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D-homothetic Deformations of Lorentzian Para-Sasakian Manifold

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Abstract: The aim of the present paper is to prove some results on the properties of LP-Sasakian manifolds under D-homothetic deformations. In the later sections we give several results on some properties which are conformal under the mentioned deformations. Lastly, we illustrate the main theorem by giving a detailed example.

Key Words: D-homothetic deformation, LP-Sasakian manifold, ϕ-section, sectional curvature.

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§1. Introduction

The notion of Lorentzian almost para-contact manifolds was introduced by K. Matsumoto [3]. Later on, a large number of geometers studied Lorentzian almost para-contact manifold and their different classes, viz., Lorentzian para-Sasakian manifolds and Lorentzian special para-Sasakian manifolds [4], [5], [6], [7]. In brief, Lorentzian para-Sasakian manifolds are called LP-Sasakian manifolds. The study of LP-Sasakian manifolds has vast applications in the theory of relativity.

In an n-dimensional differentiable manifold M, (ϕ, ξ, η) is said to be an almost paracontact structure if it admits a $(1,1)$ tensor field ϕ , a timelike contravariant vector field ξ and a 1-form η which satisfy the relations:

$$\eta(\xi) = -1, \tag{1.1}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{1.2}$$

for any vector field X on M. In an n-dimensional almost paracontact manifold with structure (ϕ, ξ, η), the following conditions hold:

$$\phi\xi = 0, \tag{1.3}$$

$$\eta \circ \phi = 0, \tag{1.4}$$

$$rank\ \phi = n - 1. \tag{1.5}$$

Let M^n be differentiable manifold with an almost paracontact structure (ϕ, ξ, η). If there exists a Lorentzian metric which makes ξ a timelike unit vector field, then there exists a

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Lorentzian metric g satisfying

$$g(X,\xi) = \eta(X), \tag{1.6}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{1.7}$$

$$(\nabla_X \phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{1.8}$$

for all vector fields X, Y on $\tilde{M}$ [2].

If a differentiable manifold M admits the structure (ϕ, ξ, η, g) such that g is an associated Lorentzian metric of the almost paracontact structure (ϕ, ξ, η, g) then we say that M^n has a Lorentzian almost paracontact structure (ϕ, ξ, η, g) and M^n is said to be Lorentzian almost paracontact manifold (LAP) with structure (ϕ, ξ, η, g) .

In a LAP-manifold with structure (ϕ, ξ, η, g) if we put

$$\Omega(X,Y) = g(\phi X, Y), \tag{1.9}$$

then the tensor field Ω is a symmetric $(0,2)$ tensor field [**?**], that is

$$\Omega(X,Y) = \Omega(Y,X), \tag{1.10}$$

for all vector fields X, Y on M^n. A LAP-manifold with structure (ϕ, ξ, η, g) is said to be Lorentzian paracontact manifold if it satisfies

$$\Omega(X,Y) = \frac{1}{2}\{(\nabla_X \eta)Y + (\nabla_Y \eta)X\} \tag{1.11}$$

and (ϕ, ξ, η, g) is said to be Lorentzian paracontact structure. Here ∇ denotes the operator of covariant differentiation w.r.t the Lorentzian metric g.

In a LP-Sasakian manifold we have the following results from [9]:

$$\nabla_X \xi = \phi X, \tag{1.12}$$

$$(\nabla_X \eta)Y = \Omega(X,Y) = g(\phi X, Y), \tag{1.13}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{1.14}$$

$$\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{1.15}$$

$$R(\xi, X)\xi = \eta(X)\xi - \eta(\xi)X = X + \eta(X)\xi, \tag{1.16}$$

$$S(X,\xi) = (n-1)\eta(X), \tag{1.17}$$

$$S(\xi,\xi) = -(n-1), \tag{1.18}$$

$$Q\xi = -(n-1), \tag{1.19}$$

where R is the curvature tensor of manifold of type $(1,3)$, S is Ricci tensor of type $(0,2)$ and Q being the Ricci operator. An example of a five-dimensional Lorentzian para-Sasakian manifold has been given by Matsumoto, Mihai and Rosaca in [5].

§2. D-homothetic Deformations of LP-Sasakian Manifolds

Let $M(\phi,\xi,\eta,g)$ be an Lorentzian almost paracontact structure. By D-homothetic deformation [8], we mean a change of structure tensors of the form

$$\overline{\eta}=a\eta,\ \overline{\xi}=\frac{1}{a}\xi,\ \overline{\phi}=\phi,\ \overline{g}=ag+a(a-1)\eta\otimes\eta,$$

where a is a positive constant.

Theorem 2.1 *Under D-homothetic deformation $M(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ is also an LP-Sasakian manifold $M(\phi,\xi,\eta,g)$.*

Proof Calculation shows that

$$\begin{aligned}
\overline{\eta}(\overline{\xi}) &= \overline{\eta}(\frac{1}{a}\xi)=a\eta(\frac{1}{a}\xi)=\eta(\xi)=-1,\\
\overline{\phi}^2(X) &= \phi^2(X)=X+\eta(X)\xi,\\
\overline{\phi}\circ\overline{\xi} &= \overline{\phi}(\frac{1}{a}\xi)=\phi(\frac{1}{a}\xi)=\frac{1}{a}\phi\xi=0,\\
\overline{\eta}\circ\overline{\phi} &= \overline{\eta}(\phi(X))=a\eta(\phi(X))=0,\\
rank\ \overline{\phi} &= rank\ \phi=n-1,\\
\overline{\eta}(X) &= a\eta(X)=ag(X,\xi),\\
\overline{g}(\overline{\phi}X,\overline{\phi}Y) &= \overline{g}(\phi X,\phi Y)=(ag+a(a-1)\eta\otimes\eta)(\phi X,\phi Y)=ag(\phi X,\phi Y),\\
(\nabla_X\overline{\phi})Y=(\nabla_X\phi)Y &= g(X,Y)\xi+\epsilon\eta(Y)X+2\eta(X)\eta(Y)\xi.
\end{aligned}$$

□

Theorem 2.2 *Under D-homothetic deformation of a LP Sasakian manifold the following relation holds*

$$(L_{\overline{\xi}}\overline{g})(X,Y)=a(L_\xi g)(X,Y),$$

where L_ξ is the Lie derivative.

Proof For an LP-Sasakian manifold we know $(L_\mu g)(X,Y)=2g(\phi X,Y)$ since $g(\phi X,Y)=g(X,\phi Y)$. Under D-homothetic deformation

$$\begin{aligned}
(L_{\overline{\xi}}\overline{g})(X,Y) &= 2\overline{g}(\overline{\phi}X,Y)\\
&= a(L_\xi g)(X,Y)+2(a^2-a)\eta(\phi X)\eta(Y)\\
&= a(L_\xi g)(X,Y).
\end{aligned}$$

□

§3. D-homothetic Deformations of Curvature Tensors on LP-Sasakian Manifolds

In this section we consider conformally flat LP-Sasakian manifold $M^n(\phi,\xi,\eta,g)$ $(n>3)$. The

Weyl conformal curvature tensor C is given by

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &\quad - g(X,Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{3.1}$$

For conformally flat manifold we have $C(X,Y)Z = 0$. So from (3.1) we have

$$\begin{aligned} R(X,Y)Z &= \frac{1}{n-2}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} \\ &\quad - \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{3.2}$$

Putting $Z = \xi$ in (3.2), we obtain from (1.14)

$$\begin{aligned} \eta(Y)X - \eta(X)Y &= \frac{1}{n-2}\{S(Y,\xi)X - S(X,\xi)Y + S(Y,\xi)QX - g(X,\xi)QY\} \\ &\quad - \frac{r}{(n-1)(n-2)}\{g(Y,\xi)X - g(X,\xi)Y\}. \end{aligned} \tag{3.3}$$

Putting $Y = \xi$ in (3.3) we calculate

$$\begin{aligned} \eta(\xi)X - \eta(X)\xi &= \frac{1}{n-2}\{S(\xi,\xi)X - S(X,\xi)\xi + S(\xi,\xi)QX - g(X,\xi)Q\xi\} \\ &\quad - \frac{r}{(n-1)(n-2)}\{g(\xi,\xi)X - g(X,\xi)\xi\}. \end{aligned} \tag{3.4}$$

After some steps of calculations we obtain

$$QX = (-1 + \frac{r}{n-1})X + (-1 + \frac{r}{n-1})\eta(X)\xi - (n-1)\eta(X). \tag{3.5}$$

Taking inner product with Y, above equation can be written as

$$S(X,Y) = (1 + \frac{r}{n-1})g(X,Y) + (-1 + \frac{r}{n-1})\eta(X)g(Y,\xi) - (n-1)\eta(X). \tag{3.6}$$

In view of (3.5), (3.6) equation (3.2) takes the form

$$\begin{aligned} R(X,Y)Z &= [g(Y,Z)X - g(X,Z)Y]\left[\left(-1 + \frac{r}{n-1}\right)\frac{1}{n-2} \right. \\ &\quad \left. + \frac{1}{n-2}\left(1 + \frac{r}{n-1}\right) - \frac{r}{(n-1)(n-2)}\right] \\ &\quad + g(Y,Z)\eta(X)\left[(\frac{r}{n-1} - 1)\frac{1}{n-2}\xi - (n-1)\right] + g(X,Z)\eta(Y) \\ &\quad \times \left[(\frac{r}{n-1} - 1)\frac{1}{n-2}\xi - (n-1)\right] + X\eta(Y)\left[(\frac{r}{n-1} - 1)\frac{1}{n-2}\eta(Z) - \frac{n-1}{n-2}\right] \\ &\quad + Y\eta(X)\left[(\frac{r}{n-1} - 1)\frac{1}{n-2}\eta(Z) - \frac{n-1}{n-2}\right]. \end{aligned} \tag{3.7}$$

For a conformally flat LP-Sasakian manifold, $R(X,Y)Z$ is given by the equation (3.7). Again in a LP-Sasakian manifold the following relation holds [9]

$$\begin{aligned}R(X,Y)\phi Z &= \phi(R(X,Y)Z)+2\{\eta(X)Y-\eta(Y)X\}\eta(Z)+2\{g(Y,Z)\eta(X)\\&-g(X,Z)\eta(Y)\}\xi-g(\phi X,Z)\phi Y+g(\phi Y,Z)\phi X-g(Y,Z)X\\&+g(X,Z)Y. \qquad (3.8)\end{aligned}$$

Again, on using equations (1.15), (1.18) and (1.4) in (3.8) we calculate

$$g(\phi R(\phi X,\phi Y)Z,\phi W)=g(R(Z,W)\phi X,\phi Y).$$

Using (3.8) and then (1.7), (1.15) in the above equation we obtain

$$\begin{aligned}g(\phi R(\phi X,\phi Y)Z,\phi W) &= g(R(X,Y)Z,W)-g(W,X)\eta(Z)\eta(Y)+g(Z,X)\eta(W)\eta(Y)\\&+2\eta(Z)\eta(X)g(W,\phi Y)-2\eta(W)\eta(X)g(Z,\phi Y)-g(\phi Z,X)g(\phi W,\phi Y)\\&+g(\phi W,X)g(\phi Z,\phi Y)-g(W,X)g(Z,\phi Y)+g(Z,X)g(W,\phi Y). \qquad (3.9)\end{aligned}$$

Replacing X, Y by ϕX and ϕY respectively in (3.8) and taking inner product with ϕW we obtain on using (1.4) and (3.9) we get

$$\begin{aligned}g(R(\phi X,\phi Y)\phi Z,\phi W) &= g(R(X,Y)Z,W)-g(W,X)\eta(Z)\eta(Y)+g(Z,X)\eta(W)\eta(Y)\\&+3g(Y,\phi W)\eta(Z)\eta(X)-3g(Z,\phi Y)\eta(W)\eta(X)+2g(\phi W,X)g(Z,Y)\\&+2g(\phi W,X)\eta(Z)\eta(Y)-2g(W,X)g(Z,\phi Y). \qquad (3.10)\end{aligned}$$

Now we shall recall the definition of ϕ-section. A plane section in the tangent space $T_p(M)$ is called a ϕ-section if there exists a unit vector X in T_pM orthogonal to ξ such that $\{X,\phi X\}$ is an orthonormal basis of the plane section. Then the sectional curvature

$$K(X,\phi X)=g(R(X,\phi X)X,\phi X) \qquad (3.11)$$

is called a ϕ-sectional curvature. A contact metric manifold $M(\phi,\xi,\eta,g)$ is said to be of constant ϕ-sectional curvature if at any point $P\in M$, the sectional curvature $K(X,\phi X)$ is independent of the choice of non-zero $X\in D_p$, where D denotes the contact distributions of the contact metric manifold defined by $\eta=0$. The definition is valid for Lorentzian manifolds also [10].

We give the following theorem.

Theorem 3.1 *In a LP-Sasakian manifold $M(\phi,\xi,\eta,g)$ the relation $(Q\phi-\phi Q)X=4n\phi X$ holds for any vector field X on M.*

Proof Let $\{X_i,\phi X_i,\xi\}$ $(i=1,2,\cdots,m)$ be a local $\phi-$basis at any point of the manifold. Now putting $Y=Z=X_i$ in (3.10) and taking summation over i, we obtain by virtue of $\eta(X_i)=0$,

$$\Sigma\phi R(\phi X,\phi X_i)\phi X_i=\Sigma R(X,X_i)X_i+2\phi Xg(X_i,X_i). \qquad (3.12)$$

Again setting $Y=Z=\phi X_i$ in (3.10) we have

$$\Sigma\phi R(\phi X,\phi^2X_i)\phi^2X_i=\Sigma R(X,\phi X_i)\phi X_i+2\phi Xg(X_i,X_i). \qquad (3.13)$$

Adding (3.12) and (3.13) and using the definition of Ricci operator, we calculate

$$\phi(Q(\phi X) - R(\phi X, \xi)\xi) = QX - R(X, \xi)\xi + 4n\phi X. \tag{3.14}$$

We can write from (1.16)

$$R(\phi X, \xi)\xi = \phi X. \tag{3.15}$$

Using (3.13) and (3.14)

$$\phi(Q(\phi X)) = QX + 4n\phi X. \tag{3.16}$$

Operating ϕ on both sides and using (1.17)

$$Q(\phi X) - \phi(QX) = 4n\phi X. \tag{3.17}$$

By virtue of (3.17) theorem (3.1) is proved. □

For the next proof we consider the symbol W^i_{jk} where W^i_{jk} denotes the difference $\overline{\Gamma}^i_{jk} - \Gamma^i_{jk}$ of Christoffel symbols in an LP-Sasakian manifold [8]. In global notation we can write

$$W(Y, Z) = (1 - a)[\eta(Z)\phi Y + \eta(Y)\phi Z] + \frac{1}{2}(1 - \frac{1}{a})[(\nabla_Y \eta)Z + (\nabla_Z \eta)Y]\xi, \tag{3.18}$$

for all $Y, Z \in \chi(M)$. We state our next theorem.

Theorem 3.2 *Under a D-homothetic deformation, the operator $Q\phi - \phi Q$ of a LP-Sasakian manifold $M(\phi, \xi, \eta, g)$ is conformal.*

Proof If R and $\overline{R}$ denote the curvature tensors of the LP-Sasakian manifold $M(\phi, \xi, \eta, g)$ and $M(\overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})$ respectively then we know from [8]

$$\begin{aligned} \overline{R}(X, Y)Z &= R(X, Y)Z + (\nabla_X W)(Z, Y) - (\nabla_Y W)(Z, X) \\ &\quad + W(W(Z, Y), X) - W(W(Z, X), Y). \end{aligned} \tag{3.19}$$

Using (1.13) in (3.18) we calculate

$$W(Y, Z) = (1 - a)[\eta(Z)\phi Y + \eta(Y)\phi Z] + (1 - \frac{1}{a})g(\phi Y, Z)\xi. \tag{3.20}$$

Taking covariant differentiation w.r.t. X and after using (1.8), (3.2), we obtain,

$$\begin{aligned} (\nabla_X W)(Y, Z) &= (1 - a)[g(\phi X, Z)\phi Y + g(X, Y)\eta(Z)\xi + 2\eta(Z)\eta(Y)X \\ &\quad + 4\eta(X)\eta(Y)\eta(Z)\xi + g(\phi X, Y)\phi Z + g(X, Z)\eta(Y)\xi] \\ &\quad + (1 - \frac{1}{a})g(\phi Y, Z)\phi X. \end{aligned} \tag{3.21}$$

Using (3.21) in (3.19) we obtain

$$\begin{aligned} \overline{R}(X, Y)Z &= R(X, Y)Z + (1 - a)\eta(Y)g(X, Z)\xi \\ &\quad + 2(1 - a)\eta(Z)\eta(Y)X + (1 - a)g(\phi X, Z)\phi Y + (1 - \frac{1}{a})g(\phi Z, Y)\phi X \\ &\quad - (1 - a)g(Y, Z)\eta(X)\xi \\ &\quad - 2(1 - a)\eta(X)\eta(Z)Y - (1 - a)g(\phi Y, Z)\phi X - (1 - \frac{1}{a})g(\phi Z, X)\phi Y \end{aligned}$$

$$+(1-a)\eta(Y)[(1-\frac{1}{a})g(\phi^2 Z,X)\xi]+(1-a)\eta(Z)[(1-a)\eta(X)\phi^2 Y$$
$$+(1-\frac{1}{a})g(\phi^2 Y,X)\xi]+(1-\frac{1}{a})g(\phi Z,X)[-(1-a)\phi X]$$
$$-(1-a)\eta(X)[(1-\frac{1}{a})g(\phi^2 Z,Y)\xi]-(1-a)\eta(Z)[(1-a)\eta(Y)\phi^2 X$$
$$+(1-\frac{1}{a})g(\phi^2 X,Y)\xi]-(1-\frac{1}{a})g(\phi Z,X)[-(1-a)\phi Y]. \quad (3.22)$$

From (3.22) we get

$$a\overline{S}(Y,Z)=S(Y,Z)+\frac{(1-a)^2}{a}. \quad (3.23)$$

Using the properties of Ricci operator

$$a\overline{Q}Y=QY+\frac{(1-a)^2}{a}.$$

Operating $\phi=\overline{\phi}$ on both sides from left hand side

$$a\overline{\phi}\,\overline{Q}Y=\phi QY+\frac{(1-a)^2}{a}.$$

Operating $\phi=\overline{\phi}$ on both sides from right hand side

$$a\overline{Q}\,\overline{\phi}Y=Q\phi Y+\frac{(1-a)^2}{a}.$$

Subtracting the above two equations we obtain

$$a(\overline{\phi}\,\overline{Q}-\overline{Q}\,\overline{\phi})=(\phi Q-Q\phi). \quad (3.24)$$

The equation (3.24) proves our theorem. □

We can also prove the following theorems as a consequence of D-homothetic deformation.

Theorem 3.3 *Under D-homothetic deformation, an $\eta-$Einstein LP-Sasakian manifold remains invariant.*

Proof In an η-Einstein LP-Sasakian manifold [9]

$$S(X,Y)=[\frac{r}{n-1}-1]g(X,Y)+[\frac{r}{n-1}-n]\eta(X)\eta(Y).$$

Under D-homothetic deformation we get

$$\overline{S}(X,Y)=[a(\frac{r}{n-1}-1)]g(X,Y)+[a(a-1)(\frac{r}{n-1}-1)+a^2(\frac{r}{n-1}-n)]\eta(X)\eta(Y).$$

Hence the result is proved. □

Theorem 3.4 *Under D-homothetic deformation, the ϕ-sectional curvature of a LP-Sasakian manifold is conformal.*

Proof Putting $Y=\phi X, Z=X$ in (3.12) and taking inner product with ϕX, we obtain on using (1.4) and the orthogonality property we get

$$ag(\overline{R}(X,\phi X)X,\phi X)=g(R(X,\phi X)X,\phi X)+(a-\frac{1}{a}) \quad (3.25)$$

$$a\overline{K}(X,\phi X) - K(X,\phi X) = (a - \frac{1}{a}). \qquad \square$$

Theorem 3.5 *There exists LP-Sasakian manifold with non-zero and non-constant ϕ-sectional curvature.*

Proof If the LP-Sasakian manifold satisfies $R(X,Y)\xi = 0$, then it can be proved easily that $K(X,\phi X) = 0$ and therefore from (3.25) we can conclude that $\overline{K}(X,\phi X) \neq 0$ for $a \neq 1$ where X is a unit vector field orthogonal to ξ. Hence the result is proved. $\square$

§4. An Example of a LP-Sasakian Manifold

In this section we shall prove the equality (3.25) by taking an example of LP-Sasakian manifold [1]. Let us consider a 5-dimensional manifold $\tilde{M} = \{(x,y,z,u,v) \in R^5 : ((x,y,z,u,v) \neq (0,0,0,0,0)\}$ where (x,y,z,u,v) are the standard coordinate in R^5. The vector fields

$$e_1 = -2\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}, \quad e_4 = -2\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial z}, \quad e_5 = \frac{\partial}{\partial v}$$

are linearly independent at each point of M. Let g be the Lorentzian metric defined by

$$\begin{aligned} g(e_i,e_j) &= 1, \text{ for } i = j \neq 3, \\ g(e_i,e_j) &= 0, \text{ for } i \neq j, \\ g(e_3,e_3) &= -1. \end{aligned}$$

Here i and j runs from 1 to 5. Let η be the 1-form defined by $\eta(Z) = g(Z,e_3)$, for any vector field Z tangent to $\tilde{M}$. Let ϕ be the $(1,1)$ tensor field defined by

$$\phi e_1 = e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0, \quad \phi e_4 = e_5, \quad \phi e_5 = e_4.$$

Then using the linearity of ϕ and g we have

$$\eta(e_3) = -1, \quad \phi^2 Z = Z + \eta(Z)e_3,$$

for any vector fields Z, W tangent to $\tilde{M}$. Thus for $e_3 = \xi$, $\tilde{M}(\phi,\xi,\eta,g)$ forms a LP-Sasakian manifold.

Let ∇ be the Levi-Civita connection on $\tilde{M}$ with respect to the metric g. Then the followings can be obtained

$$[e_1,e_2] = -2e_3, \quad [e_1,e_3] = 0, \quad [e_2,e_3] = 0.$$

On taking $e_3 = \xi$ and using Koszul's formula for the metric g, we calculate

$$\begin{aligned} &\nabla_{e_1}e_3 = e_2, \quad \nabla_{e_1}e_2 = -e_3, \quad \nabla_{e_1}e_1 = 0, \\ &\nabla_{e_2}e_3 = e_1, \quad \nabla_{e_2}e_2 = 0, \quad \nabla_{e_2}e_1 = e_3, \\ &\nabla_{e_3}e_3 = 0, \quad \nabla_{e_3}e_2 = e_1, \quad \nabla_{e_3}e_1 = e_2. \end{aligned}$$

Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

$$R(e_1,e_2)e_2 = 3e_1, \quad R(e_1,e_2)e_1 = 3e_2, \quad R(e_2,e_3)e_3 = -e_2,$$
$$R(e_1,e_3)e_2 = 0, \quad R(e_1,e_3)e_1 = -e_3, \quad R(e_2,e_3)e_2 = e_3,$$
$$R(e_1,e_2)e_3 = 0.$$

In equation (3.22) we put $X = e_1, Y = \phi e_1, Z = e_1$. Taking inner product with ϕe_1 we obtain

$$a\overline{K}(e_1,\phi e_1) - K(e_1,\phi e_1) = a - \frac{1}{a}.$$

Hence, by this example Theorem 3.4 is verified.

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