International J.Math. Combin. Vol.2(2019), 23-33

On M-Projective Curvature Tensor of a $(LCS)_n$-Manifold

Divyashree G. and Venkatesha

(Department of Mathematics, Kuvempu University, Shankaraghatta-577451, Karnataka, India)

E-mail: gdivyashree9@gmail.com, vensmath@gmail.com

Abstract: In the present paper, we characterize the M-projective curvature tensor in a $(LCS)_n$-manifold. Geometric properties on the curvature tensor such as ϕ-M-projective flat, M-projective pseudosymmetric, ϕ-M-projective semisymmetric and generalized M-projective ϕ-recurrent are studied on $(LCS)_n$-manifold.

Key Words: $(LCS)_n$-Manifold, M-projective curvature tensor, ϕ-M-projective semisymmetric, ϕ-M-projective flat, generalized M-projective ϕ-recurrent, M-projective pseudosymmetric.

AMS(2010): 53C25, 53C50, 53D10.

§1. Introduction

In 2003, Shaikh [18] introduced and studied Lorentzian concircular structure manifolds (briefly $(LCS)_n$-manifolds) with an example, which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto [11]. Also Shaikh et al. ([19,20,21,22]), Prakasha [16], Yadav [29] studied various types of $(LCS)_n$-manifolds by imposing curvature restrictions.

In 1926, the concept of local symmetry of a Riemannian manifold was started by Cartan [3]. This notion has been used in several directions by many authors such as recurrent manifolds by Walker [28], semi-symmetric manifold by Szabo [24], pseudosymmetric manifold by Chaki [4], pseudosymmetric spaces by Deszcz [10], weakly symmetric manifold by Tamassy and Binh [26], weakly symmetric Riemannian spaces by Selberg [17]. The notions of pseudo-symmetric and weak symmetry by Chaki and Deszcz and Selberge and Tamassy and Binh respectively are different. As a mild version of local symmetry, Takahashi [25] introduced the notion of ϕ-symmetry on a Sasakian manifold. In 2003, De et al. [7] introduced the concept of ϕ-recurrent Sasakian manifold, which generalizes the notion of ϕ-symmetry.

In 1971, Pokhariyal and Mishra [15] defined a tensor field W^* on a Riemannian manifold given by

$$W^*(X,Y)Z = R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]. \tag{1.1}$$

[1]Supported by Rajiv Gandhi National Fellowship F1-17.1/2015-16/RGNF-2015-17-SC-KAR-26367.

[2]Received December 09, 2018, Accepted May 21, 2019.

Such a tensor field W^* is known as M-projective curvature tensor. Ojha [13,14] studied M-projective curvature tensor on Sasakian and Kaehler manifold. The properties of M-projective curvature tensor were also studied on different manifolds by Chaubey [5,6], Venkatesha [27] and others.

Motivated by the above studies, we made an attempt to study M-projective curvature tensor on $(LCS)_n$-manifold.

The present paper is organized as follows: Section 2 is equipped with some preliminaries of $(LCS)_n$ manifold. In section 3, we proved that if (M^n, g) is an n-dimensional ϕ-M-projective flat $(LCS)_n$-manifold, then the manifold M^n is η-Einstein manifold. We have shown that if an n-dimensional $(LCS)_n$-manifold M^n is M-projective pseudosymmetric then either $L_{W^*} = (\alpha^2 - \rho)$ or the manifold is Einstein manifold, provided $(\alpha^2 - \rho) \neq 0$, in section 4. Section 5 deals with the study of ϕ-M-projective semisymmetric $(LCS)_n$-manifold and proved that the manifold is generalized η-Einstein manifold, provided $(\alpha^2 - \rho) \neq 0$. In the last section, we have studied generalized M-projective ϕ-recurrent $(LCS)_n$-manifold and gave the relations between the associated 1-forms A and B.

§2. **Preliminaries**

An n-dimensional Lorentzian manifold M^n is a smooth connected para-compact Hausdorff manifold with a Lorentzian metric g of type $(0,2)$ such that for each point $p \in M$, the tensor $g_p : T_p(M^n) \times T_p(M^n) \to R$ is a non-degenerate inner product of signature $(-,+,+,\cdots,+)$, where $T_p(M^n)$ denotes the tangent space of M^n at p and R is the real number space [18,12].

In a Lorentzian manifold (M^n, g), a vector field P defined by

$$g(X, P) = A(X),$$

for any vector field $X \in \chi(M^n)$, ($\chi(M^n)$ being the Lie algebra of vector fields on M^n) is said to be a concircular vector field [23] if

$$(\nabla_X A)(Y) = \alpha[g(X, Y) + \omega(X)A(Y)],$$

where α is a non-zero scalar function, A is a 1-form and ω is a closed 1-form.

Let M^n be a Lorentzian manifold admitting a unit time like concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

$$g(\xi, \xi) = -1. \tag{2.1}$$

Since ξ is a unit concircular vector field, there exists a non-zero 1-form η such that for

$$g(X, \xi) = \eta(X), \tag{2.2}$$

the equation of the following form holds

$$(\nabla_X\eta)(Y) = \alpha[g(X,Y) + \eta(X)\eta(Y)], \quad (\alpha \neq 0) \tag{2.3}$$

for all vector fields X and Y. Here ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non zero scalar function satisfying

$$(\nabla_X\alpha) = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{2.4}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

$$\phi X = \frac{1}{\alpha}\nabla_X\xi, \tag{2.5}$$

then from (2.3) and (2.5) we have

$$\phi^2 X = X + \eta(X)\xi, \tag{2.6}$$

$$\eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.7}$$

from which it follows that ϕ is a symmetric $(1,1)$ tensor, called the structure tensor of the manifold. Thus the Lorentzian manifold M together with the unit timelike concircular vector field ξ, its associated 1-form η and $(1,1)$ tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly $(LCS)_n$-manifold) [18]. Especially, if we take $\alpha = 1$, then we obtain the LP-Sasakian structure of Matsumoto [11].

In a $(LCS)_n$-manifold, the following relations hold [18]:

$$\eta(R(X,Y)Z) = (\alpha^2 - \rho)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.8}$$

$$R(X,Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{2.9}$$

$$R(X,\xi)Z = (\alpha^2 - \rho)[\eta(Z)X - g(X,Z)\xi], \tag{2.10}$$

$$R(\xi,X)Y = (\alpha^2 - \rho)[g(X,Y)\xi - \eta(Y)X], \tag{2.11}$$

$$R(\xi,X)\xi = (\alpha^2 - \rho)[X + \eta(X)\xi], \tag{2.12}$$

$$S(X,\xi) = (n-1)(\alpha^2 - \rho)\eta(X), \quad Q\xi = (n-1)(\alpha^2 - \rho)\xi, \tag{2.13}$$

$$(\nabla_X\phi)(Y) = \alpha[g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X], \tag{2.14}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)(\alpha^2 - \rho)\eta(X)\eta(Y) \tag{2.15}$$

for all vector fields X, Y, Z and R, S respectively denotes the curvature tensor and the Ricci tensor of the manifold.

A $(LCS)_n$ manifold M^n is said to be a generalized η-Einstein manifold [30] if the following condition

$$S(X,Y) = \lambda g(X,Y) + \mu\eta(X)\eta(Y) + \nu\Omega(X,Y) \tag{2.16}$$

holds on M^n. Here λ, μ and ν are smooth functions and $\Omega(X,Y) = g(\phi X, Y)$. If $\nu = 0$ then the manifold reduces to an η-Einstein manifold.

From (1.1), we have

$$\eta(W^*(\xi, Y)Z) = \frac{1}{2(n-1)}S(Y,Z) - \frac{1}{2}g(Y,Z), \tag{2.17}$$

$$\eta(W^*(X,\xi)Z) = -\frac{1}{2(n-1)}S(X,Z) - \frac{3}{2}(\alpha^2 - \rho)g(X,Z), \tag{2.18}$$

$$\eta(W^*(X,Y)\xi) = 0, \tag{2.19}$$

$$W^*(X,Y)\xi = 0, \quad W^*(X,\xi)\xi = 0, \quad W^*(\xi,\xi)Z = 0, \tag{2.20}$$

$$\begin{aligned} W^*(X,\xi,Z,T) = {} & \frac{1}{2(n-1)}S(X,Z)\eta(T) - \frac{1}{2(n-1)}S(X,T)\eta(Z) \\ & + \frac{1}{2}(\alpha^2 - \rho)g(X,T)\eta(Z) - \frac{1}{2}(\alpha^2 - \rho)g(X,Z)\eta(T), \end{aligned} \tag{2.21}$$

$$W^*(X,\xi,Z,\xi) = -\frac{1}{2(n-1)}S(X,Z) + \frac{1}{2}(\alpha^2 - \rho)g(X,Z), \tag{2.22}$$

$$W^*(X,\xi)Z = \frac{1}{2(n-1)}S(X,Z)\xi - \frac{1}{2}(\alpha^2 - \rho)g(X,Z)\xi, \tag{2.23}$$

$$(\nabla_U S)(X,\xi) = (n-1)\alpha(\alpha^2 - \rho)[g(U,X) + \eta(U)\eta(X)] - \alpha S(X,\phi U). \tag{2.24}$$

§3. ϕ-M-projectively Flat $(LCS)_n$-Manifold

Definition 3.1 *An n-dimensional $(LCS)_n$-manifold (M^n, g), $(n > 3)$ is called ϕ-M-projective flat if it satisfies the condition*

$$\phi^2 W^*(\phi X, \phi Y)\phi Z = 0, \tag{3.1}$$

for all vector fields X, Y, Z on the manifold.

Theorem 3.1 *If (M^n, g) is an n-dimensional ϕ-M-projective flat $(LCS)_n$-manifold, then the manifold M^n is η-Einstein manifold.*

Proof Let M^n be ϕ-M-projective flat. It is easy to define that $\phi^2(W^*(\phi X, \phi Y)\phi Z) = 0$ holds if and only if

$$g(W^*(\phi X, \phi Y)\phi Z, \phi U) = 0, \tag{3.2}$$

for any vector fields $X, Y, Z, U \in TM^n$.

By virtue of (1.1) and (3.2), one can obtain

$$\begin{aligned} g(R(\phi X, \phi Y)\phi Z, \phi U) = {} & \frac{1}{2(n-1)}[S(\phi Y, \phi Z)g(\phi X, \phi U) - S(\phi X, \phi Z)g(\phi Y, \phi U) \\ & + g(\phi Y, \phi Z)S(\phi X, \phi U) - g(\phi X, \phi Z)S(\phi Y, \phi U)]. \end{aligned} \tag{3.3}$$

Let $\{e_1, e_2, \cdots, e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M^n. By using the fact that $\{\phi e_1, \phi e_2, \cdots, \phi e_{n-1}, \xi\}$ is also a local orthonormal basis, if we put $X = U = e_i$ in (3.3) and sum up with respect to i, we get

$$\begin{aligned}\sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) &= \frac{1}{2(n-1)} \sum_{i=1}^{n-1} [S(\phi Y, \phi Z) g(\phi e_i, \phi e_i) \\ &\quad - S(\phi e_i, \phi Z) g(\phi Y, \phi e_i) + g(\phi Y, \phi Z) S(\phi e_i, \phi e_i) \\ &\quad - g(\phi e_i, \phi Z) S(\phi Y, \phi e_i)]. \end{aligned} \tag{3.4}$$

It can be easily verify by a straight forward calculation that [1],

$$\sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) = S(\phi Y, \phi Z) + g(\phi Y, \phi Z), \tag{3.5}$$

$$\sum_{i=1}^{n-1} S(\phi e_i, \phi e_i) = r - (n-1)(\alpha^2 - \rho), \tag{3.6}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi Z) S(\phi Y, \phi e_i) = S(\phi Y, \phi Z), \tag{3.7}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = (n-1) \tag{3.8}$$

and

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi Z) g(\phi Y, \phi e_i) = g(\phi Y, \phi Z). \tag{3.9}$$

By virtue of (3.5) - (3.8), the equation (3.4) becomes

$$S(\phi Y, \phi Z) = \{\frac{r - (n-1)(\alpha^2 - \rho) - 2(n-1)}{n+1}\} g(\phi Y, \phi Z). \tag{3.10}$$

On substituting (2.7) and (2.15), (3.10) yields

$$S(Y, Z) = k_1 g(Y, Z) + k_2 \eta(Y)\eta(Z), \tag{3.11}$$

where $k_1 = \{\frac{r-(n-1)(\alpha^2-\rho)-2(n-1)}{n+1}\}$ and $k_2 = \{\frac{r-2(n-1)-(n-1)(n+1)(\alpha^2-\rho)}{n+1}\}$. Thus we proved the theorem. □

§4. M-Projective Pseudosymmetric $(LCS)_n$-Manifold

Definition 4.1 *An $(LCS)_n$-manifold (M^n, g) $(n > 3)$ is said to be M-projective pseudosymmetric if it satisfies*

$$(R(X, Y) \cdot W^*)(U, V)E = L_{W^*}((X \wedge Y) \cdot W^*)(U, V)E, \tag{4.1}$$

for any vector fields $X, Y, U, V, E \in TM^n$.

Theorem 4.2 *If an* n*-dimensional* $(LCS)_n$*-manifold* M^n *is* M*-projective pseudosymmetric then either* $L_{W^*} = (\alpha^2 - \rho)$ *or the manifold is Einstein manifold, provided* $(\alpha^2 - \rho) \neq 0$.

Proof Let M^n be M-projective pseudosymmetric. Putting $Y = \xi$ in (4.1), we get

$$\begin{aligned}(R(X,\xi)\cdot W^*)(U,V)E &= L_{W^*}[(X\wedge\xi)W^*(U,V)E - W^*((X\wedge\xi)U,V)E \\ &\quad -W^*(U,(X\wedge\xi)V)E - W^*(U,V)(X\wedge\xi)E]. \end{aligned} \tag{4.2}$$

Now the left hand side of (4.2) reduces to

$$R(X,\xi)\cdot W^*(U,V)E - W^*(R(X,\xi)U,V)E - W^*(U,R(X,\xi)V)E - W^*(U,V)R(X,\xi)E. \tag{4.3}$$

In view of (2.10), (4.3) becomes

$$\begin{aligned}&(\alpha^2-\rho)[W^*(U,V,E,\xi)X - W^*(U,V,E,X)\xi - \eta(U)W^*(X,V)E + g(X,U)W^*(\xi,V)E \\ &-\eta(V)W^*(U,X)E + g(X,V)W^*(U,\xi)E - \eta(T)W^*(U,V)X + g(X,E)W^*(U,V)\xi].\end{aligned} \tag{4.4}$$

Similarly, right hand side of (4.2) reduces to

$$\begin{aligned}&L_{W^*}[W^*(U,V,E,\xi)X - W^*(U,V,E,X)\xi - \eta(U)W^*(X,V)E + g(X,U)W^*(\xi,V)E \\ &-\eta(V)W^*(U,X)E + g(X,V)W^*(U,\xi)E - \eta(E)W^*(U,V)X + g(X,E)W^*(U,V)\xi].\end{aligned} \tag{4.5}$$

On replacing the expressions (4.4) and (4.5) in (4.2), we get

$$\begin{aligned}&[L_{W^*} - (\alpha^2-\rho)]\{W^*(U,V,E,\xi)X - W^*(U,V,E,X)\xi - \eta(U)W^*(X,V)E \\ &+g(X,U)W^*(\xi,V)E - \eta(V)W^*(U,X)E + g(X,V)W^*(U,\xi)E - \eta(E)W^*(U,V)X \\ &+g(X,E)W^*(U,V)\xi\} = 0.\end{aligned} \tag{4.6}$$

Taking $V = \xi$ and using (2.2) and (2.7) in the above equation, we obtain

$$\begin{aligned}&[L_{W^*} - (\alpha^2-\rho)]\{W^*(U,\xi,E,\xi)X - W^*(U,\xi,E,X)\xi - \eta(U)W^*(X,\xi)E \\ &+g(X,U)W^*(\xi,\xi)E + W^*(U,X)E + \eta(X)W^*(U,\xi)E - \eta(E)W^*(U,\xi)X \\ &+g(X,E)W^*(U,\xi)\xi\} = 0.\end{aligned} \tag{4.7}$$

On using (2.21) - (2.23), (4.7) gives either $L_{W^*} = (\alpha^2 - \rho)$ or

$$\begin{aligned}W^*(U,X)E &= \frac{1}{2(n-1)}S(U,E)X + \frac{1}{2(n-1)}S(X,E)\eta(U)\xi \\ &\quad -\frac{1}{2}(\alpha^2-\rho)g(U,E)X - \frac{1}{2}(\alpha^2-\rho)g(X,E)\eta(U)\xi.\end{aligned} \tag{4.8}$$

The above equation implies

$$\begin{aligned} W^*(U,X,E,G) &= \frac{1}{2(n-1)}S(U,E)g(X,G) + \frac{1}{2(n-1)}S(X,E)\eta(U)\eta(G) \\ &\quad -\frac{1}{2}(\alpha^2-\rho)g(U,E)g(X,G) - \frac{1}{2}(\alpha^2-\rho)g(X,E)\eta(U)\eta(G). \end{aligned} \tag{4.9}$$

Contracting (4.9) gives

$$W^*(e_i,X,E,e_i) = 0. \tag{4.10}$$

Simplifying (4.10), we finally obtain

$$S(X,E) = (n-1)(\alpha^2-\rho)g(X,E). \tag{4.11}$$

Thus the proof of the theorem is completed. □

§5. ϕ-M-Projectively Semisymmetric $(LCS)_n$-Manifold

Definition 5.1 *An n-dimensional $(n > 3)$ $(LCS)_n$-manifold is said to be ϕ-M-projective semisymmetric if it satisfies the condition*

$$W^*(X,Y)\cdot\phi = 0, \tag{5.1}$$

which turns into

$$(W^*(X,Y)\cdot\phi)Z = W^*(X,Y)\phi Z - \phi W^*(X,Y)Z = 0. \tag{5.2}$$

Before we state our theorem we need the following lemma which was proved in [19].

Lemma 5.2([19]) *If M^n is an $(LCS)_n$-manifold, then for any X,Y,Z on M^n, the following relation holds:*

$$\begin{aligned} R(X,Y)\phi Z - \phi R(X,Y)Z &= (\alpha^2-\rho)[\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi \\ &\quad +\eta(Z)\{\eta(X)Y - \eta(Y)X\}]. \end{aligned} \tag{5.3}$$

Theorem 5.3 *If an n-dimensional $(LCS)_n$-manifold is ϕ-M-projective semisymmetric then it is a generalized η-Einstein manifold, provided $(\alpha^2-\rho) \neq 0$.*

Proof By virtue of (1.1), we have

$$\begin{aligned} W^*(X,Y)\phi Z - \phi W^*(X,Y)Z &= R(X,Y)\phi Z - \phi R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,\phi Z)X \\ &\quad -S(X,\phi Z)Y + g(Y,\phi Z)QX - g(X,\phi Z)QY \\ &\quad +S(Y,Z)\phi X - S(X,Z)\phi Y + g(Y,Z)\phi QX - g(X,Z)\phi QY]. \end{aligned} \tag{5.4}$$

On using (2.13) and (5.3) in (5.4), we obtain

$$\begin{aligned}&(\alpha^2-\rho)[\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}\xi+\eta(Z)\{\eta(X)Y-\eta(Y)X\}]\\&-\frac{1}{2(n-1)}[S(Y,\phi Z)X-S(X,\phi Z)Y+(n-1)(\alpha^2-\rho)g(Y,\phi Z)X\\&-(n-1)(\alpha^2-\rho)g(X,\phi Z)Y+S(Y,Z)\phi X-S(X,Z)\phi Y\\&+(n-1)(\alpha^2-\rho)g(Y,Z)\phi X-(n-1)(\alpha^2-\rho)g(X,Z)\phi Y]=0.\end{aligned}\tag{5.5}$$

Taking inner product of (5.5) with T, we get

$$\begin{aligned}&(\alpha^2-\rho)[\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}\eta(T)+\eta(Z)\{\eta(X)g(Y,T)-\eta(Y)g(X,T)\}]\\&-\frac{1}{2(n-1)}[S(Y,\phi Z)g(X,T)-S(X,\phi Z)g(Y,T)+(n-1)(\alpha^2-\rho)g(Y,\phi Z)g(X,T)\\&-(n-1)(\alpha^2-\rho)g(X,\phi Z)g(Y,T)+S(Y,Z)g(\phi X,T)-S(X,Z)g(\phi Y,T)\\&+(n-1)(\alpha^2-\rho)g(Y,Z)g(\phi X,T)-(n-1)(\alpha^2-\rho)g(X,Z)g(\phi Y,T)]=0.\end{aligned}\tag{5.6}$$

Contracting (5.6) gives

$$\begin{aligned}S(X,\phi Z)&=\frac{(n-1)(n-2)}{(2-n)}(\alpha^2-\rho)g(X,\phi Z)+\frac{2(n-1)}{(2-n)}(\alpha^2-\rho)g(X,Z)\\&\quad+\frac{2n(n-1)}{(2-n)}(\alpha^2-\rho)\eta(X)\eta(Z).\end{aligned}\tag{5.7}$$

Replacing X by ϕX in the above equation, we finally obtain

$$S(X,Z)=\lambda g(X,Z)+\mu\eta(X)\eta(Z)+\nu g(\phi X,Z),\tag{5.8}$$

where $\lambda=-(n-1)(\alpha^2-\rho)$, $\mu=-2(n-1)(\alpha^2-\rho)$ and $\nu=-\frac{2(n-1)}{(n-2)}$.

This completes the proof. □

§6. Generalized M-Projective ϕ-Recurrent $(LCS)_n$-Manifold

In 2008, Basari and Murathan [2] introduced generalized ϕ-recurrent Kenmotsu manifold. Later, De [8] and Pal [9] studied generalized concircularly recurrent and generalized M-projectively recurrent Riemannian manifold.

Definition 6.1 *A $(LCS)_n$-manifold M^n $(n>3)$ is said to be generalized M-projective ϕ-recurrent if it satisfies*

$$\phi^2((\nabla_U W^*)(X,Y,Z))=A(U)W^*(X,Y)Z+B(U)[g(Y,Z)X-g(X,Z)Y],\tag{6.1}$$

where A and B are two 1-forms, B is non-zero and are defined by $A(U)=g(U,\rho_1)$ and $B(U)=g(U,\rho_2)$. Here ρ_1 and ρ_2 are vector fields associated to the 1-forms A and B respectively.

Theorem 6.2 *If the $(LCS)_n$-manifold M^n is generalized M-projective ϕ-recurrent, then the associated 1-forms A and B are related as follows;*

$$[\frac{n}{2}(\alpha^2-\rho)+r]A(U)+(n-1)B(U)+\frac{1}{2(n-1)}dr(U)=0. \tag{6.2}$$

Proof Suppose that M^n is generalized M-projective ϕ-recurrent $(LCS)_n$-manifold. Then, by using (2.6), (6.1) takes the form

$$(\nabla_U W^*)(X,Y)Z+\eta((\nabla_U W^*)(X,Y)Z)\xi = A(U)W^*(X,Y)Z+B(U)[g(Y,Z)X-g(X,Z)Y]. \tag{6.3}$$

From (6.3), it follows that

$$\begin{aligned}
&g((\nabla_U W^*)(X,Y,Z),T)+g((\nabla_U W^*)(X,Y,Z),\xi)g(T,\xi)\\
&-\frac{1}{2(n-1)}[(\nabla_U S)(Y,Z)g(X,T)-(\nabla_U S)(X,Z)g(Y,T)+g(Y,Z)(\nabla_U S)(X,T)\\
&-g(X,Z)(\nabla_U S)(Y,T)]-\frac{1}{2(n-1)}[(\nabla_U S)(Y,Z)\eta(X)-(\nabla_U S)(X,Z)\eta(Y)\\
&+g(Y,Z)(\nabla_U S)(X,\xi)-g(X,Z)(\nabla_U S)(Y,\xi)]\eta(T)=A(U)[g(R(X,Y)Z,T)\\
&-\frac{1}{2(n-1)}\{S(Y,Z)g(X,T)-S(X,Z)g(Y,T)+g(Y,Z)S(X,T)-g(X,Z)S(Y,T)\}]\\
&+B(U)[g(Y,Z)g(X,T)-g(X,Z)g(Y,T)].
\end{aligned} \tag{6.4}$$

On contraction, the above equation yields

$$\begin{aligned}
&(\nabla_U S)(Y,Z)+\eta((\nabla_U R)(\xi,Y,Z))-\frac{1}{2(n-1)}[(n-2)(\nabla_U S)(Y,Z)\\
&+dr(U)g(Y,Z)]-\frac{1}{2(n-1)}[-(\nabla_U S)(Y,Z)-(\nabla_U S)(Z,\xi)\eta(Y)+g(Y,Z)(\nabla_U S)(\xi,\xi)\\
&-(\nabla_U S)(Y,\xi)\eta(Z)]=A(U)[S(Y,Z)-\frac{1}{2(n-1)}\{(n-2)S(Y,Z)+rg(Y,Z)\}]\\
&+(n-1)B(U)g(Y,Z).
\end{aligned} \tag{6.5}$$

In (6.5), setting $Z=\xi$ and then using (2.2), (2.3), (2.7), (2.12) and (2.13) one can get

$$\begin{aligned}
&(\nabla_U S)(Y,\xi)[1-\frac{n-2}{2(n-1)}]-\frac{dr(U)}{2(n-1)}\eta(U)=A(U)[S(Y,\xi)\\
&-\frac{1}{2(n-1)}\{(n-2)S(Y,\xi)+r\eta(Y)\}]+(n-1)B(U)\eta(Y).
\end{aligned} \tag{6.6}$$

Now, taking $Y=\xi$ in (6.6) and using (2.2), (2.7) and (2.24), we finally obtain (6.2). □

References

[1] Amit Prakash, ϕ-Conformally flat $(LCS)_n$-manifolds, *Global Journal of Advanced Research*, 4(2), 136-143, 2015.

[2] A. Basari and C. Murathan, On generalized ϕ-recurrent Kenmotsu manifolds, *Sdu Fen Edebiyat Fakultesi Fen Dergisi (E-DERGI)*, 3(10), 91-97, 2008.

[3] E. Cartan, Sur une classes remarquable despaces de Riemann, *Bull. Soc. Math. France*, 54, 214-264, 1926.

[4] M. C. Chaki, On weakly symmetric manifolds, *Analel stiintificae Ale Univeritatii Alenandru Ioan CuzzaIasi Romania*, 33, 53-58, 1987.

[5] Chaubey S. K. and Ojha R. H, On the M-projective curvature tensor of a Kenmotsu manifold, *Differential Geomerty-Dynamical Systems*, 12, 52-60, 2010.

[6] Chaubey S. K, Some properties of LP-Sasakian manifolds equipped with M-Projective curvature tensor, *Bulletin of Mathematical Analysis and Applications*, 3(4), 50-58, 2011.

[7] U. C. De, A. A. Shaikh and S. Biswas, On ϕ-recurrent Sasakian manifolds, *Novi Sad J. Math.*, 33, 13-48, 2003.

[8] U. C. De and A. K. Gazi, On generalized concircularly recurrent manifolds, *Studia Sci. Math. Hungar.*, 46, 278-296, 2009.

[9] U. C. De and P. Pal, On generalized M-projectively recurrent manifolds, *Ann. Univ. Paedagog. Crac. Stud. Math.* , 13, 77-101, 2014.

[10] R. Deszcz, On pseudosymmetric spaces, *Acta Math. Hungar.* , 53, 481- 482, 1992.

[11] K. Matsumoto, On Lorentzian almost paracontact manifolds, *Bull. Yamagata Univ. Nat. Sci.*, 12, 151-156, 1989.

[12] B. O. Neill, *Semi-Riemannian Geometry,* Academic Press, New York, 1983.

[13] Ojha R. H, A note on the m-projective curvature tensor, *Indian J. Pure Applied Math.*, 8(12), 1531-1534, 1975.

[14] Ojha R. H, On Sasakian manifold, *Kyungpook Math. J.*, 13, 211-215, 1973.

[15] Pokhariyal G. P. and Mishra R. S, Curvature tensor and their relativistic significance II, *Yokohama Mathematical Journal*, 19, 97-103, 1971.

[16] D. G. Prakasha, On Ricci η-recurrent $(LCS)_n$-manifolds, *Acta Universitatis Apulensis*, 24, 109-118, 2010.

[17] A. Selberg, Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, *Indian Math. Soc.*, 20, 47-87, 1956.

[18] A. A. Shaikh, On Lorentzian almost paracontact manifolds with a structure of the concircular type, *Kyungpook Math.J.*, 43(2), 305-314, 2003.

[19] A. A. Shaikh, Some results on $(LCS)_n$-manifolds, *Journal of the Korean Mathematical Society*, vol. 46(3), 449-461, 2009.

[20] A. A. Shaikh and S. K. Hui, On generalized ϕ-recurrent $(LCS)_n$-manifolds, *AIP Conference Proceedings*, 1309, 419-429, 2010.

[21] A. A. Shaikh, T. Basu and S. Eyasmin, On the existence of ϕ-recurrent $(LCS)_n$-manifolds, *Extracta Mathematicae*, 23(1), 71-83, 2008.

[22] A. A. Shaikh and T. Q. Binh, On weakly symmetric $(LCS)_n$-manifolds, *Journal of Advanced Mathematical Studies*, 2(2), 103-118, 2009.

[23] Schouten J.A, *Ricci Calculas* (Second Edition), Springer-Verlag., 322, 1954.

[24] Z. I. Szabo, Structure theorems on Riemannian spaces satisfying $R(X, Y) \cdot R = 0$. I. The local version, *J. Differential Geom.*, 17(4), 531-582, 1982.

[25] T. Takahashi, Sasakian ϕ-symmetric space, *Tohoku Math. J.*, 29, 91-113, 1977.

[26] L. Tamassy and T. Q. Binh, On weakly symmetric and weakly projective sym- metric Riemannian manifold, *Colloquia Math. Soc.*, 50, 663-667, 1989.

[27] Venkatesha and B. Sumangala, On m-projective curvature tensor of a generalized Sasakian space form, *Acta Math. Univ. Comenianae*, Vol. LXXXII, 2, 209-217, 2013.

[28] A. A. Walker, On Ruses spaces of recurrent curvature, *Proc. London. Math.Soc.*, 52, 36-64, 1950.

[29] S. K. Yadav, P. K. Dwivedi and D. Suthar, On $(LCS)_{2n+1}$-manifolds satisfying certain conditions on the concircular curvature tensor, *Thai Jounal of Mathematics*, 9(3), 597-603, 2011.

[30] A. Yildiz, U. C. De and Erhan Ata, On a type of Lorentzian para-Sasakian manifolds, *Math. Reports*, 16(66), 61-67, 1 (2014).