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Computing Zagreb Polynomials of Generalized xyz-Point-Line Transformation Graphs $T^{xyz}(G)$ With $z = -$

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Abstract: In this paper, we discuss relations among Zagreb polynomials of a graph G and generalized xyz-point-line transformation graphs $T^{xyz}(G)$ when $z = -$. Zagreb polynomials of xyz-point-line transformation graphs are obtained in terms of Zagreb polynomials of the graph G.

Key Words: Zagreb indices, Zagreb polynomials, xyz-point-line transformation graphs.

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§1. Introduction

By a Graph $G = (V, E)$ we mean a nontrivial, finite, simple, undirected graph with vertex set V and an edge set E of order n and size m. The *degree* $d_G(v)$ of a vertex v in G is the number of edges incident to it in G. Let $\overline{G}$, $L(G)$ and $S(G)$ of a graph G are complement, line graph and subdivision graph of a graph G respectively. The *partial complement of subdivision graph* $\overline{S}(G)$ of a graph G whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if one is a vertex of G and the other is an edge of G non incident with it.

In this paper, we denote $u \sim v$ ($u \nsim v$) for vertices u and v are adjacent (resp., nonadjacent), $e \sim f$ ($e \nsim f$) for the adjacent (resp., nonadjacent) edges e and f and $u \sim e$ ($u \nsim e$) for the vertex u and an edge e are incident (resp., nonincident) in G. Other undefined notations and terminologies can be found in [17] or [19].

Polynomials are one of the graph invariants which does not depend on the labeling or pictorial representation of the graph. A topological index is also one such graph invariant. The topological indices have their applications in several branches of science and technology.

The first and second Zagreb indices are amongst the oldest and best known topological indices defined in 1972 by Gutman [15] as follows:

$$M_1(G) = \sum_{v \in V(G)} d_G(v)^2 \quad \text{and} \quad M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v),$$

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respectively. These are widely studied degree based topological indices due to their applications in chemistry, for details refer to [10,11,14,16,23]. The first Zagreb index [21] can also be expressed as

$$M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)].$$

Ashrafi et al. [1] defined respectively the first and second Zagreb coindices as

$$\overline{M_1}(G) = \sum_{uv \notin E(G)} [d_G(u) + d_G(v)] \text{ and } \overline{M_2}(G) = \sum_{uv \notin E(G)} [d_G(u) d_G(v)].$$

In 2004, Milićević et al. [20] reformulated the Zagreb indices in terms of edge-degrees instead of vertex-degrees. The first and second reformulated Zagreb indices are defined respectively by

$$EM_1(G) = \sum_{e \in E(G)} d_G(e)^2 \text{ and } EM_2(G) = \sum_{e \sim f} [d_G(e) d_G(f)].$$

In [18], Hosamani and Trinajstić defined the first and second reformulated Zagreb coindices respectively as

$$\overline{EM_1}(G) = \sum_{e \nsim f} [d_G(e) + d_G(f)] \text{ and } \overline{EM_2}(G) = \sum_{e \nsim f} [d_G(e) d_G(f)].$$

Considering the Zagreb indices, Fath-Tabar [13] defined first and the second *Zagreb polynomials* as

$$M_1(G, x) = \sum_{v_i, v_j \in E(G)} x^{d_G(v_i) + d_G(v_j)} \text{ and } M_2(G, x) = \sum_{v_i, v_j \in E(G)} x^{d_G(v_i) \cdot d_G(v_j)}$$

respectively, where x is a variable. In addition, Shuxian [22] defined two polynomials related to the first Zagreb index in the form

$$M_1^*(G, x) = \sum_{v_i \in V(G)} d_G(v_i) x^{d_G(v_i)} \text{ and } M_0(G, x) = \sum_{v_i \in V(G)} x^{d_G(v_i)}.$$

A. R. Bindusree et al. defined the following polynomials in [9],

$$M_4(G, x) = \sum_{v_i, v_j \in E(G)} x^{d_G(v_i)((d_G(v_i) + d_G(v_j))}, \; M_5(G, x) = \sum_{v_i, v_j \in E(G)} x^{d_G(v_j)((d_G(v_i) + d_G(v_j))},$$

$$M_{a,b}(G, x) = \sum_{v_i, v_j \in E(G)} x^{a d_G(v_i) + b d_G(v_j)}, \; M'_{a,b}(G, x) = \sum_{v_i, v_j \in E(G)} x^{(d_G(v_i) + a)(d_G(v_j) + b)}.$$

§2. Generalized xyz-Point-Line Transformation Graph $T^{xyz}(G)$

For a graph $G = (V, E)$, let G^0 be the graph with $V(G^0) = V(G)$ and with no edges, G^1 the complete graph with $V(G^1) = V(G)$, $G^+ = G$, and $G^- = \overline{G}$. Let $\mathcal{G}$ denotes the set of simple graphs. The graph operations depending on $x, y, z \in \{0, 1, +, -\}$ induce functions $T^{xyz} : \mathcal{G} \to \mathcal{G}$. These operations were introduced by Deng et al. in [12] and named them as xyz-transformations of G, denoted by $T^{xyz}(G) = G^{xyz}$. In [2], Wu Bayoindureng et al. introduced the total transformation graphs and studied their basic properties. Motivated by this, Basavanagoud [3] studied the basic properties of the xyz-transformation graphs by changing them as *xyz-point-line transformation graphs* and denoted as $T^{xyz}(G)$ to avoid confusion between various transformations.

Definition 2.1([12]) *Given a graph G with vertex set $V(G)$ and edge set $E(G)$ and three variables $x, y, z \in \{0, 1, +, -\}$, the xyz-point-line transformation graph $T^{xyz}(G)$ of G is the graph with vertex set $V(T^{xyz}(G)) = V(G) \cup E(G)$ and the edge set $E(T^{xyz}(G)) = E((G)^x) \cup E((L(G))^y) \cup E(W)$ where $W = S(G)$ if $z = +$, $W = \overline{S}(G)$ if $z = -$, W is the graph with $V(W) = V(G) \cup E(G)$ and with no edges if $z = 0$ and W is the complete bipartite graph with parts $V(G)$ and $E(G)$ if $z = 1$.*

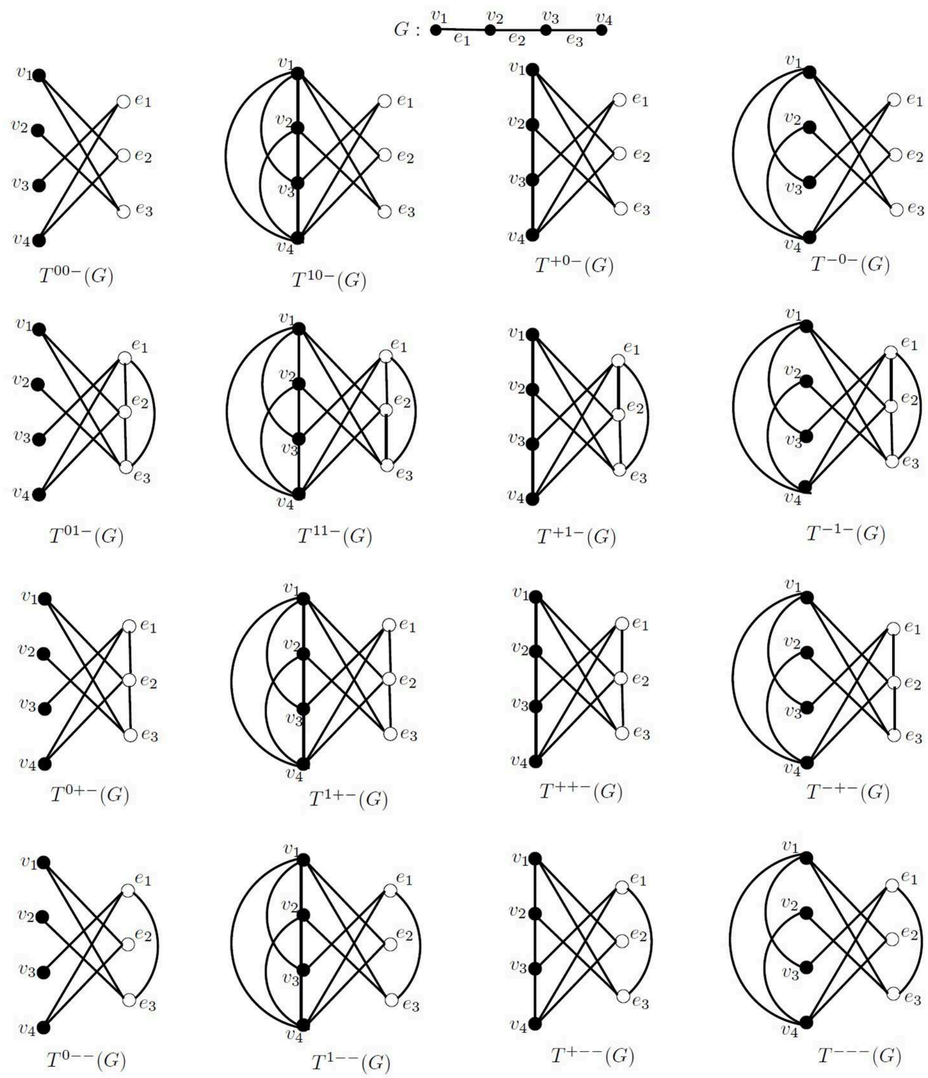

Figure 1. P_4 and its generalized xyz-polint-line transformation graphs $T^{xy-}(P_4)$.

Since there are 64 distinct 3 - permutations of $\{0, 1, +, -\}$. Thus 64 kinds of generalized xyz-point-line transformation graphs are obtained. There are 16 different graphs for each case when $z = 0$, $z = 1$, $z = +$, $z = -$. In this paper, we consider the xyz-point-line transformation graph $T^{xyz}(G)$ with $z = -$. The self-explanatory examples of the path P_4 and its *xyz-point-line transformation graphs* $T^{xy-}(P_4)$ are depicted in Figure 1. For more on generalized transformation graphs refer to [2]-[8].

The following Observations are useful in proving the theorems.

Observation 2.1([4]) Let G be a graph of order n and size m. Let v be a vertex of G and $Y = \{0, 1, +, -\}$. Then

$$d_{T^{xy-}}(v) = \begin{cases} m - d_G(v) & if\ x = 0, y \in Y, \\ n + m - 1 - d_G(v) & if\ x = 1, y \in Y, \\ m & if\ x = +, y \in Y, \\ n + m - 1 - 2d_G(v) & if\ x = -, y \in Y. \end{cases}$$

Observation 2.2([4]) Let G be a graph of order n and size m. Let e be an edge of G and $Y = \{0, 1, +, -\}$. Then

$$d_{T^{xy-}}(e) = \begin{cases} n - 2 & if\ y = 0, x \in Y, \\ n + m - 3 & if\ y = 1, x \in Y, \\ n - 2 + d_G(e) & if\ y = +, x \in Y, \\ n + m - 3 - d_G(e) & if\ y = -, x \in Y. \end{cases}$$

§3. Results on the Zagreb Polynomials of $T^{xy-}(G)$

In this section, we obtain the Zagreb polynomials of the xyz-point-line transformation graph $T^{xyz}(G)$ with $z = -$. In this process, to cover the edges in the complements $\overline{G}, \overline{S}(G)$ and $\overline{L(G)}$ we need the degrees of nonadjacent vertices (or edges) in the graph. Degrees of these nonadjacent vertices (or edges) gives Zagreb coindices. To overcome from this problem Basavanagoud and Jakkannavar [7] defined the first, second and third Zagreb co-polynomials of a graph G by using the concept of Zagreb coindices as

$$\overline{M_1}(G, x) = \sum_{v_i, v_j \notin E(G)} x^{d_G(v_i) + d_G(v_j)}, \quad \overline{M_2}(G, x) = \sum_{v_i, v_j \notin E(G)} x^{d_G(v_i) \cdot d_G(v_j)}$$

and

$$\overline{M_3}(G, x) = \sum_{v_i, v_j \notin E(G)} x^{|d_G(v_i) - d_G(v_j)|}$$

respectively, where x is a variable.

In addition, in [7] they defined

$$\overline{M_4}(G,x) = \sum_{v_i,v_j \notin E(G)} x^{d_G(v_i)(d_G(v_i)+d_G(v_j))}, \quad \overline{M_5}(G,x) = \sum_{v_i,v_j \notin E(G)} x^{d_G(v_j)(d_G(v_i)+d_G(v_j))},$$
$$\overline{M_{a,b}}(G,x) = \sum_{v_i,v_j \notin E(G)} x^{ad_G(v_i)+bd_G(v_j)}, \quad \overline{M'}_{a,b}(G,x) = \sum_{v_i,v_j \notin E(G)} x^{(d_G(v_i)+a)(d_G(v_j)+b)}.$$

The following theorems give results on Zagreb polynomials of the generalized xyz-point-line transformation graphs $T^{xy-}(G)$.

Theorem 3.1 *Let G be a graph of order n and size m. Then Zagreb polynomials of $T^{00-}(G)$ are*

$$\begin{aligned}
M_1(T^{00-}(G),x) &= mx^{m+n-2}M_0(G,x^{-1}) - x^{m+n-2}M_1^*(G,x^{-1})\\
M_2(T^{00-}(G),x) &= mx^{m(n-2)}M_0(G,x^{-(n-2)}) - x^{m(n-2)}M_1^*(G,x^{-(n-2)})\\
M_3(T^{00-}(G),x) &= mx^{|n-m-2|}M_0(G,x) - x^{|n-m-2|}M_1^*(G,x).
\end{aligned}$$

Proof From Observations (2.1) and (2.2) we have

$$d_{T^{00-}(G)}(v) = \begin{cases} m - d_G(v) & if\ v \in V(G),\\ n-2 & if\ v \in E(G).\end{cases}$$

By using definition of $M_1(G,x)$, we have

$$\begin{aligned}
M_1(T^{00-}(G),x) &= \sum_{uv \in E(T^{00-}(G))} x^{d_{T^{00-}(G)}(u)+d_{T^{00-}(G)}(v)}\\
&= \sum_{u \nsim v} x^{d_{T^{00-}(G)}(u)+d_{T^{00-}(G)}(v)} = \sum_{u \nsim v} x^{m-d_G(v)+n-2}\\
&= x^{2-m-n} \sum_{v \in V(G)} (m-d_G(v))x^{-d_G(u)}.\\
&= mx^{m+n-2}M_0(G,x^{-1}) - x^{m+n-2}M_1^*(G,x^{-1}).
\end{aligned}$$

By using definition of $M_2(G,x)$, we have

$$\begin{aligned}
M_2(T^{00-}(G),x) &= \sum_{uv \in E(T^{00-}(G))} x^{d_{T^{00-}(G)}(u)d_{T^{00-}(G)}(v)}\\
&= \sum_{u \nsim v} x^{d_{T^{00-}(G)}(u)d_{T^{00-}(G)}(v)}\\
&= \sum_{u \nsim v} x^{(m-d_G(u))(n-2)}\\
&= x^{-m(n-2)} \sum_{v \in V(G)} (m-d_G)x^{-m(n-2)}d_G(v)\\
&= mx^{m(n-2)}M_0(G,x^{-(n-2)}) - x^{m(n-2)}M_1^*(G,x^{-(n-2)})
\end{aligned}$$

By using definition of $M_3(G,x)$, we have

$$\begin{aligned}
M_3(T^{00-}(G),x) &= \sum_{uv\in E(T^{00-}(G))} x^{|d_{T^{00-}(G)}(u)-d_{T^{00-}(G)}(v)|}\\
&= \sum_{u\nsim v} x^{|d_{T^{00-}(G)}(u)-d_{T^{00-}(G)}(v)|}\\
&= \sum_{u\nsim v} x^{|m-d_G(u)-n+2|}\\
&= x^{|n-m-2|}\sum_{u\in V(G)}(m-d_G(v))x^{|d_G(u)|}\\
&= mx^{|n-m-2|}M_0(G,x)-x^{|n-m-2|}M_1^*(G,x) \qquad \square
\end{aligned}$$

Theorem 3.2 *Let G be a graph of order n and size m. Then, the Zagreb polynomials of $T^{01-}(G)$ are*

$$\begin{aligned}
M_1(T^{01-}(G),x) &= \binom{m}{2}x^{2m+n-3}+mx^{2m+n-3}M_0(G,x^{-1})-x^{2m+n-3}M_1^*(G,x^{-1})\\
M_2(T^{01-}(G),x) &= \binom{m}{2}x^{(n+m-3)^2}+mx^{m(n+m-3)}M_0(G,x^{-(n+m-3)})\\
&\quad -x^{m(n+m-3)}M_1^*(G,x^{-(n+m-3)})\\
M_3(T^{01-}(G),x) &= \binom{m}{2}+mx^{n-3}M_0(G,x^{-1})-x^{n-3}M_1^*(G,x^{-1}).
\end{aligned}$$

Proof From Observations (2.1) and (2.2) we have

$$d_{T^{01-}(G)}(v)=\begin{cases} m-d_G(v) & if\ v\in V(G)\\ n+m-3 & if\ v\in E(G)\end{cases}$$

By using the definition of $M_1(G,x)$, we have

$$\begin{aligned}
M_1(T^{01-}(G),x) &= \sum_{uv\in E(T^{01-}(G))} x^{d_{T^{01-}(G)}(u)+d_{T^{01-}(G)}(v)}\\
&= \sum_{uv\in E(L(G))} x^{d_{T^{01-}(G)}(u)+d_{T^{01-}(G)}(v)}\\
&\quad +\sum_{u,v\notin E(LG)} x^{d_{T^{01-}(G)}(u)+d_{T^{01-}(G)}(v)}\\
&\quad +\sum_{u\nsim v} x^{d_{T^{01-}(G)}(u)+d_{T^{01-}(G)}(v)}\\
&= \sum_{uv\in E(L(G))} x^{2(n+m-3)}+\sum_{u,v\notin E(L(G)} x^{2(n+m-3)}+\sum_{u\nsim v} x^{m-d_G(v)+n+m-3}\\
&= \binom{m}{2}x^{2m+n-3}+mx^{2m+n-3}M_0(G,x^{-1})-x^{2m+n-3}M_1^*(G,x^{-1}).
\end{aligned}$$

Similarly, we known

$$\begin{aligned}
M_2(T^{01-}(G),x) &= \sum_{uv\in E(T^{01-}(G))} x^{d_{T^{01-}(G)}(u)d_{T^{01-}(G)}(v)} \\
&= \sum_{uv\in E(L(G))} x^{d_{T^{01-}(G)}(u)d_{T^{01-}(G)}(v)} + \sum_{u,v\notin E(LG)} x^{d_{T^{01-}(G)}(u)d_{T^{01-}(G)}(v)} \\
&\quad + \sum_{u\nsim v} x^{d_{T^{01-}(G)}(u)d_{T^{01-}(G)}(v)} \\
&= \binom{m}{2} x^{(n+m-3)^2} + m x^{m(n+m-3)} M_0(G, x^{-(n+m-3)}) \\
&\quad - x^{m(n+m-3)} M_1^*(G, x^{-(n+m-3)})
\end{aligned}$$

$$\begin{aligned}
M_3(T^{01-}(G),x) &= \sum_{uv\in E(T^{01-}(G))} x^{|d_{T^{01-}(G)}(u)-d_{T^{01-}(G)}(v)|} \\
&= \sum_{uv\in E(L(G))} x^{|d_{T^{01-}(G)}(u)-d_{T^{01-}(G)}(v)|} + \sum_{u,v\notin E(L(G)} x^{|d_{T^{01-}(G)}(u)-d_{T^{01-}(G)}(v)|} \\
&\quad + \sum_{u\nsim v} x^{|d_{T^{01-}(G)}(u)-d_{T^{01-}(G)}(v)|} \\
&= \binom{m}{2} + m x^{n-3} M_0(G, x^{-1}) - x^{n-3} M_1^*(G, x^{-1}). \qquad \square
\end{aligned}$$

Theorem 3.3 *Let G be a graph of order n and size m. Then Zagreb polynomials of $T^{0+-}(G)$ are*

$$\begin{aligned}
M_1(T^{0+-}(G),x) &= x^{2m} M_1(L(G), x^{-1}) + x^{m+n-2} \sum_{u\nsim v} x^{-d_G(u)+d_G(v)} \\
M_2(T^{0+-}(G),x) &= M'_{(n-2,n-2)}(L(G),x) + \sum_{u\nsim v} x^{(m-d_G(u))(n-2+d_G(v))} \\
M_3(T^{0+-}(G),x) &= M_3(L(G),x) - x^{m-n+2} \sum_{u\nsim v} x^{|d_G(u)+d_G(v)}
\end{aligned}$$

Proof From Observations (2.1) and (2.2) we have

$$d_{T^{0+-}(G)}(v) = \begin{cases} m - d_G(v) & if\ v \in V(G) \\ n - 2 + d_G(v) & if\ v \in E(G) \end{cases}$$

Applying the definition of $M_1(G,x)$, we have

$$\begin{aligned}
M_1(T^{0+-}(G),x) &= \sum_{uv\in E(T^{0+-}(G))} x^{d_{T^{0+-}(G)}(u)+d_{T^{0+-}(G)}(v)} \\
&= \sum_{uv\in E(L(G))} x^{d_{T^{0+-}(G)}(u)+d_{T^{0+-}(G)}(v)} + \sum_{u\nsim v} x^{d_{T^{0+-}(G)}(u)+d_{T^{0+-}(G)}(v)} \\
&= x^{2m} M_1(L(G), x^{-1}) + x^{m+n-2} \sum_{u\nsim v} x^{-d_G(u)+d_G(v)}.
\end{aligned}$$

Similarly, we know

$$\begin{aligned} M_2(T^{0+-}(G),x) &= \sum_{uv\in E(T^{0+-}(G))} x^{d_{T^{0+-}(G)}(u)d_{T^{0+-}(G)}(v)} \\ &= \sum_{uv\in E(L(G))} x^{d_{T^{0+-}(G)}(u)d_{T^{0+-}(G)}(v)} + \sum_{u\nsim v} x^{d_{T^{0+-}(G)}(u)d_{T^{0+-}(G)}(v)} \\ &= M'_{(n-2,n-2)}(L(G),x) + \sum_{u\nsim v} x^{(m-d_G(u))(n-2+d_G(v))} \end{aligned}$$

and

$$\begin{aligned} M_3(T^{0+-}(G),x) &= \sum_{uv\in E(T^{0+-}(G))} x^{|d_{T^{0+-}(G)}(u)-d_{T^{0+-}(G)}(v)|} \\ &= \sum_{uv\in E(L(G))} x^{|d_{T^{0+-}(G)}(u)-d_{T^{0+-}(G)}(v)|} + \sum_{u\nsim v} x^{|d_{T^{0+-}(G)}(u)-d_{T^{0+-}(G)}(v)|} \\ &= M_3(L(G),x) - x^{m-n+2}\sum_{u\nsim v} x^{|d_G(u)+d_G(v)|}. \end{aligned}$$ □

Theorem 3.4 *Let G be a graph of order n and size m. Then Zagreb polynomials of $T^{0--}(G)$ are*

$$\begin{aligned} M_1(T^{0--}(G),x) &= x^{2(n+m-3)}\overline{M_1}(L(G),x^{-1}) + x^{2m+n-3}\sum_{u,v\notin E(G)} x^{-(d_G(u)+d_G(v))} \\ M_2(T^{0--}(G),x) &= M'_{(-n-m+3,-n-m+3)}(L(G),x) + \sum_{u\nsim v} x^{(m-d_G(u))(m+n-3-d_G(v))} \\ M_3(T^{0--}(G),x) &= M_3(L(G),x) - x^{|n-3|}\sum_{u\nsim v} x^{|d_G(u)+d_G(v)} \end{aligned}$$

Proof From Observations (2.1) and (2.2) we have

$$d_{T^{0--}(G)}(v) = \begin{cases} m-d_G(v) & if\ v\in V(G) \\ n+m-3-d_G(v) & if\ v\in E(G) \end{cases}$$

By the definition of $M_1(G,x)$, we have

$$\begin{aligned} M_1(T^{0--}(G),x) &= \sum_{uv\in E(T^{0--}(G))} x^{d_{T^{0--}(G)}(u)+d_{T^{0+-}(G)}(v)} \\ &= \sum_{uv\notin E(L(G))} x^{d_{T^{0--}(G)}(u)+d_{T^{0--}(G)}(v)} + \sum_{u\nsim v} x^{d_{T^{0--}(G)}(u)+d_{T^{0--}(G)}(v)} \\ &= x^{2(n+m-3)}\overline{M_1}(L(G),x^{-1}) + x^{2m+n-3}\sum_{u,v\notin E(G)} x^{-(d_G(u)+d_G(v))} \end{aligned}$$

Similarly, we know

$$M_2(T^{0--}(G),x) = \sum_{uv\in E(T^{0--}(G))} x^{d_{T^{0--}(G)}(u)d_{T^{0--}(G)}(v)}$$

$$\begin{aligned}
&= \sum_{uv\notin E(L(G))} x^{d_{T^{0--}(G)}(u)d_{T^{0--}(G)}(v)} + \sum_{u\nsim v} x^{d_{T^{0--}(G)}(u)d_{T^{0--}(G)}(v)} \\
&= M'_{(-n-m+3,-n-m+3)}(L(G),x) + \sum_{u\nsim v} x^{(m-d_G(u))(m+n-3-d_G(v))}.
\end{aligned}$$

$$\begin{aligned}
M_3(T^{0--}(G),x) &= \sum_{uv\in E(T^{0--}(G))} x^{|d_{T^{0--}(G)}(u)-d_{T^{0--}(G)}(v)|} \\
&= \sum_{uv\notin E(L(G))} x^{|d_{T^{0--}(G)}(u)-d_{T^{0--}(G)}(v)|} + \sum_{u\nsim v} x^{|d_{T^{0--}(G)}(u)-d_{T^{0--}(G)}(v)|} \\
&= M_3(L(G),x) - x^{|n-3|}\sum_{u\nsim v} x^{|d_G(u)+d_G(v)|}. \qquad \square
\end{aligned}$$

Theorem 3.5 *Let G be a graph of order n and size m. Then*

$$\begin{aligned}
M_1(T^{10-}(G),x) &= x^{2(n+m-1)}M_{-1,-1}(G,x) + x^{2m}M_1(\overline{G},x) + mx^{(2n+m-3)}M_0(G,x^{-1}) \\
&\quad + M_1^*(G,x^{-1})x^{(2n+m-3)} \\
M_2(T^{10-}(G),x) &= M'_{-(n+m-1),-(n+m-1)}(G,x) + M'_{m,m}(\overline{G},x) \\
&\quad + mx^{(n+m-1)(n+2)}M_0(G,x^{-(n+2)}) - x^{(n+m-1)(n+2)}M_1^*(G,x^{-(n+2)}) \\
M_3(T^{10-}(G),x) &= M_3(G,x) + \overline{M_3}(G,x) + mx^{m+1}M_0(G,x^{-1}) - x^{m+1}M_1^*(G,x^{-1}).
\end{aligned}$$

Proof From Observations (2.1) and (2.2) we have,

$$d_{T^{10-}(G)}(v) = \begin{cases} n+m-1+d_G(v) & if\ v \in V(G) \\ n-2 & if\ v\in E(G) \end{cases}$$

By the definition of $M_1(G,x)$, we know

$$\begin{aligned}
M_1(T^{10-}(G),x) &= \sum_{uv\in E(T^{10-}(G))} x^{d_{T^{10-}(G)}(u)+d_{T^{10-}(G)}(v)} \\
&= \sum_{uv\in E(G)} x^{d_{T^{10-}(G)}(u)+d_{T^{10-}(G)}(v)} + \sum_{uv\notin E(G)} x^{d_{T^{10-}(G)}(u)+d_{T^{10-}(G)}(v)} \\
&\quad + \sum_{u\nsim v} x^{d_{T^{10-}(G)}(u)+d_{T^{10-}(G)}(v)} \\
&= \sum_{uv\in E(G)} x^{n+m-1+d_G(u)+n+m-1+d_G(v)} + \sum_{uv\notin E(G)} x^{n+m-1+d_G(u)+n+m-1+d_G(v)} \\
&\quad + \sum_{u\nsim v} x^{n+m-1-d_G(v)+n-2} \\
&= x^{2(n+m-1)}M_{-1,-1}(G,x) + x^{2m}M_1(\overline{G},x) + mx^{(2n+m-3)}M_0(G,x^{-1}) \\
&\quad + M_1^*(G,x^{-1})x^{(2n+m-3)}.
\end{aligned}$$

Similarly, by the definition of $M_2(G,x)$, we have

$$\begin{aligned}
M_2(T^{10-}(G),x) &= \sum_{uv\in E(T^{10-}(G))} x^{d_{T^{10-}(G)}(u)d_{T^{10-}(G)}(v)} \\
&= \sum_{uv\in E(G)} x^{d_{T^{10-}(G)}(u)d_{T^{10-}(G)}(v)} + \sum_{uv\notin E(G)} x^{d_{T^{10-}(G)}(u)d_{T^{10-}(G)}(v)} \\
&\quad + \sum_{u\nsim v} x^{d_{T^{10-}(G)}(u)d_{T^{10-}(G)}(v)} \\
&= M'_{-(n+m-1),-(n+m-1)}(G,x) + M'_{m,m}(\overline{G},x) \\
&\quad + mx^{(n+m-1)(n+2)}M_0(G,x^{-(n+2)}) - x^{(n+m-1)(n+2)}M_1^*(G,x^{-(n+2)}).
\end{aligned}$$

$$\begin{aligned}
M_3(T^{10-}(G),x) &= \sum_{uv\in E(T^{10-}(G))} x^{|d_{T^{10-}(G)}(u)-d_{T^{10-}(G)}(v)|} \\
&= \sum_{uv\in E(G)} x^{|d_{T^{10-}(G)}(u)-d_{T^{10-}(G)}(v)|} + \sum_{uv\notin E(G)} x^{|d_{T^{10-}(G)}(u)-d_{T^{10-}(G)}(v)|} \\
&\quad + \sum_{u\nsim v} x^{|d_{T^{10+}(G)}(u)-d_{T^{10+}(G)}(v)|} \\
&= M_3(G,x) + \overline{M_3}(G,x) + mx^{m+1}M_0(G,x^{-1}) - x^{m+1}M_1^*(G,x^{-1}). \qquad \square
\end{aligned}$$

The proof of following theorems are analogous to that of Theorems 3.1-3.5.

Theorem 3.6 *Let G be a graph of order n and size m. Then*

$$\begin{aligned}
M_1(T^{11-}(G),x) &= x^{2(n+m-1)}M_1(G,x^{-1}) + x^{2m}M_1(\overline{G},x) \\
&\quad + \binom{m}{2}x^{2(n+m-3)} + x^{2(n+m-3)}M_1^*(G,x^{-1}) \\
M_2(T^{11-}(G),x) &= M'_{-(n+m-1),-(n+m-1)}(G,x) + M'_{m,m}(\overline{G},x) + \binom{m}{2}x^{(n+m-3)^2} \\
&\quad + x^{(n+m-1)(n+m-3)}M_1^*(G,x^{(n+m-3)}) \\
M_3(T^{11-}(G),x) &= M_1(G,x) + \overline{M_1}(G,x) + \binom{m}{2} + x^2M_1^*(G,x).
\end{aligned}$$

Theorem 3.7 *Let G be a graph of order n and size m. Then*

$$\begin{aligned}
M_1(T^{1+-}(G),x) &= x^{2(n+m-1)}M_{-1,-1}(G,x) + x^{2m}M_1(\overline{G},x) \\
&\quad + x^{2(n-2)}M_1(L(G),x) + x^{2(n+m-3)}\sum_{u\nsim v} x^{-d_G(u)+d_G(v)} \\
M_2(T^{1+-}(G),x) &= M'_{-(n+m-1),(n-2)}(G,x^{-1}) + \overline{M'}_{-(n+m-1),(n-2)}(G,x^{-1}) \\
&\quad + M'_{n-2,n-2}(L(G),x) + \sum_{u\nsim v} x^{(n+m-1-d_G(u))(n-2+d_G(v))} \\
M_3(T^{1+-}(G),x) &= x^{m+1}M_3(G,x) + x^{2m}\overline{M_1}(G,x) + M_3(L(G),x) + x^{m+1}\sum_{u\nsim v} x^{|d_G(u)+d_G(v)|}.
\end{aligned}$$

Theorem 3.8 *Let G be a graph of order n and size m. Then*

$$\begin{aligned}
M_1(T^{1--}(G),x) &= x^{2(n+m-1)}M_{-1,-1}(G,x) + x^{2m}M_1(\overline{G},x) + x^{2(n+m-3)}\overline{M}_1(L(G),x) \\
&\quad + x^{2(n+m-2)}\sum_{u\nsim v} x^{-d_G(u)-d_G(v)} \\
M_2(T^{1--}(G),x) &= M'_{-(n+m-1),-(n+m-1)}(G,x) + M'_{m,m}(\overline{G},x) + M'_{n+2,n+2}(L(\overline{G}),x) \\
&\quad + \sum_{u\nsim e} x^{(n+m-1-d_G(u))(n+m-3-d_G(v))} \\
M_3(T^{1--}(G),x) &= M_1(G,x) + \overline{M_1}(G,x) + \overline{M_1}(L(G),x) + x^2\sum_{u\nsim v} x^{|d_G(u)+d_G(v)|}.
\end{aligned}$$

Theorem 3.9 *Let G be a graph of order n and size m. Then*

$$\begin{aligned}
M_1(T^{+0-}(G),x) &= mx^{2m} + m(n-2)x^{m+n-2} \\
M_2(T^{+0-}(G),x) &= mx^{m^2} + m(n-2)x^{m(n-2)} \\
M_3(T^{+0-}(G),x) &= m + m(n-2)x^{|m-n+2|}.
\end{aligned}$$

Theorem 3.10 *Let G be a graph of order n and size m. Then*

$$\begin{aligned}
M_1(T^{+1-}(G),x) &= mx^{2m} + \binom{m}{2}x^{2(n+m-3)} + m(n-2)x^{(2m+n-3)} \\
M_2(T^{+1-}(G),x) &= mx^{m^2} + \binom{m}{2}x^{(n+m-3)^2} + m(n-2)x^{m(m+n-3)} \\
M_3(T^{+1-}(G),x) &= m + \binom{m}{2} + m(n-2)x^{|n-3|}.
\end{aligned}$$

Theorem 3.11 *Let G be a graph of order n and size m. Then*

$$\begin{aligned}
M_1(T^{++-}(G),x) &= mx^{2m} + x^{2(n-2)}\overline{M}_1(L(G),x) + mx^{m+n-2}M_0(G,x) - x^{m+n-2}M_1^*(G,x) \\
M_2(T^{++-}(G),x) &= mx^{m^2} + M_{n-2,n-2}(L(G),x) + mx^{m(n-2)}M_0(G,x^m) - x^{m(n-2)}M_1^*(G,x^m) \\
M_3(T^{++-}(G),x) &= m + M_3(L(G),x) + mx^{|m+n-2|}M_0(G,x) - x^{|m+n-2|}M_1^*(G,x).
\end{aligned}$$

Theorem 3.12 *Let G be a graph of order n and size m. Then*

$$\begin{aligned}
M_1(T^{+--}(G),x) &= mx^{2m} + x^{2(n+m-3)}\overline{M}_1(L(G),x^{-1}) \\
&\quad + mx^{2m+n-3}M_0(G,x^{-1}) - x^{2m+n-3}M_1^*(G,x^{-1}) \\
M_2(T^{+--}(G),x) &= mx^{m^2} + M_{n-2,n-2}(\overline{L(G)},x) \\
&\quad + mx^{m(m+n-3)}M_0(G,x^{-m}) - x^{m(m+n-3)}M_1^*(G,x^{-m}) \\
M_3(T^{+--}(G),x) &= m + \overline{M_3}(L(G),x) + mx^{n-3}M_0(G,x) - x^{n-3}M_1^*(G,x).
\end{aligned}$$

Theorem 3.13 *Let G be a graph of order n and size m. Then*

$$\begin{aligned}
M_1(T^{-0-}(G),x) &= x^{-2(n+m-1)}\overline{M_1}(G,x^2)+mx^{2n+m-3}M_0(G,x^{-2})\\
&\quad -x^{(2n+m-3)}M_1^*(G,x^{-2})\\
M_2(T^{-0-}(G),x) &= \overline{M'}_{-(n+m-1),-(n+m-1)}(G,x^2)+mx^{(n-2)(n+m-1)}M_0(G,x^{-2(n-2)})\\
&\quad +mx^{(n-2)(n+m-1)}M_1^*(G,x^{-2(n-2)})\\
M_3(T^{-0-}(G),x^{-2}) &= \overline{M_3}(G,x^2)+mx^{m+1}M_0(G,x^{-2})-x^{m+1}M_1^*(G,x^{-2}).
\end{aligned}$$

Theorem 3.14 *Let G be a graph of order n and size m. Then*

$$\begin{aligned}
M_1(T^{-1-}(G),x) &= x^{2(n+m-1)}\overline{M_1}(G,x^{-2})+\binom{m}{2}x^{2(n+m-3)}+mx^{2(n+m-2)}M_0(G,x^{-2})\\
&\quad -x^{2(n+m-3)}M_1^*(G,x)\\
M_2(T^{-1-}(G),x) &= \overline{M'}_{-(n+m-1),-(n+m-1)}(G,x^2)+\binom{m}{2}x^{(n+m-3)^2}\\
&\quad +mx^{(n+m-3)(n+m)}M_0(G,x^{-2(n+m-3)})\\
&\quad -x^{(n+m-3)(n+m)}M_1^*(G,x^{-2(n+m-3)})\\
M_3(T^{-1-}(G),x^{-2}) &= \overline{M_3}(G,x^2)+\binom{m}{2}+mx^{-2}M_0(G,x^2)-x^{-2}M_1^*(G,x^2).
\end{aligned}$$

Theorem 3.15 *Let G be a graph of order n and size m. Then*

$$\begin{aligned}
M_1(T^{-+-}(G),x) &= x^{2(n+m-1)}\overline{M_1}(G,x^{-2})+x^{(n-2)}M_1(L(G),x)+x^{2n+m-3}\sum_{u\nsim v}x^{-2d_G(u)+d_G(v)}\\
M_2(T^{-+-}(G),x) &= \overline{M'}_{-(n+m-1),-(n+m-1)}(G,x^2)+M'_{(n+2,n+2)}(L(G),x)\\
&\quad +\sum_{u\nsim v}x^{(n+m-1-2d_G(u))(n-2+d_G(v))}\\
M_3(T^{-+-}(G),x) &= x^{|m+1|}M'_{-2,-1}(\overline{G},x)+M_3(L(G),x)+x^{m+1}\sum_{u\nsim v}x^{|2d_G(u)+d_G(v)|}.
\end{aligned}$$

Theorem 3.16 *Let G be a graph of order n and size m. Then*

$$\begin{aligned}
M_1(T^{---}(G),x) &= x^{2(n+m-1)}\overline{M_1}(G,x^{-2})+x^{2(n+m-3)}\overline{M_1}(L(G),x^{-1})\\
&\quad +x^{2(n+m-1)}\sum_{u\nsim v}x^{-2d_G(u)-d_G(v)}\\
M_2(T^{---}(G),x) &= \overline{M'}_{-(n+m-1),-(n+m-1)}(G,x^2)+\overline{M'}_{-(n+m-3),-(n+m-3)}(L(G),x)\\
&\quad +\sum_{u\nsim v}x^{(n+m-1-d_G(u))(n+m-3-d_G(v))}\\
M_3(T^{---}(G),x) &= \overline{M_3}(G,x)+\overline{M_3}(L(G),x)+x^2\sum_{u\nsim v}x^{|d_G(u)-d_G(v)|}
\end{aligned}$$

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