) + f(s) \right\} N(s) + g(s)\, B(s).$$

It follows that

$$\left\{ \begin{array}{lcl} f'(s)\, T(s) - g(s)\, \kappa_\alpha(s) & = & 1, \\ g'(s) + f(s) & = & 0, \\ g(s) & = & 0. \end{array} \right. \tag{33}$$

There is no $f(s)$ and $g(s)$ satisfying equations (33). Thus, we get the following theorem.

Theorem 4.5 *There is no curve in A_3 whose position vector always lies in the sp$\{T(s), N(s)\}$ according to intrinsic equi-affine frame.*

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