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Neighbourhood V_4-Magic Labeling of Some Shadow Graphs

Vineesh K.P.

(Department of Mathematics, Sree Narayana Guru College, Chelannur, Kozhikode, Kerala 673 616, India)

Anil Kumar V.

(Department of Mathematics, University of Calicut, Malappuram, Kerala 673 635, India)

E-mail: kpvineeshmaths@gmail.com, anil@uoc.ac.in

Abstract: The Klein 4-group,denoted by V_4 is an abelian group of order 4. It has elements $V_4 = \{0, a, b, c\}$ with $a + a = b + b = c + c = 0$ and $a + b = c, b + c = a, c + a = b$. A graph $G(V(G), E(G)$ is said to be neighbourhood V_4-magic if there exists a labeling $f : V(G) \to V_4\backslash\{0\}$ such that the induced mapping $N_f^+ : V(G) \to V_4$ defined by $N_f^+(v) = \sum_{u\in N(v)} f(u)$ is a constant map. If this constant is $p(p \neq 0)$,we say that f is a $p-$neighbourhood V_4-magic labeling of G and G a $p-$neighbourhood V_4-magic graph. If this constant is zero, we say that f is a $0-$neighghbourhood V_4-magic labeling of G and G a $0-$neighbourhood V_4-magic graph. In this paper, we discuss neighbourhood V_4-magic labeling of some shadow graphs.

Key Words: Klein-4-group, shadow graphs, a-neighbourhood V_4-magic graphs, 0-neighbourhood V_4-magic graphs, Smarandachely V_4-magic.

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§1. Introduction

Throughout this paper we consider simple, finite, connected and undirected graphs. For standard terminology and notation we follow [1] and [2]. For a detailed survey on graph labeling we refer [6]. The V_4-magic graphs were introduced by S. M. Lee et al. in 2002 [3]. We say that, a graph $G = (V(G), E(G))$, with vertex set $V(G)$ and edge set $E(G)$ is neighbourhood V_4-magic if there exists a labeling $f : V(G) \to V_4\backslash\{0\}$ such that the induced mapping $N_f^+ : V(G) \to V_4$ defined by $N_f^+(v) = \sum_{u\in N(v)} f(u)$ is a constant map. Otherwise, it is said to be *Smarandachely* V_4*-magic*, i.e., $\left|\left\{N_f^+(v), v \in V(G)\right\}\right| \geq 2$. If this constant is p, where p is any non zero element in V_4,then we say that f is a $p-$neighbourhood V_4-magic labeling of G and G is said to be a $p-$neighbourhood V_4-magic graph. If this constant is 0,then we say that f is a $0-$neighbourhood V_4-magic labeling of G and G is said to be a $0-$neighbourhood V_4-magic graph. We divide the class of neighbourhood V_4-magic graphs into the following three categories:

(1) $\Omega_a :=$ the class of all $a-$neighbourhood V_4-magic graphs;
(2) $\Omega_0 :=$ the class of all $0-$neighbourhood V_4-magic graphs, and
(3) $\Omega_{a,0} := \Omega_a \cap \Omega_0$.

The shadow graph $Sh(G)$ of a connected graph G is constructed by taking two copies of G say G_1 and G_2, join each vertex u in G_1; to the neighbours of the corresponding vertex v in G_2.The Bistar

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$B_{m,n}$is the graph obtained by joining the central vertex $K_{1,m}$ and $K_{1,n}$ by an edge [6]. The wheel graph W_n is defined as $W_n \simeq C_n + K_1$, where C_n for $n \geq 3$ is a cycle of length n.The Helm H_n is a graph obtained from the wheel graph W_n by attaching a pendant edge at each vertex of the cycle C_n [7].The Sunflower SF_n is obtained from a wheel with the central vertex w_0 and cycle $C_n = w_1w_2w_3 \cdots w_nw_1$ and additional vertices $v_1, v_2, v_3, \cdots, v_n$ where v_i is joined by edges to w_i and w_{i+1} where $i+1$ is taken over modulo n [8].Jelly fish graph$J(m,n)$is obtained from a $4-$cycle $w_1w_2w_3w_4w_1$ by joining w_1 and w_3 with an edge and appending the central vertex of $K_{1,m}$ to w_2 and appending the central vertex of $K_{1,n}$ to w_4 [6]. The graph $P_2\Box P_n$ is called Ladder, it is denoted by L_n [5].The graph with vertex set$\{u_i, v_i : 0 \leq i \leq n+1\}$and edge set $\{u_iu_{i+1}, v_iv_{i+1} : 0 \leq i \leq n\} \cup \{u_iv_i : 1 \leq i \leq n\}$ is called the ladder L_{n+2}.The corona $P_n \odot K_1$ is called the comb graph CB_n.The Book graph B_n is the graph $S_n\Box P_2$, where S_n is the star with $n+1$ vertices and P_2 is the path on 2 vertices [5]. A gear graph G_n is obtained from the wheel graph by adding a vertex between every pair of adjacent vertices of the cycle. G_n has $2n+1$vertices and $3n$ edges [9]. This paper investigate neighbourhood V_4-magic labeling of shadow graphs of the above said graphs.

§2. Main Results

Theorem 2.1 *The graph $Sh(C_n) \in \Omega_a$ if and only if $n \equiv 0(mod\ 4)$.*

Proof Considering the shadow graph $Sh(C_n)$, let $\{u_1, u_2, u_3, \cdots, u_n\}$be the vertex set of first copy of C_n and let $\{v_1, v_2, v_3, \cdots, v_n\}$be the corresponding vertex set of second copy of C_n in order. Assume that $n \not\equiv 0(mod\ 4)$.Then either $n \equiv 1(mod\ 4)$ or $n \equiv 2(mod\ 4)$ or $n \equiv 3(mod\ 4)$. We show that in each these cases $Sh(C_n) \notin \Omega_a$.

Case 1. $n \equiv 1(mod\ 4)$

In this case $n = 4k+1$ for some $k \in \mathbb{N}$. Then $V(Sh(C_n)) = \{u_i, v_i : 1 \leq i \leq 4k+1\}$. If possible, let $Sh(C_n) \in \Omega_a$ with a labeling f. Then $N_f^+(u_2) = a$ implies that $f(u_1) + f(v_1) + f(u_3) + f(v_3) = a$, $N_f^+(u_4) = a$ implies that $f(u_3) + f(v_3) + f(u_5) + f(v_5) = a$. Proceeding like this, $N_f^+(u_{4k}) = a$ implies that $f(u_{4k-1}) + f(v_{4k-1}) + f(u_{4k+1}) + f(v_{4k+1}) = a$. Now consider $f(u_1) + f(v_1)$,then either $f(u_1) + f(v_1) = 0$ or $f(u_1) + f(v_1) = a$ or $f(u_1) + f(v_1) = b$ or $f(u_1) + f(v_1) = c$.

Subcase 1.1 $f(u_1) + f(v_1) = 0$

If $f(u_1) + f(v_1) = 0$,then $f(u_3) + f(v_3) = a, f(u_5) + f(v_5) = 0, f(u_7) + f(v_7) = a$, which implies that $f(u_{4k+1}) + f(v_{4k+1}) = 0$. Now $N_f^+(u_1) = a$ implies that $f(u_2) + f(v_2) = a, f(u_4) + f(v_4) = 0, f(u_6) + f(v_6) = a$. Proceeding like this we get $f(u_{4k}) + f(v_{4k}) = 0$. Therefore,$N_f^+(u_{4k+1}) = f(u_1) + f(v_1) + f(u_{4k}) + f(v_{4k}) = 0 + 0 = 0$, a contradiction.

Subcase 1.2 $f(u_1) + f(v_1) = a$

If $f(u_1) + f(v_1) = a$, then proceeding as in Subcase 1.1 we get $N_f^+(u_{4k+1}) = f(u_1) + f(v_1) + f(u_{4k}) + f(v_{4k}) = a + a = 0$, a contradiction.

Subcase 1.3 $f(u_1) + f(v_1) = b$

If $f(u_1)+f(v_1) = b$,then $f(u_3)+f(v_3) = c, f(u_5)+f(v_5) = b, f(u_7)+f(v_7) = c$, which implies that $f(u_{4k+1})+f(v_{4k+1}) = b$. Now,$N_f^+(u_1) = a$ gives $f(u_2)+f(v_2) = c, f(u_4)+f(v_4) = b, f(u_{4k})+f(v_{4k}) = b$. Therefore,$N_f^+(u_{4k+1}) = f(u_1) + f(v_1) + f(u_{4k}) + f(v_{4k}) = b + b = 0$, which is a contradiction.

Subcase 1.4 $f(u_1) + f(v_1) = c$

If $f(u_1) + f(v_1) = c$, then proceeding as in Subcase 1.3 we get $N_f^+(u_{4k+1}) = f(u_1) + f(v_1) + f(u_{4k}) + f(v_{4k}) = c + c = 0$, a contradiction.

Thus if $n \equiv 1(mod\ 4)$, we have $Sh(C_n) \notin \Omega_a$.

Case 2. $n \equiv 2(mod\ 4)$

In this case $n = 4k + 2$ for some $k \in \mathbb{N}$. Then $V(Sh(C_n)) = \{u_i, v_i : 1 \leq i \leq 4k + 2\}$. If possible let $Sh(C_n) \in \Omega_a$ with a labeling f. Considering $f(u_1) + f(v_1)$, then either $f(u_1) + f(v_1) = 0$ or $f(u_1) + f(v_1) = a$ or $f(u_1) + f(v_1) = b$ or $f(u_1) + f(v_1) = c$.

Subcase 2.1 $f(u_1) + f(v_1) = 0$

If $f(u_1) + f(v_1) = 0$,then $N_f^+(u_2) = a, f(u_3) + f(v_3) = a, f(u_5) + f(v_5) = 0$, which implies that $f(u_{4k+1}) + f(v_{4k+1}) = 0$. Therefore, $N_f^+(u_{4k+2}) = f(u_1) + f(v_1) + f(u_{4k+1}) + f(v_{4k+1}) = 0 + 0 = 0$, a contradiction.

Subcase 2.2 $f(u_1) + f(v_1) = a$

If $f(u_1) + f(v_1) = a$, then proceeding as in Subcase 2.1 we get $N_f^+(u_{4k+2}) = f(u_1) + f(v_1) + f(u_{4k+1}) + f(v_{4k+1}) = a + a = 0$, which is a contradiction.

Subcase 2.3 $f(u_1) + f(v_1) = b$

If $f(u_1) + f(v_1) = b$,then $N_f^+(u_2) = a$ implies that $f(u_3) + f(v_3) = c, f(u_5) + f(v_5) = b$, implies that $f(u_{4k+1}) + f(v_{4k+1}) = b$. Therefore, $N_f^+(u_{4k+2}) = f(u_1) + f(v_1) + f(u_{4k+1}) + f(v_{4k+1}) = b + b = 0$, which is a contradiction.

Subcase 2.4 $f(u_1) + f(v_1) = c$

If $f(u_1) + f(v_1) = c$, then proceeding as in Subcase 2.3 we get $N_f^+(u_{4k+2}) = f(u_1) + f(v_1) + f(u_{4k+1}) + f(v_{4k+1}) = c + c = 0$, a contradiction.

Thus if $n \equiv 2(mod\ 4)$, $Sh(C_n) \notin \Omega_a$.

Case 3. $n \equiv 3(mod\ 4)$

In this case $n = 4k + 3$ for some $k \in \mathbb{N}$. Then $V(Sh(C_n)) = \{u_i, v_i : 1 \leq i \leq 4k + 3\}$. If possible let $Sh(C_n) \in \Omega_a$ with a labeling f. Considering $f(u_1) + f(v_1)$, then either $f(u_1) + f(v_1) = 0$ or $f(u_1) + f(v_1) = a$ or $f(u_1) + f(v_1) = b$ or $f(u_1) + f(v_1) = c$.

Subcase 3.1 $f(u_1) + f(v_1) = 0$

If $f(u_1) + f(v_1) = 0$, then $N_f^+(u_2) = a$ gives $f(u_3) + f(v_3) = a$, $f(u_5) + f(v_5) = 0$, $f(u_{4k+1}) + f(v_{4k+1}) = 0$, $f(u_{4k+3}) + f(v_{4k+3}) = a$. Now, $N_f^+(u_1) = a$ implies that $f(u_2) + f(v_2) = 0$, $f(u_4) + f(v_4) = a$, $f(u_{4k+2}) + f(v_{4k+2}) = 0$. Therefore $N_f^+(u_{4k+3}) = f(u_1) + f(v_1) + f(u_{4k+2}) + f(v_{4k+2}) = 0 + 0 = 0$, which is a contradiction.

Subcase 3.2 $f(u_1) + f(v_1) = a$

If $f(u_1) + f(v_1) = a$, then proceeding as in Subcase 3.1 we get $N_f^+(u_{4k+3}) = f(u_1) + f(v_1) + f(u_{4k+2}) + f(v_{4k+2}) = a + a = 0$, a contradiction.

Subcase 3.3 $f(u_1) + f(v_1) = b$

If $f(u_1) + f(v_1) = b$, then $N_f^+(u_2) = a$ implies that $f(u_3) + f(v_3) = c, f(u_5) + f(v_5) = b, f(u_{4k+1}) + f(v_{4k+1}) = b, f(u_{4k+3}) + f(v_{4k+3}) = c$. Now, $N_f^+(u_1) = a$ implies that $f(u_2) + f(v_2) = b, f(u_4) + f(v_4) =$

$c, f(u_{4k+2}) + f(v_{4k+2}) = b$. Therefore, $N_f^+(u_{4k+3}) = f(u_1) + f(v_1) + f(u_{4k+2}) + f(v_{4k+2}) = b + b = 0$, which is a contradiction.

Subcase 3.4 $f(u_1) + f(v_1) = c$

If $f(u_1) + f(v_1) = c$, then proceeding as in Subcase 3.3 we get $N_f^+(u_{4k+3}) = f(u_1) + f(v_1) + f(u_{4k+2}) + f(v_{4k+2}) = c + c = 0$, a contradiction.

Thus if $n \equiv 3(mod\ 4)$, we also have $Sh(C_n) \notin \Omega_a$. Therefore, $n \not\equiv 0(mod\ 4)$ implies that $Sh(C_n) \notin \Omega_a$.

Conversely if $n \equiv 0(mod\ 4)$, We define $f : V(Sh(C_n)) \to V_4 \backslash \{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i \equiv 1, 2(mod\ 4), \\ c & \text{if} \quad i \equiv 0, 3(mod\ 4) \end{cases} \qquad \text{and} \quad f(v_i) = a \quad \text{for} \quad 1 \leq i \leq n.$$

Then, f is an $a-$neighbourhood V_4-magic labeling for $Sh(C_n)$. This completes the proof of the theorem. □

Theorem 2.2 *$Sh(C_n) \in \Omega_0$ for all $n \geq 3$.*

Proof The degree of each vertex in $Sh(C_n)$is 4. By labeling all the vertices by a, we get $N_f^+(u) = 0$ for all $u \in V(Sh(C_n))$. □

Corollary 2.3 *$Sh(C_n) \in \Omega_{a,0}$ if and only if $n \equiv 0(mod\ 4)$.*

Proof The proof is obviously follows from Theorems 2.1 and 2.2. □

Theorem 2.4 *The graph $Sh(P_n) \in \Omega_0$ for all $n \geq 2$.*

Proof If we label all the vertices by a,we get $G \in \Omega_0$. □

Theorem 2.5 *$Sh(P_n) \in \Omega_a$ for $n \equiv 0, 2, 3(mod\ 4)$.*

Proof Let G be the shadow graph $Sh(P_n)$, and let $\{u_i : 1 \leq i \leq n\}and\{v_i : 1 \leq i \leq n\}$ be the vertex sets of first and second copy of P_n respectively.

Case 1. $n \equiv 0(mod\ 4)$

Define $f : V(G) \to V_4 \backslash \{0\}$ as:

$$f(u_i) = \begin{cases} a & \text{if} \quad i \equiv 0, 1(mod\ 4), \\ b & \text{if} \quad i \equiv 2, 3(mod\ 4), \end{cases}$$

$$f(v_i) = \begin{cases} a & \text{if} \quad i \equiv 0, 1(mod\ 4), \\ c & \text{if} \quad i \equiv 2, 3(mod\ 4). \end{cases}$$

Case 2. $n \equiv 2(mod\ 4)$

Define $f : V(G) \to V_4 \backslash \{0\}$ as:

$$f(u_i) = \begin{cases} a & \text{if} \quad i \equiv 0, 3(mod\ 4), \\ b & \text{if} \quad i \equiv 1, 2(mod\ 4), \end{cases}$$

$$f(v_i) = \begin{cases} a & \text{if} \quad i \equiv 0, 3(mod\ 4), \\ c & \text{if} \quad i \equiv 1, 2(mod\ 4). \end{cases}$$

Case 3. $n \equiv 3(mod\ 4)$

Define $f : V(G) \to V_4 \backslash \{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i \equiv 1, 2(mod\ 4), \\ a & \text{if} \quad i \equiv 0, 3(mod\ 4), \end{cases}$$

$$f(v_i) = \begin{cases} c & \text{if} \quad i \equiv 1, 2(mod\ 4), \\ a & \text{if} \quad i \equiv 0, 3(mod\ 4). \end{cases}$$

In all the above cases, we have $N_f^+(u_i) = N_f^+(v_i) = a$ for $1 \le i \le n$. Therefore, $Sh(P_n) \in \Omega_a$ for $n \equiv 0, 2, 3(mod\ 4)$. □

Theorem 2.6 *$Sh(P_n) \notin \Omega_a$ for $n \equiv 1(mod\ 4)$.*

Proof Consider the shadow graph $Sh(P_n)$ with $n \equiv 1(mod\ 4)$. Let $\{u_i : 1 \le i \le 4k + 1\}$ and$\{v_i : 1 \le i \le 4k + 1\}$ be the vertex sets of first and second copy of P_n respectively. Assume that $Sh(P_n) \in \Omega_a$ with a labeling f. Since $N_f^+(u_1) = a$, we have either $f(u_2) = b$ and $f(v_2) = c$ or $f(u_2) = c$ and $f(v_2) = b$. Without loss of generality assume that $f(u_2) = b$ and $f(v_2) = c$. Then $f(u_{4k}) = f(v_{4k})$ implies that $N_f^+(u_{4k+1}) = 0$, a contradiction. Therefore, $Sh(P_n) \notin \Omega_a$. □

Corollary 2.7 *$Sh(P_n) \in \Omega_{a,0}$ for $n \equiv 0, 2, 3(mod\ 4)$.*

Proof The proof directly follows from theorems 2.4 and 2.5. □

Theorem 2.8 *$Sh(K_{1,n}) \in \Omega_a$ for all $n \in \mathbb{N}$.*

Proof Let $V = \{u_i, v_i : 0 \le i \le n\}$ be the vertex set of $Sh(K_{1,n})$ where $\{u_i : 0 \le i \le n\}$ and $\{v_i : 0 \le i \le n\}$ are the vertex sets of first and second copy of $K_{1,n}$ with apex u_0, v_0 respectively. Define $f : V \to V_4 \backslash \{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 0, 1, \\ a & \text{if} \quad i = 2, 3, \cdots, n, \end{cases}$$

$$f(v_i) = \begin{cases} c & \text{if} \quad i = 0, 1, \\ a & \text{if} \quad i = 2, 3, \cdots, n. \end{cases}$$

Then, $N_f^+(u_i) = N_f^+(v_i) = a$ for all $0 \le i \le n$. This completes the proof. □

Theorem 2.9 *$Sh(K_{1,n}) \in \Omega_0$ for all $n \in \mathbb{N}$.*

Proof If we label all the vertices by a,we get $Sh(K_{1,n}) \in \Omega_0$. □

Corollary 2.10 *$Sh(K_{1,n}) \in \Omega_{a,0}$ for all $n \in \mathbb{N}$.*

Proof The proof obviously follows from Theorems 2.8 and 2.9. □

Theorem 2.11 *$Sh(B_{m,n}) \in \Omega_0$ for all m and n.*

Proof Labeling all the vertices by a, we get $Sh(B_{m,n}) \in \Omega_0$ for all m and n. □

Theorem 2.12 *$Sh(B_{m,n}) \in \Omega_a$ for all $m > 1$ and $n > 1$.*

Proof Let $V_1 = \{u, v, u_1, u_2, \cdots, u_m, v_1, v_2, v_3, \cdots, v_n\}$ be the vertex set of first copy of $B_{m,n}$ and $V_2 = \{u', v', u'_1, u'_2, \cdots, u'_m, v'_1, v'_2, v'_3, \cdots, v'_n\}$ be the corresponding vertex set of second copy of $B_{m,n}$, where u_i, v_i are pendant vertices adjacent to u, v respectively. Then $V(Sh(B_{m,n})) = V_1 \cup V_2$.

Define $f : V(Sh(B_{m,n})) \to V_4 \backslash \{0\}$ as:

$f(u) = f(v) = b$;

$f(u') = f(v') = c$;

$f(u_i) = f(u'_i) = a$ for $1 \leq i \leq m$;

$f(v_i) = f(v'_i) = a$ for $1 \leq i \leq n$.

Then, f is an $a-$neighbourhod labeling of $Sh(B_{m,n})$. This completes the proof. □

Corollary 2.13 *$Sh(B_{m,n}) \in \Omega_{a,0}$ for all $m > 1$ and $n > 1$.*

Proof The proof follows from Theorems 2.11 and 2.12. □

Theorem 2.14 *$Sh(W_n) \in \Omega_0$ for all $n \geq 3$.*

Proof The degree of a vertex in $Sh(W_n)$ is either 6 or $2n$. If we label all the vertices by a, we get $N_f^+(u) = 0$ for all $u \in V(Sh(W_n))$. □

Theorem 2.15 *$Sh(W_n) \in \Omega_a$ for all $n \equiv 1(mod\ 2)$.*

Proof Let $V_1 = \{u_0, u_1, u_2, \cdots, u_n\}$ be the vertex set of first copy of W_n with central vertex u_0 and let $V_2 = \{v_0, v_1, v_2, \cdots, v_n\}$ be the corresponding vertex set of second copy of $W_n$with central vertex v_0. Then, $V = V(Sh(W_n)) = V_1 \cup V_2$. Define $f : V \to V_4 \backslash \{0\}$ as:

$$f(u_i) = b \quad \text{if} \quad i = 0, 1, 2, 3, \cdots, n,$$

$$f(v_i) = c \quad \text{if} \quad i = 0, 1, 2, 3, \cdots, n.$$

Then, $N_f^+(u_i) = N_f^+(v_i) = a$ for all $i = 0, 1, 2, \cdots, n$. □

Corollary 2.16 *$Sh(W_n) \in \Omega_{a,0}$ for all $n \equiv 1(mod\ 2)$.*

Proof The proof directly follows from Theorems 2.14 and 2.15. □

Theorem 2.17 *$Sh(W_n) \in \Omega_a$ for all $n \equiv 2(mod\ 4)$.*

Proof Let $V_1 = \{u_0, u_1, u_2, \cdots, u_n\}$ be the vertex set of first copy of W_n with central vertex u_0 and let $V_2 = \{v_0, v_1, v_2, \cdots, v_n\}$ be the vertex set of second copy with central vertex v_0.Then $V(Sh(W_n)) = V_1 \cup V_2$. Define $f : V(Sh(W_n)) \to V_4 \backslash \{0\}$ as:

$$f(u_i) = \begin{cases} a & \text{if} \quad i \equiv 1, 3(mod\ 4), \\ c & \text{if} \quad i \equiv 0, 2(mod\ 4), \end{cases}$$

$$f(v_i) = \begin{cases} a & \text{if} \quad i \equiv 1,3(mod\ 4), \\ b & \text{if} \quad i \equiv 0,2(mod\ 4). \end{cases}$$

Clearly, $N_f^+(u_i) = N_f^+(v_i) = a$ for all $i = 0, 1, 2, \ldots, n$. Hence $Sh(W_n) \in \Omega_a$. □

Corollary 2.18 *$Sh(W_n) \in \Omega_{a,0}$ for all $n \equiv 2(mod\ 4)$.*

Proof The proof directly follows from Theorems 2.14 and 2.17. □

Theorem 2.19 *$Sh(H_n) \in \Omega_0$ for all $n \geq 3$.*

Proof In $Sh(H_n)$, degree of vertices are either 2 or 8 or $2n$. If we label all the vertices by a, we get $N_f^+(u) = 0$ for all $u \in V(Sh(H_n))$. □

Theorem 2.20 *$Sh(H_n)$ admits $a-$neighbourhood V_4-magic labeling for all $n \equiv 1(mod\ 2)$.*

Proof Consider the shadow graph $Sh(H_n)$. Let v be central vertex,$v_1, v_2, v_3, \cdots, v_n$ be the rim vertices and $u_1, u_2, u_3, \cdots, u_n$ be the pendant vertices adjacent to $v_1, v_2, v_3, \cdots, v_n$ in the first copy of H_n and let $v', v'_1, v'_2, v'_3, \cdots, v'_n, u'_1, u'_2, u'_3, \cdots, u'_n$ be the corresponding vertices in the second copy of H_n. Then $V(Sh(H_n)) = \{v, v', v_i, v'_i, u_i, u'_i : 1 \leq i \leq n\}$. We define $f : V(Sh(H_n)) \to V_4 \backslash \{0\}$ as:

$$f(v) = a \quad \text{and} \quad f(v_i) = f(u_i) = b \quad \text{for} \quad i = 1, 2, 3, \ldots, n,$$
$$f(v') = a \quad \text{and} \quad f(v'_i) = f(u'_i) = c \quad \text{for} \quad i = 1, 2, 3, \ldots, n.$$

Obviously, f is an $a-$neighbourhood V_4-magic labeling of $Sh(H_n)$. □

Corollary 2.21 *$Sh(H_n) \in \Omega_{a,0}$ for all $n \equiv 1(mod\ 2)$.*

Proof The proof directly follows from Theorems 2.19 and 2.20. □

Theorem 2.22 *$Sh(SF_n)$ admits $a-$neighbourhood V_4-magic labeling for all $n \equiv 2(mod\ 4)$.*

Proof Considering $Sh(SF_n)$, let the vertex set of first copy of SF_n be $V_1 = \{w, w_i, v_i : 1 \leq i \leq n\}$ where w is the central vertex, $w_1, w_2, w_3, \cdots, w_n$ are vertices of the cycle and v_i is the vertex joined by edges to w_i and w_{i+1} where $i + 1$ is taken over modulo n. Let $V_2 = \{w', w'_i, v'_i : 1 \leq i \leq n\}$ be the corresponding vertex set of second copy of SF_n. Then $V(Sh(SF_n)) = V_1 \cup V_2$. Define $f : V(Sh(SF_n)) \to V_4 \backslash \{0\}$ as:

$$f(w_i) = \begin{cases} b & \text{if} \quad i \equiv 1(mod\ 2), \\ c & \text{if} \quad i \equiv 0(mod\ 2), \end{cases}$$

$$f(v_i) = \begin{cases} b & \text{if} \quad i \equiv 1(mod\ 2), \\ c & \text{if} \quad i \equiv 0(mod\ 2), \end{cases}$$

$$f(w) = f(w') = f(w'_i) = f(v'_i) = a \text{ for } i = 1, 2, 3, \cdots, n.$$

Then f is an $a-$neighbourhood V_4-magic labeling of $Sh(SF_n)$. □

Theorem 2.23 *$Sh(SF_n)$ admits $0-$neighbourhood V_4-magic labeling for all n.*

Proof If we label all the vertices by a, we get $N_f^+(u) = 0$ for all $u \in V(Sh(SF_n))$. □

Theorem 2.24 *$Sh(SF_n) \in \Omega_{a,0}$ for all $n \equiv 2(mod\ 4)$.*

Proof The proof is obviously follows from Theorems 2.22 and 2.23. □

Theorem 2.25 *$Sh(C_n \odot K_2) \in \Omega_a$ for all $n \equiv 0(mod\ 4)$.*

Proof Let G bet the shadow graph $Sh(C_n \odot K_2)$. Let $V_1 = \{u_i, v_i, w_i : 1 \leq i \leq n\}$ be the vertex set of first copy of $C_n \odot K_2$, where $u_i's$ are vertices of C_n and v_j, w_j are the vertices on j^{th} copy of K_2 and let $V_2 = \{u'_i, v'_i, w'_i : 1 \leq i \leq n\}$ be the corresponding vertex set of second copy of $C_n \odot K_2$. Then $V(G) = V_1 \cup V_2$. Define $f : V(G) \to V_4\backslash\{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i \equiv 1, 2(mod\ 4), \\ c & \text{if} \quad i \equiv 0, 3(mod\ 4), \end{cases}$$

$$f(v_i) = \begin{cases} c & \text{if} \quad i \equiv 1, 2(mod\ 4), \\ b & \text{if} \quad i \equiv 0, 3(mod\ 4), \end{cases}$$

$$f(w_i) = \begin{cases} c & \text{if} \quad i \equiv 1, 2(mod\ 4), \\ b & \text{if} \quad i \equiv 0, 3(mod\ 4), \end{cases}$$

$$f(u'_i) = f(v'_i) = f(w'_i) = a \text{ for } i = 1, 2, 3, \cdots, n.$$

Then f is an $a-$neighbourhood V_4-magic labeling of $Sh(C_n \odot K_2)$. □

Theorem 2.26 *$Sh(C_n \odot K_2) \in \Omega_0$ for all n.*

Proof By labeling all the vertices of $Sh(C_n \odot K_2)$ by a,we get $N_f^+(u) = 0$. □

Corollary 2.27 *$Sh(C_n \odot K_2) \in \Omega_{a,0}$ for all $n \equiv 0(mod\ 4)$.*

Proof The proof follows from Theorems 2.25 and 2.26. □

Theorem 2.28 *$Sh(C_n \odot \overline{K}_m) \in \Omega_a$ for all m and $n \geq 3$.*

Proof Let G be the shadow graph $Sh(C_n \odot \overline{K}_m)$. Let $u_1, u_2, u_3, \cdots, u_n$ be the rim vertices of first copy of $C_n \odot \overline{K}_m$ and $\{u_{i1}, u_{i2}, u_{i3}, \cdots, u_{im}\}$ be the set of pendant vertices adjacent to u_i for $1 \leq i \leq n$ in $C_n \odot \overline{K}_m$ and let $u'_1, u'_2, u'_3, \cdots, u'_n$ be the rim vertices of second copy of $C_n \odot \overline{K}_m$ and $\{u'_{i1}, u'_{i2}, u'_{i3}, \cdots, u'_{im}\}$ be the set of pendant vertices adjacent to u'_i for $1 \leq i \leq n$ in second copy of $C_n \odot \overline{K}_m$. Here we consider two cases.

Case 1. $m = 1$

Define $f : V(G) \to V_4\backslash\{0\}$ as:

$$f(u_i) = f(u_{i,1}) = b \quad \text{for} \quad i = 1, 2, 3, \cdots, n.$$

$$f(u'_i) = f(u'_{i,1}) = c \quad \text{for} \quad i = 1, 2, 3, \cdots, n.$$

Case 2. $m \geq 2$

Define $f : V(G) \to V_4\backslash\{0\}$ as:

$$f(u_i) = b \quad \text{for} \quad i = 1, 2, 3, \cdots, n.$$

$$f(u_i') = c \quad \text{for} \quad i = 1, 2, 3, \cdots, n.$$
$$f(u_{ij}') = a \quad \text{for} \quad i = 1, 2, 3, \cdots, n.$$
$$f(u_{ij}) = \begin{cases} b & \text{if} \quad j = 1, \\ c & \text{if} \quad j = 2, \\ a & \text{if} \quad j > 2. \end{cases}$$

Obviously, f is an $a-$neighbourhood V_4-magic labeling of $Sh(C_n \odot \overline{K}_m)$. □

Theorem 2.29 *$Sh(C_n \odot \overline{K}_m) \in \Omega_0$ for all m and $n \geq 3$.*

Proof Labeling all the vertices by a,we get $Sh(C_n \odot \overline{K}_m) \in \Omega_0$. □

Corollary 2.30 *$Sh(C_n \odot \overline{K}_m) \in \Omega_{a,0}$ for all m and $n \geq 3$.*

Proof The proof directly follows from Theorems 2.28 and 2.29. □

Theorem 2.31 *$Sh(J(m,n)) \in \Omega_0$ for all m and n.*

Proof Labeling all the vertices by a, we get $Sh(J(m,n)) \in \Omega_0$. □

Theorem 2.32 *$Sh(J(m,n)) \in \Omega_a$ for all m and n.*

Proof Let G be the graph $Sh(J(m,n))$. Let $V_1 = \{w_i, u_j, v_k : 1 \leq i \leq 4, 1 \leq j \leq m, 1 \leq k \leq n\}$ and $E_1 = \{w_1w_2, w_2w_3, w_3w_4, w_4w_1, w_1w_3\} \cup \{w_2u_j : 1 \leq j \leq m\} \cup \{w_4v_j : 1 \leq j \leq n\}$ be the vertex and edge set of first copy of $J(m,n)$ and let $V_2 = \{w_i', u_j', v_k' : 1 \leq i \leq 4, 1 \leq j \leq m, 1 \leq k \leq n\}$ be the corresponding vertex set of second copy of $J(m,n)$. Then $V(G) = V_1 \cup V_2$. Define $f : V(G) \to V_4 \backslash \{0\}$ as:

$$f(w_i) = b \quad \text{for} \quad i = 1, 2, 3, 4;$$
$$f(w_i') = c \quad \text{for} \quad i = 1, 2, 3, 4;$$

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 1, \\ a & \text{if} \quad i \geq 2, \end{cases} \qquad f(u_i') = \begin{cases} c & \text{if} \quad i = 1, \\ a & \text{if} \quad i \geq 2, \end{cases}$$

$$f(v_i) = \begin{cases} b & \text{if} \quad i = 1, \\ a & \text{if} \quad i \geq 2, \end{cases} \qquad f(v_i') = \begin{cases} c & \text{if} \quad i = 1, \\ a & \text{if} \quad i \geq 2. \end{cases}$$

Then, f is an $a-$neighbourhood V_4-magic labeling of $Sh(J(m,n))$. □

Corollary 2.33 *$Sh(J(m,n)) \in \Omega_{a,0}$ for all m and n.*

Proof The proof directly follows from Theorems 2.31 and 2.32. □

Theorem 2.34 *$Sh(L_n) \in \Omega_0$ for all n.*

Proof By labeling all the vertices by a,we get $Sh(L_n) \in \Omega_0$ for all n. □

Theorem 2.35 *$Sh(L_n) \in \Omega_a$ for all $n \equiv 2(mod\ 3)$.*

Proof Consider $Sh(L_n)$ with $n \equiv 2(mod\ 3)$. Let $V_1 = \{u_i, v_i : 1 \leq i \leq n\}$ be the vertex set of first copy of L_n with edge set $E_1 = \{u_iu_{i+1}, v_iv_{i+1}, u_jv_j : 1 \leq i \leq n-1, 1 \leq j \leq n\}$. Also let $V_2 = \{u_i', v_i' : 1 \leq i \leq n\}$ be the corresponding set of vertices in second copy of L_n. Then $V = V(Sh(L_n)) = V_1 \cup V_2$. Define $f : V \to V_4\backslash\{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i \equiv 1, 2(mod\ 6), \\ c & \text{if} \quad i \equiv 4, 5(mod\ 6), \\ a & \text{if} \quad i \equiv 0, 3(mod\ 6), \end{cases}$$

$$f(v_i) = \begin{cases} c & \text{if} \quad i \equiv 1, 2(mod\ 6), \\ b & \text{if} \quad i \equiv 4, 5(mod\ 6), \\ a & \text{if} \quad i \equiv 0, 3(mod\ 6), \end{cases}$$

$$f(u_i') = a \quad \text{for} \quad i = 1, 2, 3, \cdots, n,$$

$$f(v_i') = a \quad \text{for} \quad i = 1, 2, 3, \cdots, n.$$

Then, f is an a-neighbourhood V_4-magic labeling of $Sh(L_n)$. □

Corollary 2.36 *$Sh(L_n) \in \Omega_{a,0}$ for all $n \equiv 2(mod\ 3)$.*

Proof The proof directly follows from Theorems 2.34 and 2.35. □

Theorem 2.37 *$Sh(L_{n+2}) \in \Omega_0$ for all $n \in \mathbb{N}$.*

Proof By labeling all the vertices by a,we get $Sh(L_{n+2}) \in \Omega_0$ for all n. □

Theorem 2.38 *$Sh(L_{n+2}) \in \Omega_a$ for all $n \in \mathbb{N}$.*

Proof Let G be the shadow graph $Sh(L_{n+2})$. Let $V_1 = \{u_i, v_i : 0 \leq i \leq n+1\}$ and $E_1 = \{u_iu_{i+1}, v_iv_{i+1} : 0 \leq i \leq n\} \cup \{u_iv_i : 1 \leq i \leq n\}$ be the vertex and edge set of first copy of L_{n+2} and let $V_2 = \{u_i', v_i' : 0 \leq i \leq n+1\}$ be the corresponding set of vertices in second copy of L_{n+2}. Define $f : V(Sh(L_{n+2})) \to V_4\backslash\{0\}$ as:

$$f(u_i) = f(v_i) = b \quad \text{for} \quad i = 0, 1, 2, 3, \cdots, n+1,$$

$$f(u_i') = f(v_i') = c \quad \text{for} \quad i = 0, 1, 2, 3, \cdots, n+1,$$

Then, $N_f^+(u) = a$ for all vertices u in $Sh(L_{n+2})$. □

Corollary 2.39 *$Sh(L_{n+2}) \in \Omega_{a,0}$ for all $n \in \mathbb{N}$.*

Proof The proof directly follows from Theorems 2.37 and 2.38. □

Theorem 2.40 *$Sh(CB_n) \in \Omega_a$ for all $n > 1$.*

Proof Let $\{u_i, v_i : 1 \leq i \leq n\}$ be the vertex set of first copy of CB_n where v_i $(1 \leq i \leq n)$ are the pendant vertices adjacent to u_i $(1 \leq i \leq n)$. Let $\{u_i', v_i' : 1 \leq i \leq n\}$ be the corresponding set of vertices in second copy of CB_n. Define $f : V(Sh(CB_n)) \to V_4\backslash\{0\}$ as

$$\begin{aligned} f(u_i) &= b \quad \text{if} \quad 1 \le i \le n; \\ f(u_i') &= c \quad \text{if} \quad 1 \le i \le n; \\ f(v_i) &= \begin{cases} a & \text{if} \quad i = 1 \text{ or } n, \\ b & \text{if} \quad 1 < i < n, \end{cases} \\ f(v_i') &= \begin{cases} a & \text{if} \quad i = 1 \text{ or } n, \\ c & \text{if} \quad 1 < i < n. \end{cases} \end{aligned}$$

Then f is an $a-$neighbourhood V_4-magic labeling of CB_n. □

Theorem 2.41 *$Sh(CB_n) \in \Omega_0$ for all $n \in \mathbb{N}$.*

Proof By labeling all the vertices by a,we get $Sh(CB_n) \in \Omega_0$. □

Corollary 2.42 *$Sh(CB_n) \in \Omega_{a,0}$ for all $n > 1$.*

Proof The proof directly follows from Theorems 2.40 and 2.41. □

Theorem 2.43 *$Sh(K_{m,n}) \in \Omega_a$ for all $m > 1$ and $n > 1$.*

Proof Let G be the shadow graph $Sh(K_{m,n})$. Let $X = \{u_1, u_2, u_3, \cdots, u_m\}$ and $Y = \{v_1, v_2, v_3, \cdots, v_n\}$ be the bipartition of the first copy of $K_{m,n}$ and let $X' = \{u_1', u_2', u_3', \ldots, u_m'\}$ and $Y' = \{v_1', v_2', v_3', \ldots, v_n'\}$ be the corresponding bipartition second copy of $K_{m,n}$. Define $f : V(G) \to V_4\backslash\{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 1, \\ c & \text{if} \quad i = 2, \\ a & \text{if} \quad i > 2, \end{cases} \qquad f(v_j) = \begin{cases} b & \text{if} \quad j = 1, \\ c & \text{if} \quad j = 2, \\ a & \text{if} \quad j > 2, \end{cases}$$

$$f(u_i') = a \ \text{ for } 1 \le i \le m \text{ and } f(v_j') = a \ \text{ for } 1 \le j \le n.$$

Then f is an $a-$neighbourhood V_4-magic labeling of $Sh(K_{m,n})$. This completes the proof of the theorem. □

Theorem 2.44 *$Sh(K_{m,n}) \in \Omega_0$ for all $m, n \in \mathbb{N}$.*

Proof Labeling all the vertices by a, we get $Sh(K_{m,n}) \in \Omega_0$. □

Corollary 2.45 *$Sh(K_{m,n}) \in \Omega_{a,0}$ for all $m > 1$ and $n > 1$.*

Proof The proof directly follows from Theorems 2.43 and 2.44. □

Theorem 2.46 *$Sh(B_n) \in \Omega_a$ for all $n \equiv 1(mod\ 2)$.*

Proof Let G be the shadow graph $Sh(B_n)$. Let vertex set of first copy of B_n be $V_1 = \{(u, v_j), (u_i, v_j) : 1 \le i \le n, 1 \le j \le 2\}$, where $\{u, u_1, u_2, u_3, \cdots, u_n\}$ and $\{v_1, v_2\}$ be the vertex sets of S_n and P_2 respectively, and u be the central vertex, $u_i's$ are pendant vertices in S_n. Also let $V_2 = \{(u', v_j'), (u_i', v_j') : 1 \le i \le n, 1 \le j \le 2\}$ be the corresponding vertex set of second copy of B_n. Then $V(G) = V_1 \cup V_2$.

Define $f : V(G) \to V_4\backslash\{0\}$ as:

$$f(u, v_j) = \begin{cases} b & \text{if } \quad j = 1, \\ c & \text{if } \quad j = 2, \end{cases} \qquad \text{and} \qquad f(u_i, v_j) = \begin{cases} b & \text{if } \quad j = 1 \text{ and } 1 \le i \le n, \\ c & \text{if } \quad j = 2 \text{ and } 1 \le i \le n, \end{cases}$$

$$f(u', v_j') = a \text{ for } j = 1, 2 \text{ and } f(u_i', v_j') = a \ \text{ for } \ 1 \le i \le n, \ 1 \le j \le 2.$$

Clearly, f is an $a-$neighbourhood V_4-magic labeling of $Sh(B_n)$. □

Theorem 2.47 *$Sh(B_n) \in \Omega_0$ for all $n \in \mathbb{N}$.*

Proof By labeling all the vertices by a, we get $Sh(B_n) \in \Omega_0$. □

Corollary 2.48 *$Sh(B_n) \in \Omega_{a,0}$ for all $n \equiv 1(mod\ 2)$.*

Proof The proof follows from Theorems 2.46 and 2.47. □

Theorem 2.49 *$Sh(G_n) \in \Omega_0$ for all n.*

Proof The degree of vertices in $Sh(B_n)$ is either 4 or 6 or $2n$. If we label all the vertices by a, we get $N_f^+(u) = 0$ for all $u \in V(Sh(G_n))$. □

Theorem 2.50 *$Sh(G_n) \in \Omega_a$ for all $n \equiv 2(mod\ 4)$.*

Proof Let G be the shadow graph $Sh(G_n)$. Let $V_1 = \{u, u_i : 1 \le i \le 2n\}$ and $E_1 = \{uu_{2i-1} : 1 \le i \le n\} \cup \{u_i u_{i+1} : 1 \le i \le 2n-1\} \cup \{u_{2n}u_1\}$ be the vertex and edge set of first copy of G_n. Let $V_2 = \{u', u_i' : 1 \le i \le 2n\}$ be the corresponding vertex set of second copy of G_n.Then $V(G) = V_1 \cup V_2$. Define $f : V(G) \to V_4\backslash\{0\}$ as:

$$f(u) = b, \ f(u') = c \text{ and } f(u_i) = a \text{ for } 1 \le i \le 2n,$$

$$f(u_i') = \begin{cases} a & \text{if } \quad i \equiv 0(mod\ 4), \\ b & \text{if } \quad i \equiv 1(mod\ 4), \\ a & \text{if } \quad i \equiv 2(mod\ 4), \\ c & \text{if } \quad i \equiv 3(mod\ 4). \end{cases}$$

Then f is an $a-$neighbourhood V_4-magic labeling for $Sh(G_n)$. □

Corollary 2.51 *$Sh(G_n) \in \Omega_{a,0}$ for all $n \equiv 2(mod\ 4)$.*

Proof The proof directly follows from Theorems 2.49 and 2.50. □

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