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On (j, m) Symmetric Convex Harmonic Functions

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Abstract: In the present paper we define and investigate a new class of sense preserving harmonic univalent functions $HCV^{j,m}(k, \alpha)$ related to uniformly convex analytic functions. We obtain co-efficient bounds, distortion theorem and extreme points.

Key Words: (j, m) Symmetric functions, harmonic Functions, uniformly convex analytic functions.

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§1. Introduction

Let $\mathcal{U} = \{z : |z| < 1\}$ denote an open unit disc and let H denote the class of all complex valued, harmonic and sense preserving univalent functions f in $\mathcal{U}$ normalized by $f(0) = f_z(0) - 1 = 0$. Each $f \in H$ can be expressed by $f = h + \bar{g}$ where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad , \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \qquad |b_1| < 1, \tag{1.1}$$

are analytic in $\mathcal{U}$. A necessary and sufficient condition for f to be locally univalent and sense - preserving in $\mathcal{U}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{U}$. Clunie and Sheil-Small [3] studied H together with some geometric sub-classes of H. We note that the family H of orientation preserving, normalized harmonic univalent functions reduces to the well known class S of normalized univalent functions in $\mathcal{U}$, if the co-analytic part of f is identically zero, that is $g \equiv 0$. Harmonic functions are famous for their use in the study of minimal surfaces and also play important roles in a variety of problems in applied mathematics. We can find more details in [1, 2, 4, 5]. Also let $\overline{H}$ denote the subclass of H consisting of functions $f = h + \bar{g}$ so that the functions h and g take the form

$$h(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad , \qquad g(z) = -\sum_{n=1}^{\infty} b_n z^n, \qquad |b_1| < 1. \tag{1.2}$$

Definition 1.1 *Let m be any positive integer. A domain $\mathcal{D}$ is said to be m-fold symmetric if a rotation of $\mathcal{D}$ about the origin through an angle $\frac{2\pi}{m}$ carries $\mathcal{D}$ onto itself. A function f is said*

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to be m-fold symmetric in $\mathcal{D}$ if for every z in $\mathcal{D}$ we have

$$f\left(e^{\frac{2\pi i}{m}} z\right) = e^{\frac{2\pi i}{m}} f(z),\ z \in \mathcal{D}.$$

The family of all k-fold symmetric functions is denoted by $\mathcal{S}^k$, and for $k = 2$ we get class of odd univalent functions. The notion of (j, m)-symmetrical functions ($m = 2, 3, \cdots$, and $j = 0, 1, 2, \cdots, m-1$) is a generalization of the notion of even, odd, k-symmetrical functions and also generalizes the well-known result that each function defined on a symmetrical subset can be uniquely expressed as the sum of an even function and an odd function. The theory of (j, m)-symmetrical functions has many interesting applications; for instance, in the investigation of the set of fixed points of mappings, for the estimation of the absolute value of some integrals, and for obtaining some results of the type of Cartan's uniqueness theorem for holomorphic mappings, see [8]. Denote the family of all (j, m)-symmetrical functions by $\mathcal{S}^{(j,m)}$. We observe that, $\mathcal{S}^{(0,2)}$, $\mathcal{S}^{(1,2)}$ and $\mathcal{S}^{(1,m)}$ are the classes of even, odd and m-symmetric functions respectively. We have the following decomposition theorem.

Theorem 1.2([8]) *For every mapping $f : \mathcal{U} \mapsto \mathbb{C}$, and a m-fold symmetric set, there exists exactly one sequence of (j, m)-symmetrical functions $f_{j,m}$ such that*

$$f(z) = \sum_{j=0}^{m-1} f_{j,m}(z),$$

where

$$f_{j,m}(z) = \frac{1}{m} \sum_{v=0}^{m-1} \varepsilon^{-vj} f\left(\varepsilon^v z\right),\ z \in \mathcal{U}. \tag{1.3}$$

Remark 1.3 Equivalently, (1.3) may be written as

$$f_{j,m}(z) = \sum_{n=1}^{\infty} \delta_{n,j} a_n z^n, \quad a_1 = 1, \tag{1.4}$$

where

$$\delta_{n,j} = \frac{1}{m} \sum_{v=0}^{m-1} \varepsilon^{(n-j)v} = \begin{cases} 1, & n = lm + j; \\ 0, & n \neq lm + j; \end{cases}, \tag{1.5}$$

$$(l \in \mathbb{N},\ m = 1, 2, \cdots,\ j = 0, 1, 2, \cdots, m-1).$$

Yong Chan Kim et al [7] discussed the class $HCV(k, \alpha)$ of complex valued, sense preserving harmonic univalent functions. f of the form (1.1) and satisfying

$$R\left\{1 + (1 + ke^{i\phi}) \frac{z^2 h''(z) + \overline{2zg'(z) + z^2 g''(z)}}{zh'(z) - zg'(z)}\right\} \geq \alpha, 0 \leq \alpha < 1. \tag{1.6}$$

Now, using the concept of (j, m) symmetric points we define the following.

Definition 1.4 *For $0 \leq \alpha < 1$ and $m = 1,2,3,\cdots, j = 0,1,2,\cdots, m-1$. Let $HCV^{j,m}(k,\alpha)$ which denote the class of sense-preserving, harmonic univalent functions f of the form* (1.1) *which satisfy the condition*

$$\begin{aligned}\frac{\partial}{\partial\theta}\left(argf(re^{i\theta})\right) &= Im\left(\frac{\frac{\partial}{\partial\theta}f'(re^{i\theta})}{f'_{j,m}(re^{i\theta})}\right) \\ &= Re\left\{1+(1+ke^{i\phi})\frac{z^2h''(z)+\overline{2zg'(z)+z^2g''(z)}}{zh'_{j,m}(z)-\overline{zg'_{j,m}(z)}}\right\} \geq \alpha. \qquad (1.7)\end{aligned}$$

where $z = re^{i\theta}, 0 \leq r < 1, 0 \leq \theta < 2\pi$, $0 \leq k < \infty$ and $f_{j,m} = h_{j,m} + \overline{g_{j,m}}$ where $h_{j,m}, g_{j,m}$ given by

$$h_{j,m}(z) = \frac{1}{m}\sum_{v=0}^{m-1}\varepsilon^{-vj}h(\varepsilon^v z), g_{j,m}(z) = \frac{1}{m}\sum_{v=0}^{m-1}\varepsilon^{-vj}g(\varepsilon^v z). \qquad (1.8)$$

We need the following result due to Jahangiri [6] to prove our main results.

Theorem 1.5 *Let $f = h + \bar{g}$ with h and g of the form* (1.1)*. If*

$$\sum_{n=1}^{\infty}\frac{n(n-\alpha)}{1-\alpha}|a_n| + \sum_{n=1}^{\infty}\frac{n(n+\alpha)}{1-\alpha}|b_n| \leq 2, \quad a_1 = 1,\ 0 \leq \alpha < 1, \qquad (1.9)$$

then f is harmonic, sense-preserving, univalent in $\mathcal{U}$, and f is convex harmonic of order α denoted by $HK(\alpha)$. Notice that the condition (1.9) *is also necessary if $f \in \overline{H}K(\alpha) \equiv HK(\alpha) \cap \overline{H}$.*

§2. Main Results

Theorem 2.1 *Let $f = h + \overline{g}$ of the form* (**??**) *and $f_{j,m} = h_{j,m} + \overline{g}_{j,m}$ with $h_{j,m}$ and $\overline{g}_{j,m}$ given by* (1.8)*. If $0 \leq k < \infty, 0 \leq \alpha < 1, m = 1,2,3,\cdots, j = 0,1,2,\cdots, m-1$ and*

$$\sum_{n=1}^{\infty}\frac{n[n(k+1)-k-\alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})} \mid a_n \mid + \sum_{n=1}^{\infty}\frac{n[n(k+1)+k+\alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})} \mid b_n \mid \leq 2, \qquad (2.1)$$

then f is harmonic, sense- preserving, univalent in $\mathcal{U}$, and $f \in HCV^{j,m}(k,\alpha)$, where $\delta_{n,j}$ given by (1.5).

Proof Since $n - \alpha \leq n + nk - k - \alpha\delta_{n,j}$ and $n + \alpha \leq n + nk + k + \alpha\delta_{n,j}$ for $0 \leq k < \infty$, it follows from Theorem 1.5 that $f \in HK(\alpha)$ and hence f is sense- preserving and convex univalent in $\mathcal{U}$. Now we need to show that if (2.1) holds then

$$\begin{aligned}&Re\left\{\frac{zh'(z)+(1+ke^{i\phi})z^2h''(z)+(1+2ke^{i\phi})\overline{zg'(z)}+(1+ke^{i\phi})\overline{z^2g''(z)}}{zh'_{j,m}(z)-\overline{zg'_{j,m}(z)}}\right\} \\ &= Re\left(\frac{A(z)}{B(z)}\right) \geq \alpha. \qquad (2.2)\end{aligned}$$

Using the fact that $Re(w) \geq \alpha$ if and only if $\mid 1-\alpha+w \mid \geq \mid 1+\alpha-w \mid$ it suffices to show that

$$\mid A(z)+(1-\alpha)B(z) \mid - \mid A(z)-(1+\alpha)B(z) \mid \quad \geq 0, \tag{2.3}$$

where $A(z) = zh'(z)+(1+ke^{i\phi})z^2h''(z)+(1+2ke^{i\phi})\overline{zg'(z)}+(1+ke^{i\phi})\overline{z^2g''(z)}$ and $B(z) = zh'_{j,m}(z) - \overline{zg'_{j,m}(z)}$. substituting for A(z) and B(z) in (2.3), we obtain

$$\begin{aligned}
&|A(z)+(1-\alpha)B(z)| - |A(z)-(1+\alpha)B(z)| \\
&= \Big| zh'(z)+(1+ke^{i\phi})z^2h''(z)+(1+2ke^{i\phi})\overline{zg'(z)}+(1+ke^{i\phi})\overline{z^2g''(z)} \\
&\qquad\qquad + (1-\alpha)[zh'_{j,m}(z)-\overline{zg'_{j,m}(z)}] \Big| \\
&\quad - \Big| zh'(z)+(1+ke^{i\phi})z^2h''(z)+(1+2ke^{i\phi})\overline{zg'(z)}+(1+ke^{i\phi})\overline{z^2g''(z)} \\
&\qquad\qquad -(1+\alpha)[zh'_{j,m}(z)-\overline{zg'_{j,m}(z)}] \Big| \\
&= \Big| [1+(1-\alpha)\delta_{1,j}]z + \sum_{n=2}^{\infty} n[n+(n-1)ke^{i\phi}+(1-\alpha)\delta_{n,j}]a_nz^n \\
&\qquad\qquad + \sum_{n=1}^{\infty} n[n+k(n+1)ke^{i\phi}-(1-\alpha)\delta_{n,j}]\ \overline{b_n}\overline{z^n} \Big| \\
&\quad - \Big| [1-(1+\alpha)\delta_{1,j}]z + \sum_{n=2}^{\infty} nn[n+(n-1)ke^{i\phi}-(1+\alpha)\delta_{n,j}]a_nz^n \\
&\qquad\qquad - \sum_{n=1}^{\infty} n[n+k(n+1)ke^{i\phi}+(1+\alpha)\delta_{n,j}]\overline{b_n}\overline{z^n} \Big| \\
&\geq [1+(1-\alpha)\delta_{1,j}] \mid z \mid - \sum_{n=2}^{\infty} n[n(k+1)-k-(1-\alpha)\delta_{n,j}] \mid a_n \mid\mid z^n \mid \\
&\qquad\qquad - \sum_{n=1}^{\infty} n[n(k+1)+k+(1-\alpha)\delta_{n,j}] \mid b_n \mid\mid z^n \mid \\
&= (2(1-\alpha)\delta_{1,j}) \mid z \mid \left\{ 1 - \sum_{n=2}^{\infty} \frac{n[n(k+1)-k-\alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})} \mid a_n \mid\mid z \mid^{n-1} \right. \\
&\qquad\qquad \left. - \sum_{n=1}^{\infty} \frac{n[n(k+1)+k+\alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})} \mid b_n \mid\mid z \mid^{n-1} \right\} \\
&\geq (2(1-\alpha)\delta_{1,j}) \mid z \mid \left\{ 1 - \left(\sum_{n=2}^{\infty} \frac{n[n(k+1)-k-\alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})} \mid a_n \mid \right. \right. \\
&\qquad\qquad \left. \left. + \sum_{n=1}^{\infty} \frac{n[n(k+1)+k+\alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})} \mid b_n \mid \right) \right\} \geq 0
\end{aligned}$$

by (2.1). The harmonic functions

$$f(z) = z + \sum_{n=1}^{\infty} \frac{(1-\alpha\delta_{1,j})}{n[n(k+1)-k-\alpha\delta_{n,j}]}|a_n| \ + \sum_{n=1}^{\infty} \frac{(1-\alpha\delta_{1,j})}{n[n(k+1)+k+\alpha\delta_{n,j}]}|b_n| \leq 2, \tag{2.4}$$

where

$$\sum_{n=1}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=2,$$

show that the coefficient bound given in Theorem 2.1 is sharp. The functions of the form (2.4) are in $HCV^{j,m}(k,\alpha)$ because

$$\sum_{n=1}^{\infty}\frac{n[n(k+1)-k-\alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})}\mid a_n\mid+\sum_{n=1}^{\infty}\frac{n[n(k+1)+k+\alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})}\mid b_n\mid$$
$$=\sum_{n=1}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=2. \tag{2.5}$$

This completes the proof. □

If $j=m=1$ we get the following result proved by Yong Chan Kim et al in [7].

Corollary 2.2 *Let $f=h+\overline{g}$ of the form (??). If $0\le k<\infty,\quad 0\le\alpha<1$ and*

$$\sum_{n=1}^{\infty}\frac{n(n+nk-k-\alpha)}{(1-\alpha)}|a_n|+\sum_{n=1}^{\infty}\frac{n(n+nk+k+\alpha)}{(1-\alpha)}|b_n|\le 2,$$

then f is harmonic, sense- preserving, univalent in $\mathcal{U}$, and $f\in HCV(k,\alpha)$.

Now we show that the bound (2.1) is also necessary for functions in $\overline{H}CV(k,\alpha)$.

Theorem 2.3 *Let $f=h+\overline{g}$ of the form (1.2) and $f_{j,m}=h_{j,m}+\overline{g}_{j,m}$ with $h_{j,m}$ and $\overline{g}_{j,m}$ given by (1.8). Then $f\in\overline{H}CV^{j,m}(k,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty}\frac{n[n(k+1)-k-\alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})}\mid a_n\mid+\sum_{n=1}^{\infty}\frac{n[n(k+1)+k+\alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})}\mid b_n\mid\le 2 \tag{2.6}$$

where $0\le k<\infty, 0\le\alpha<1, m=1,2,3,\cdots,j=0,1,2,\cdots,m-1,$ and $\delta_{n,j}$ given by (1.5).

Proof In view of Theorem 2.3, we only need to show that $\overline{H}CV^{j,m}(k,\alpha)$ if condition (2.6) does not hold. We note that a necessary and sufficient condition for $f=h+\overline{g}$ of the form (1.1) to be satisfied. Equivalently, we must have

$$Re\left\{\frac{A(z)}{B(z)}-\alpha\right\}$$
$$=Re\left\{\frac{zh'(z)+(1+ke^{i\phi})z^2h''(z)+(1+2ke^{i\phi})(\overline{zg'(z)})+(1+ke^{i\phi})(\overline{z^2g''(z)})}{zh'_{j,k}(z)-\overline{zg'_{j,k}(z)}}-\alpha\right\}\ge 0.$$

Therefore,

$$Re\left\{\frac{(1-\delta_{1,j}\alpha)z-\sum_{n=2}^{\infty}n[n(k+1)-k-\alpha\delta_{n,j}]|a_n|z^n+\sum_{n=1}^{\infty}n[n(k+1)+k+\alpha\delta_{n,j}]\overline{|b_n|z^n}}{\delta_{1,j}z-\sum_{n=2}^{\infty}n\delta_{n,j}|a_n|z^n+\sum_{n=1}^{\infty}n\delta_{n,j}|b_n|z^n}\right\}\ge 0 \tag{2.7}$$

upon choosing the value of z on the positive real axis where $0\le z=r<1$ the above inequality reduces to

$$\frac{(1-\delta_{1,j}\alpha)-\left\{\sum_{n=2}^{\infty} n[n(k+1)-k-\alpha\delta_{n,j}]|a_n|+\sum_{n=1}^{\infty} n[n(k+1)+k+\alpha\delta_{n,j}]\overline{|b_n|}\right\} r^{n-1}}{\delta_{1,j}-\sum_{n=2}^{\infty} n\delta_{n,j}|a_n|r^{n-1}+\sum_{n=1}^{\infty} n\delta_{n,j}|b_n|r^{n-1}} \geq 0. \tag{2.8}$$

If condition (2.6) does not hold then the numerator in (2.8) is negative for r sufficiently close to 1. Thus there exists $z_0 = r_0$ in (0,1) for which the quotient (2.8) is negative. This contradicts the required condition for $f \in \overline{H}CV^{j,m}(k,\alpha)$and so proof is complete. □

§3. Extreme Points and Distortion Bounds

Theorem 3.1 *Let f be of the form of* (1.2). *Then $f \in clco\overline{H}CV(k,\alpha)$ if and only if* $f(z) = \sum_{n=1}^{\infty}(\tau_n h_n(z)+\lambda_n g_n(z))$ *where* $h_1(z)=z$, $h_n(z) = z - \frac{(1-\alpha\delta_{1,j})}{n[n(k+1)-k-\alpha\delta_{n,j}]}z^n, (n=2,3,4,\cdots,)$ *and* $g_n(z) = z - \frac{(1-\alpha\delta_{1,j})}{n[n(k+1)+k+\alpha\delta_{n,j}]}\bar{z}^n, (n=1,2,3,\cdots,), \sum_{n=1}^{\infty}(\tau_n+\lambda_n)=1, \tau_n \geq 0$ *and* $\lambda_n \geq 0$, *In particular, the extreme points of* $\overline{H}CV^{j,m}(k,\alpha)$ *are* $\{h_n\}$*and*$\{g_n\}$, *and* $\delta_{n,j}$ *given by* (1.5).

Proof For functions of f of the form $f(z) = \sum_{n=1}^{\infty}(\tau_n h_n(z)+\lambda_n g_n(z))$, we have

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty}(\tau_n+\lambda_n)z - \sum_{n=2}^{\infty}\frac{(1-\alpha\delta_{1,j})}{n[n(k+1)-k-\alpha\delta_{n,j}]}\tau_n z^n \\ &\quad -\sum_{n=1}^{\infty}\frac{(1-\alpha\delta_{1,j})}{n[n(k+1)+k+\alpha\delta_{n,j}]}\lambda_n z^{-n} = z-\sum_{n=2}^{\infty}a_n z^n - \sum_{n=1}^{\infty} b_n z^{-n}. \end{aligned} \tag{3.1}$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{n[n(k+1)-k-\alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})}|a_n|+\sum_{n=1}^{\infty}\frac{n[n(k+1)+k+\alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})}|b_n| = \sum_{n=2}^{\infty}\tau_n+\sum_{n=1}^{\infty}\lambda_n = 1-\tau_1 \leq 1$$

and so $f \in \overline{H}CV^{j,m}(k,\alpha)$.

Conversely, Suppose that $f \in \overline{H}CV^{j,m}(k,\alpha)$. We set $\tau_n = \frac{n[n(k+1)-k-\alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})}|a_n|, n = 2,3,4,\cdots,$ $\lambda_n = \frac{n[n(k+1)+k+\alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})}|b_n|, n=1,2,3,\cdots,$ and $\tau_1 = 1-\sum_{n=2}^{\infty}\tau_n - \sum_{n=1}^{\infty}\lambda_n$. Then $\sum_{n=1}^{\infty}(\tau_n+\lambda_n)=1$, $0 \leq \tau_n \leq 1$, $0 \leq \lambda_n \leq 1, (n=1,2,3,\cdots,)$ thus by simple calculations we get $f(z) = \sum_{n=1}^{\infty}(\tau_n h_n(z)+\lambda_n g_n(z))$ and the proof is complete. □

Theorem 3.2 *If* $f \in \overline{H}CV^{j,m}(k,\alpha)$ *then*

$$|f(z) \leq (1+|b_1|)r + \frac{1}{2}\left[\frac{1-\alpha\delta_{1,j}}{k+2-\alpha\delta_{2,j}} - \frac{1+2k+\alpha\delta_{1,j}}{k+2-\alpha\delta_{2,j}}|b_1|\right]r^2, \quad |z|=r<1$$

and

$$|f(z) \geq (1-|b_1|)r - \frac{1}{2}\left[\frac{1-\alpha\delta_{1,j}}{k+2-\alpha\delta_{2,j}} - \frac{1+2k+\alpha\delta_{1,j}}{k+2-\alpha\delta_{2,j}}|b_1|\right]r^2, \quad |z|=r<1$$

where $0 \leq \alpha < 1$, *and* $\delta_{n,j}$ *given by* (1.5).

Proof Calculation shows that

$$\begin{aligned}
|f(z)| &\leq (1+|b_1|)r+\sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n \leq (1+|b_1|)r+\sum_{n=2}^{\infty}(|a_n|+|b_n|)r^2 \\
&\leq (1+|b_1|)r \\
&\quad +\frac{1-\alpha\delta_{1,j}}{2[(k+2)-\alpha\delta_{2,j}]}\left\{\sum_{n=2}^{\infty}\frac{2[(k+2)-\alpha\delta_{2,j}]}{1-\alpha\delta_{1,j}}|a_n|+\sum_{n=2}^{\infty}\frac{2[(k+2)-\alpha\delta_{2,j}]}{1-\alpha\delta_{1,j}}|b_n|\right\}r^2 \\
&\leq (1+|b_1|)r+\frac{1-\alpha\delta_{1,j}}{2[(k+2)-\alpha\delta_{2,j}]}\left\{\sum_{n=2}^{\infty}\frac{n[n(k+1)-k-\alpha\delta_{n,j}]}{1-\alpha\delta_{1,j}}|a_n|\right. \\
&\qquad \left.+\sum_{n=2}^{\infty}\frac{n[n(k+1)+k+\alpha\delta_{n,j}]}{1-\alpha\delta_{1,j}}|b_n|\right\}r^2 \\
&\leq (1+|b_1|)r+\frac{1-\alpha\delta_{1,j}}{2[(k+2)-\alpha\delta_{2,j}]}\left\{1-\frac{1+2k+\alpha\delta_{1,j}}{1-\alpha\delta_{1,j}}|b_1|\right\}r^2 \\
&\leq (1+|b_1|)r+\frac{1}{2}\left[\frac{1-\alpha\delta_{1,j}}{k+2-\alpha\delta_{2,j}}-\frac{1+2k+\alpha\delta_{1,j}}{k+2-\alpha\delta_{2,j}}|b_1|\right]r^2
\end{aligned}$$

and

$$\begin{aligned}
|f(z)| &\geq (1+|b_1|)r-\sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n \geq (1+|b_1|)r-\sum_{n=2}^{\infty}(|a_n|+|b_n|)r^2 \\
&\geq (1-|b_1|)r \\
&\quad -\frac{1-\alpha\delta_{1,j}}{2[(k+2)-\alpha\delta_{2,j}]}\left\{\sum_{n=2}^{\infty}\frac{2[(k+2)-\alpha\delta_{2,j}]}{1-\alpha\delta_{1,j}}|a_n|+\sum_{n=2}^{\infty}\frac{2[(k+2)-\alpha\delta_{2,j}]}{1-\alpha\delta_{1,j}}|b_n|\right\}r^2 \\
&\geq (1-|b_1|)r-\frac{1-\alpha\delta_{1,j}}{2[(k+2)-\alpha\delta_{1,j}]}\left\{\sum_{n=2}^{\infty}\frac{n[n(k+1)-k-\alpha\delta_{n,j}]}{1-\alpha\delta_{1,j}}|a_n|\right. \\
&\qquad \left.+\sum_{n=2}^{\infty}\frac{n[n(k+1)+k+\alpha\delta_{n,j}]}{1-\alpha\delta_{1,j}}|b_n|\right\}r^2 \\
&\geq (1-|b_1|)r-\frac{1-\alpha\delta_{1,j}}{2[(k+2)-\alpha\delta_{2,j}]}\left\{1-\frac{1+2k+\alpha\delta_{1,j}}{1-\alpha\delta_{1,j}}|b_1|\right\}|b_1|r^2 \\
&\geq (1-|b_1|)r-\frac{1}{2}\left[\frac{1-\alpha\delta_{2,j}}{k+2-\alpha\delta_{2,j}}-\frac{1+2k+\alpha\delta_{1,j}}{k+2-\alpha\delta_{2,j}}|b_1|\right]r^2.
\end{aligned}$$

This completes the proof. □

If $j=m=1$ we get the following result proved by Yong Chan Kim et al. in [7]

Corollary 3.3 *If $f\in\overline{H}CV(k,\alpha)$ then*

$$|f(z)\leq(1+|b_1|)r+\frac{1}{2}\left[\frac{1-\alpha}{k+2-\alpha}-\frac{1+2k+\alpha}{k+2-\alpha}|b_1|\right]r^2,\quad |z|=r<1$$

and

$$|f(z)\geq(1-|b_1|)r-\frac{1}{2}\left[\frac{1-\alpha}{k+2-\alpha}-\frac{1+2k+\alpha}{k+2-\alpha}|b_1|\right]r^2,\quad |z|=r<1.$$

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