International J.Math. Combin. Vol.2(2019), 69-85

C-Geometric Mean Labeling of Some Cycle Related Graphs

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Abstract: In a study of traffic, the labeling problems in graph theory can be used by considering the crowd at every junction as the weights of a vertex and expected average traffic in each street as the weight of the corresponding edge. If we assume the expected traffic at each street as the arithmetic mean of the weight of the end vertices, that causes mean labeling of the graph. When we consider a geometric mean instead of arithmetic mean in a large population of a city, the rate of growth of traffic in each street will be more accurate. The geometric mean labeling of graphs have been defined in which the edge labels may be assigned by either flooring function or ceiling function. In this, the readers will get some confusion in finding the edge labels which edge is assigned by flooring function and which edge is assigned by ceiling function. To avoid this confusion, we establish the C-Geometric mean labeling on graphs by considering the edge labels obtained only from the ceiling function. A C-Geometric mean labeling of a graph G with q edges, is an injective function from the vertex set of G to $\{1, 2, 3, \cdots, q+1\}$ such that the edge labels obtained from the ceiling function of geometric mean of the vertex labels of the end vertices of each edge, are all distinct and the set of edge labels is $\{2, 3, 4, \cdots, q+1\}$. A graph is said to be a C-Geometric mean graph if it admits a C-Geometric mean labeling. In this paper, we study the C-geometric meanness of some cycle related graphs such as cycle, union of a path and a cycle, union of two cycles, the graph $C_3 \times P_n$, corona of cycle, the graphs $P_{a,b}$, P_a^b and some chain graphs.

Key Words: Labeling, C-Geometric mean labeling, Smarandache $2k$-Geometric mean labeling, C-Geometric mean graph.

AMS(2010): 05C78.

§1. Introduction

Throughout this paper, by a graph we mean a finite, undirected and simple graph. Let $G(V, E)$ be a graph with p vertices and q edges. For notations and terminology, we follow [4]. For a detailed survey on graph labeling we refer to [3].

Path on n vertices is denoted by P_n. $G \odot S_m$ is the graph obtained from G by attaching m pendant vertices at each vertex of G. Let G_1 and G_2 be any two graphs with p_1 and p_2 vertices respectively. Then the cartesian product $G_1 \times G_2$ has p_1p_2 vertices which are $\{(u, v) : u \in G_1, v \in G_2\}$ and any two

[1]Received November 1, 2018, Accepted May 30, 2019.

vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \times G_2$ if either $u_1 = u_2$ and v_1 and v_2 are adjacent in G_2 or u_1 and u_2 are adjacent in G_1 and $v_1 = v_2$.

Let u and v be two fixed vertices. We connect u and v by means of $b \geq 2$ internally disjoint paths of length $a \geq 2$ each. The resulting graph embedded in the plane is denoted by $P_{a,b}$. Let a and b be integers such that $a \geq 2$ and $b \geq 2$. Let $y_1, y_2, \cdots, y_a$ be the 'a' fixed vertices. We connect y_i and y_{i+1} by means of b internally disjoint paths of length $(i+1)$ for each i, $1 \leq i \leq a-1$. The resulting graph embedded in the plane is denoted by P_a^b.

Barrientos [1] defines a chain graph as one with blocks $B_1, B_2, B_3, \cdots, B_m$ such that for every i, B_i and B_{i+1} have a common vertex in such a way that the block cut point graph is a path. The chain graph $\widehat{G}(p_1, k_1, p_2, k_2, \cdots, k_{n-1}, p_n)$ is obtained from n cycles of length $p_1, p_2, p_3, \cdots, p_n$ and $(n-1)$ paths on $k_1, k_2, k_3, \cdots, k_{n-1}$ vertices respectively by identifying a cycle and a path at a vertex alternatively as follows. If the i^{th} cycle is of odd length, then its $\left(\frac{p_i+3}{2}\right)^{th}$ vertex is identified with a pendant vertex of the i^{th} path and if the i^{th} cycle is of even length, then its $\left(\frac{p_i+2}{2}\right)^{th}$ vertex is identified with a pendant vertex of the i^{th} path while the other pendant vertex of the i^{th} path is identified with the first vertex of the $(i+1)^{th}$ cycle. The chain graph $G^*(p_1, p_2, \cdots, p_n)$ is obtained from n cycles of length $p_1, p_2, \cdots, p_n$ by identifying consecutive cycles at a vertex as follows. If the i^{th} cycle is of odd length, then its $\left(\frac{p_i+3}{2}\right)^{th}$ vertex is identified with the first vertex of $(i+1)^{th}$ cycle and if the i^{th} cycle is of even length, then its $\left(\frac{p_i+2}{2}\right)^{th}$ vertex is identified with the first vertex of $(i+1)^{th}$ cycle. The graph $G'(p_1, p_2, \cdots, p_n)$ is obtained from n cycles of length $p_1, p_2, \cdots, p_n$ by identifying consecutive cycles at an edge as follows:

The $\left(\frac{p_j+3}{2}\right)^{th}$ *edge of* j^{th} *cycle is identified with the first edge of* $(j+1)^{th}$ *cycle when* j *is odd and the* $\left(\frac{p_j+1}{2}\right)^{th}$ *edge of* j^{th} *cycle is identified with the first edge of* $(j+1)^{th}$ *cycle when* j *is even.*

The study of graceful graphs and graceful labeling methods was first introduced by Rosa [5] and many authors are working in graph labeling [2,3]. Motivated by their methods, we introduce a new type of labeling called C-Geometric mean labeling. A function f is called a C-Geometric mean labeling of a graph G if $f : V(G) \to \{1, 2, 3, \cdots, q+1\}$ is injective and the induced function $f^* : E(G) \to \{2, 3, 4, \cdots, q+1\}$ defined as

$$f^*(uv) = \left\lceil \sqrt{f(u)f(v)} \right\rceil, \quad \text{for all} \quad uv \in E(G)$$

is bijective. Furthermore, if

$$f^*(uv) = \left\lceil \sqrt[2k]{f(u)^k f(v)^k} \right\rceil, \quad \text{for all} \quad uv \in E(G)$$

is bijective, where $k \geq 1$ is an integer, such a function f is called a Smarandache $2k$-Geometric mean labeling, and C-Geometric mean labeling of a graph G if $k = 1$. A graph that admits a C-Geometric mean labeling is called a C-Geometric mean graph.

In [6], S.Somasundaram et al. defined the geometric mean labeling as follows:

A graph $G = (V, E)$ *with* p *vertices and* q *edges is said to be a geometric mean graph if it is possible to label the vertices* $x \in V$ *with distinct labels* $f(x)$ *from* $1, 2, \cdots, q+1$ *in such way that when each edge* $e = uv$ *is labeled with* $f(uv) = \left\lfloor \sqrt{f(u)f(v)} \right\rfloor$ *or* $\left\lceil \sqrt{f(u)f(v)} \right\rceil$ *then the edge labels are distinct.*

In the above definition, the readers will get some confusion in finding the edge labels which edge is assigned by flooring function and which edge is assigned by ceiling function.

In [6], the authors have given a geometric mean labeling of the graph $C_5 \cup C_7$ as in the Figure 1.

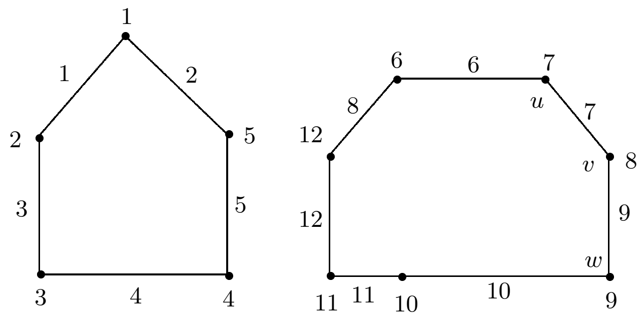

Figure 1 A geometric mean labeling of $C_5 \cup C_7$.

From the above figure, for the edge uv, they have used flooring function $\left\lfloor \sqrt{f(u)f(v)} \right\rfloor$ and for the edge vw, they have used ceiling function $\left\lceil \sqrt{f(u)f(v)} \right\rceil$ for fulfilling their requirement. To avoid the confusion of assigning the edge labels in their definition, we just consider the ceiling function $\left\lceil \sqrt{f(u)f(v)} \right\rceil$ for our discussion. Based on our definition, the C-Geometric mean labeling of the same graph $C_5 \cup C_7$ is given in Figure 2.

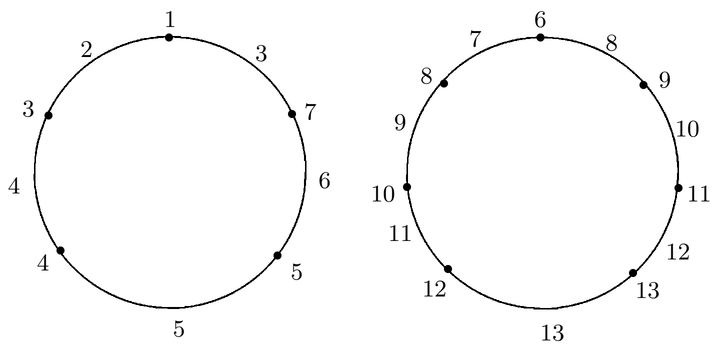

Figure 2 A C-Geometric mean labeling of $C_5 \cup C_7$

In this paper, we have discussed the C-Geometric mean labeling of the cycle for $n \geq 4$, union of any two cycles C_m and C_n, union of the cycle C_m and a path P_n, the graph $C_3 \times P_n$, corona of cycle, the graphs $P_{a,b}$, P_a^b and some chain graphs.

§2. Main Results

Theorem 2.1 *A graph C_n is a C-Geometric mean graph only if $n \geq 4$.*

Proof The proof is divided into 2 cases following.

Case 1. $n \geq 4$.

Let $v_1, v_2, \cdots, v_n$ be the vertices of C_n. Define $f : V(C_n) \to \{1, 2, 3, \cdots, n+1\}$ as follows:

$$f(v_i) = \begin{cases} 2i-1, & 1 \leq i \leq 2, \\ 2i-2, & 3 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1, \\ n+1, & i = \left\lfloor \frac{n}{2} \right\rfloor + 2, \\ 2n+5-2i, & \left\lfloor \frac{n}{2} \right\rfloor + 3 \leq i \leq n. \end{cases}$$

Then, the induced edge labeling is obtained as follows:

$$f^*(v_iv_{i+1}) = \begin{cases} 2i, & 1 \le i \le 2, \\ 2i-1, & 3 \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 1, \\ n+1, & i = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is odd}, \\ n, & i = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is even}, \\ 2n+4-2i, & 3 \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 3 \le i \le n-1 \end{cases}$$

and $f^*(v_nv_1) = 3$.

Hence, f is a C-Geometric mean labeling of the cycle C_n. Thus the cycle C_n is a C-Geometric mean graph for $n \ge 4$.

Case 2. $n = 3$.

Let v_1, v_2 and v_3 be the vertices of C_3. To get the edge label $q+1, q$ and $q+1$ should be the vertex labels for two of the vertices of C_3, say $v_1 = q = 3$ and $v_2 = q+1 = 4$. Also to obtain the edge label 2, 1 is to be a vertex label of a vertex of C_3, say $v_3 = 1$. Since the edge labels of the edges v_1v_3 and v_2v_3 are one and the same. Hence C_3 is not a C-Geometric mean graph. □

Theorem 2.2 *A union of two cycles C_m and C_n is a C-Geometric mean graph if $m \ge 3$ and $n \ge 3$.*

Proof Let $u_1, u_2, \cdots, u_m$ and $v_1, v_2, \cdots, v_n$ be the vertices of the cycles C_m and C_n respectively.

Case 1. $m \ge 4$ or $n \ge 4$.

Define $f : V(C_m \cup C_n) \to \{1, 2, 3, \cdots, m+n+1\}$ as follows:

$$f(u_i) = \begin{cases} i, & 1 \le i \le \left\lceil \sqrt{m+2} \right\rceil - 2, \\ i+1, & \left\lceil \sqrt{m+2} \right\rceil - 1 \le i \le m-1, \end{cases}$$

$$f(u_m) = m+2,$$

$$f(v_i) = \begin{cases} m-1+2i, & 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 1, \\ m+2n+4-2i, & \left\lfloor \frac{n}{2} \right\rfloor + 2 \le i \le n. \end{cases}$$

Then, the induced edge labeling is known as follows:

$$f^*(u_iu_{i+1}) = \begin{cases} i+1, & 1 \le i \le \left\lceil \sqrt{m+2} \right\rceil - 2, \\ i+2, & \left\lceil \sqrt{m+2} \right\rceil - 1 \le i \le m-1, \end{cases}$$

$$f^*(u_1, u_m) = \left\lceil \sqrt{m+2} \right\rceil,$$

$$f^*(v_iv_{i+1}) = \begin{cases} m+2i, & 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor, \\ m+n+1, & i = \left\lfloor \frac{n}{2} \right\rfloor + 1, \\ m+2n+3-2i, & \left\lfloor \frac{n}{2} \right\rfloor + 2 \le i \le n-1 \end{cases}$$

and $f^*(v_1v_n) = m+3$.

Hence, f is a C-Geometric mean labeling of the graph $C_m \cup C_n$. Thus the graph $C_m \cup C_n$ is a C-Geometric mean graph, for $m \ge 4$ or $n \ge 4$.

Case 2. $m = 3$ and $n = 3$.

A C-Geometric mean labeling of $C_3 \cup C_3$ is shown in Figure 3.

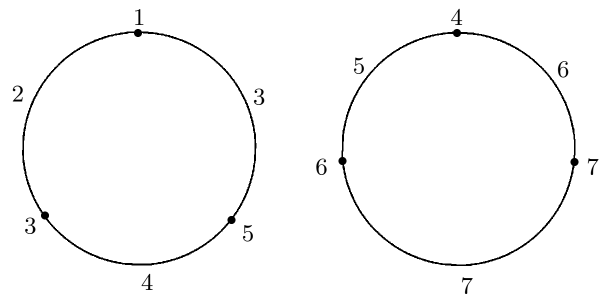

Figure 3 A C-Geometric mean labeling of $C_3 \cup C_3$.

This completes the proof. □

Theorem 2.3 *A graph $C_m \cup P_n$ is a C-Geometric mean graph if $m \geq 3$ and $n \geq 2$.*

Proof Let $u_1, u_2, \cdots, u_m$ and $v_1, v_2, \cdots, v_n$ be the vertices of the cycle C_m and the path P_n respectively.

Define $f : V(C_m \cup P_n) \to \{1, 2, 3, \cdots, m+n\}$ as follows:

$$f(u_i) = \begin{cases} n + 2i - 2, & 1 \leq i \leq \lfloor \frac{m}{2} \rfloor + 1, \\ n + 2m + 3 - 2i, & \lfloor \frac{m}{2} \rfloor + 2 \leq i \leq m, \end{cases}$$

$$f(v_i) = i, \text{ for } 1 \leq i \leq n-1 \text{ and}$$

$$f(v_n) = n + 1.$$

Then, the induced edge labeling is obtained as follows:

$$f^*(u_i u_{i+1}) = \begin{cases} n - 1 + 2i, & 1 \leq i \leq \lfloor \frac{m}{2} \rfloor, \\ m + n, & i = \lfloor \frac{m}{2} \rfloor + 1, \\ n + 2m + 2 - 2i, & \lfloor \frac{m}{2} \rfloor + 2 \leq i \leq m - 1, \end{cases}$$

$$f^*(u_1 u_m) = n + 2 \text{ and}$$

$$f^*(v_i v_{i+1}) = i + 1, \text{ for } 1 \leq i \leq n - 1.$$

Hence, f is a C-Geometric mean labeling of the graph $C_m \cup P_n$. Thus the graph $C_m \cup P_n$ is a C-Geometric mean graph, for $m \geq 3$ and $n \geq 2$. □

Theorem 2.4 *A graph $C_3 \times P_n$ is a C-Geometric mean graph if $n \geq 4$.*

Proof Let $V(C_3 \times P_n) = \{v_1^{(i)}, v_2^{(i)}, v_3^{(i)}; 1 \leq i \leq n\}$ be the vertex set of $C_3 \times P_n$ and $E(C_3 \times P_n) = \{v_1^{(i)} v_2^{(i)}, v_2^{(i)} v_3^{(i)}, v_1^{(i)} v_3^{(i)}; 1 \leq i \leq n\} \cup \{v_1^{(i)} v_1^{(i+1)}, v_2^{(i)} v_2^{(i+1)}, v_3^{(i)} v_3^{(i+1)}; 1 \leq i \leq n-1\}$ be the edge set of $C_3 \times P_n$.

Define $f : V(C_3 \times P_n) \to \{1, 2, 3, \cdots, 6n-2\}$ as follows

$$f(v_1^{(j)}) = \begin{cases} 9j-8, & 1 \le j \le 2, \\ 8j-11, & 3 \le j \le 4, \end{cases}$$

$$f(v_2^{(j)}) = \begin{cases} 6j-3, & 1 \le j \le 2, \\ 2j+11, & 3 \le j \le 4, \end{cases}$$

$$f(v_3^{(j)}) = \begin{cases} 5+j, & 1 \le j \le 2, \\ 7j-6, & 3 \le j \le 4 \end{cases}$$

and $f(v_i^{(j)}) = f(v_i^{(j-3)}) + 18$ for $1 \le i \le 3$ and $5 \le j \le n$. Then, the induced edge labeling is obtained as follows:

$$f^*(v_1^{(j)}v_2^{(j)}) = \begin{cases} 2, & j = 1, \\ 5j, & 2 \le j \le 3, \\ f^*(v_1^{(j-3)}v_2^{(j-3)}) + 18, & 4 \le j \le n, \end{cases}$$

$$f^*(v_2^{(j)}v_3^{(j)}) = \begin{cases} 3j+2, & 1 \le j \le 2, \\ 5j+1, & 3 \le j \le 4, \\ f^*(v_2^{(j-3)}v_3^{(j-3)}) + 18, & 5 \le j \le n, \end{cases}$$

$$f^*(v_1^{(j)}v_3^{(j)}) = \begin{cases} 6j-3, & 1 \le j \le 2, \\ 8j-10, & 3 \le j \le 4, \\ f^*(v_1^{(j-3)}v_3^{(j-3)}) + 18, & 5 \le j \le n, \end{cases}$$

$$f^*(v_1^{(j)}v_1^{(j+1)}) = \begin{cases} 8j-4, & 1 \le j \le 2 \\ 8j-7, & 3 \le j \le 4, \\ f^*(v_1^{(j-3)}v_1^{(j-2)}) + 18, & 5 \le j \le n-1, \end{cases}$$

$$f^*(v_2^{(j)}v_2^{(j+1)}) = \begin{cases} 6, & j = 1 \\ 5j+3, & 2 \le j \le 4, \\ f^*(v_2^{(j-3)}v_2^{(j-2)}) + 18, & 5 \le j \le n-1, \end{cases}$$

$$f^*(v_3^{(j)}v_3^{(j+1)}) = \begin{cases} 4j+3, & 1 \le j \le 2, \\ 5j+4, & 3 \le j \le 4 \\ f^*(v_3^{(j-3)}v_3^{(j-2)}) + 18, & 5 \le j \le n-1. \end{cases}$$

Hence f is a C-Geometric mean labeling of $C_3 \times P_n$. Thus the graph $C_3 \times P_n$ is a C-Geometric mean graph, for $n \ge 4$. □

Theorem 2.5 *A graph $C_n \odot S_m$ is a C-Geometric mean graph if $n \ge 3$ and $m \le 2$.*

Proof Let $u_1, u_2, \cdots, u_n$ be the vertices of the cycle C_n and let $v_0^{(i)}, v_1^{(i)}, \cdots, v_m^{(i)}$ be the vertices of the star graph S_m such that $v_0^{(i)}$ is the central vertex of S_m, for $1 \leq i \leq n$.

Case 1. $m = 1$.

Subcase 1.1 $\left\lceil \sqrt{2(2n+1)} \right\rceil$ is odd and $n \geq 5$.

Define $f : V(C_n \odot S_1) \rightarrow \{1, 2, 3, \cdots, 2n+1\}$ as follows:

$$f(u_i) = \begin{cases} 2i, & 1 \leq i \leq \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor, \\ 2i+1, & \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor + 1 \leq i \leq i \leq n, \end{cases}$$

$$f(v_1^{(i)}) = \begin{cases} 2i-1, & 1 \leq i \leq \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor, \\ 2i, & \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor + 1 \leq i \leq i \leq n. \end{cases}$$

Then, the induced edge labeling is obtained as follows:

$$f^*(u_i u_{i+1}) = \begin{cases} 2i+1, & 1 \leq i \leq \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor - 1, \\ 2i+2, & \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor + 1 \leq i \leq i \leq n-1, \end{cases}$$

$$f^*(u_1 u_n) = \left\lceil \sqrt{2(2n+1)} \right\rceil,$$

$$f^*(u_i v_1^{(i)}) = \begin{cases} 2i, & 1 \leq i \leq \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor, \\ 2i+1, & \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor + 1 \leq i \leq n. \end{cases}$$

Subcase 1.2 $\left\lceil \sqrt{2(2n+1)} \right\rceil$ is even.

Define $f : V(C_n \odot S_2) \rightarrow \{1, 2, 3, \cdots, 2n+1\}$ as follows:

$$f(u_i) = \begin{cases} 2i, & 1 \leq i \leq \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor - 2, \\ 2i-1, & i = \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor - 1, \\ 2i+1, & \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor \leq i \leq i \leq n, \end{cases}$$

$$f(v_1^{(i)}) = \begin{cases} 2i-1, & 1 \leq i \leq \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor - 2, \\ 2i, & \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor - 1 \leq i \leq i \leq n. \end{cases}$$

Then, the induced edge labeling is obtained as follows:

$$f^*(u_i u_{i+1}) = \begin{cases} 2i+1, & 1 \leq i \leq \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor - 1, \\ 2i+2, & \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor \leq i \leq i \leq n-1, \end{cases}$$

$$f^*(u_1u_n) = \left\lceil \sqrt{2(2n+1)} \right\rceil$$

$$\text{and } f^*(u_iv_1^{(i)}) = \begin{cases} 2i, & 1 \leq i \leq \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor - 1, \\ 2i+1, & \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor \leq i \leq n. \end{cases}$$

Hence, the graph $C_n \odot S_1$, for $n \geq 4$ admits a C-Geometric mean labeling. For $n = 3$, a C-Geometric mean labeling of $C_3 \odot S_1$ is shown in Figure 4.

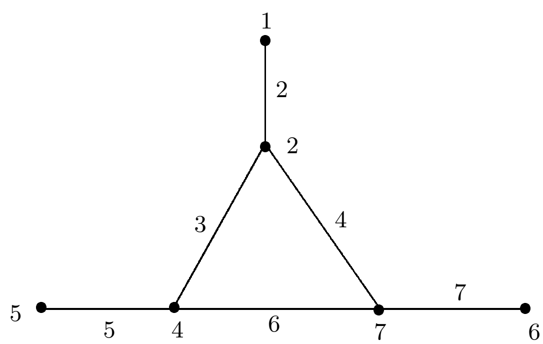

Figure 4 A C-Geometric mean labeling of $C_3 \odot S_1$.

Case 2. $m = 2$.

Subcase 2.1 $\left\lceil \sqrt{6n} \right\rceil \equiv 0 (mod\ 3)$.

Define $f : V(C_n \odot S_2) \to \{1, 2, 3, \cdots, 3n+1\}$ as follows:

$$f(u_i) = \begin{cases} 3i-1, & 1 \leq i \leq \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor - 1, \\ 3i, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor \leq i \leq n, \end{cases}$$

$$f(v_1^{(i)}) = \begin{cases} 3i-2, & 1 \leq i \leq \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor, \\ 3i-1, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \leq i \leq n, \end{cases}$$

$$f(v_2^{(i)}) = \begin{cases} 3i, & 1 \leq i \leq \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor - 1, \\ 3i+1, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor \leq i \leq n. \end{cases}$$

Then, the induced edge labeling is obtained as follows:

$$f^*(u_iu_{i+1}) = \begin{cases} 3i+1, & 1 \leq i \leq \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor - 1, \\ 3i+2, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor \leq i \leq n-1, \end{cases}$$

$$f^*(u_nu_1) = \left\lceil \sqrt{6n} \right\rceil,$$

$$f^*(u_iv_1^{(i)}) = \begin{cases} 3i-1, & 1 \leq i \leq \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor, \\ 3i, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \leq i \leq n, \end{cases}$$

$$f^*(u_iv_2^{(i)}) = \begin{cases} 3i, & 1 \leq i \leq \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor - 1, \\ 3i+1, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor \leq i \leq n. \end{cases}$$

Subcase 2.2 $\lceil \sqrt{6n} \rceil \equiv 1(mod\ 3)$.

Define $f : V(C_n \odot S_2) \to \{1, 2, 3, \cdots, 3n+1\}$ as follows:

$$f(u_i) = \begin{cases} 3i-1, & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor, \\ 3i+1, & i = \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1, \\ 3i, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 2 \le i \le n, \end{cases}$$

$$f(v_1^{(i)}) = \begin{cases} 3i-2, & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor, \\ 3i-1, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \le i \le n, \end{cases}$$

$$f(v_2^{(i)}) = \begin{cases} 3i, & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1, \\ 3i+1, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 2 \le i \le n. \end{cases}$$

Then, the induced edge labeling is obtained as follows:

$$f^*(u_iu_{i+1}) = \begin{cases} 3i+1, & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor - 1, \\ 3i+2, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor \le i \le n-1, \end{cases}$$

$$f^*(u_nu_1) = \left\lceil \sqrt{6n} \right\rceil,$$

$$f^*(u_iv_1^{(i)}) = \begin{cases} 3i-1, & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor, \\ 3i, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \le i \le n, \end{cases}$$

$$f^*(u_iv_2^{(i)}) = \begin{cases} 3i, & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor, \\ 3i+1, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \le i \le n. \end{cases}$$

Subcase 2.3 $\lceil \sqrt{6n} \rceil \equiv 2(mod\ 3)$.

Define $f : V(C_n \odot S_2) \to \{1, 2, 3, \cdots, 3n+1\}$ as follows:

$$f(u_i) = \begin{cases} 3i-1, & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor, \\ 3i, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \le i \le n, \end{cases}$$

$$f(v_1^{(i)}) = \begin{cases} 3i-2, & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor, \\ 3i-1, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \le i \le n, \end{cases}$$

$$f(v_2^{(i)}) = \begin{cases} 3i, & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor, \\ 3i+1, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \le i \le n. \end{cases}$$

Then, the induced edge labeling is obtained as follows:

$$f^*(u_iu_{i+1}) = \begin{cases} 3i+1, & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor, \\ 3i+2, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \le i \le n-1, \end{cases}$$

$$f^*(u_nu_1) = \left\lceil \sqrt{6n} \right\rceil,$$

$$f^*(u_iv_1^{(i)}) = \begin{cases} 3i-1, & 1 \leq i \leq \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor, \\ 3i, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \leq i \leq n, \end{cases}$$

$$f^*(u_iv_2^{(i)}) = \begin{cases} 3i, & 1 \leq i \leq \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor, \\ 3i+1, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \leq i \leq n. \end{cases}$$

Hence, the graph $C_n \odot S_2$, for $n \geq 3$ admits a C-Geometric mean labeling. Thus the graph $C_n \odot S_m$ is a C-Geometric mean graph, for $n \geq 3$ and $m \leq 2$. □

Theorem 2.6 *A graph $\widehat{G}(p_1, m_1, p_2, m_2, \cdots, m_{n-1}, p_n)$ is a C-Geometric mean graph if $p_1 \neq 3$.*

Proof Let $\{v_i^{(j)}; 1 \leq j \leq n, 1 \leq i \leq p_j\}$ and $\{u_i^{(j)}; 1 \leq j \leq n-1, 1 \leq i \leq m_j\}$ be the n number of cycles and $(n-1)$ number of paths respectively. For $1 \leq j \leq n-1$, the j^{th} cycle and j^{th} path are identified by a vertex $v^{(j)}_{\frac{p_j+2}{2}}$ and $u_1^{(j)}$ while p_j is even and $v^{(j)}_{\frac{p_j+3}{2}}$ and $u_1^{(j)}$ while p_j is odd and the j^{th} path and $(j+1)^{th}$ cycle are identified by a vertex $u_{m_j}^{(j)}$ and $v_1^{(j+1)}$.

Define $f : V(\widehat{G}(p_1, m_1, p_2, m_2, \cdots, m_{n-1}, p_n)) \rightarrow \left\{1, 2, 3, \cdots, \sum\limits_{j=1}^{n-1}(p_j + m_j) \ + p_n - n + 2\right\}$ as follows:

If p_1 is odd and $p_1 \neq 3$,

$$f\left(v_i^{(1)}\right) = \begin{cases} 2i-1, & 1 \leq i \leq 2, \\ 2i-2, & 3 \leq i \leq \left\lfloor \frac{p_1}{2} \right\rfloor + 2, \\ 2p_1 + 5 - 2i, & \left\lfloor \frac{p_1}{2} \right\rfloor + 3 \leq i \leq p_1. \end{cases}$$

and if p_1 is even,

$$f\left(v_i^{(1)}\right) = \begin{cases} 3, & j = 1, \\ 2i, & 2 \leq j \leq \left\lfloor \frac{p_1}{2} \right\rfloor, \\ 2p_1 + 3 - 2i, & \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \leq j \leq p_1 - 1. \end{cases}$$

$f\left(v_{p_1}^{(1)}\right) = 1$, $f\left(u_i^{(1)}\right) = p_1 + i$, for $2 \leq i \leq m_1$. For $2 \leq j \leq n$,

$$f(v_i^{(j)}) = \begin{cases} \sum\limits_{k=1}^{j-1}(p_k + m_k) + 2i - j, & 2 \leq i \leq \left\lfloor \frac{p_j}{2} \right\rfloor + 1, \\ \sum\limits_{k=1}^{j-1}(p_k + m_k) + 2i - j - 1, & i = \left\lfloor \frac{p_j}{2} \right\rfloor + 2 \text{ and } p_j \text{ is odd}, \\ \sum\limits_{k=1}^{j-1}(p_k + m_k) + 2i - j - 3, & i = \left\lfloor \frac{p_j}{2} \right\rfloor + 2 \text{ and } p_j \text{ is even}, \\ \sum\limits_{k=1}^{j-1}(p_k + m_k) + 2p_j - 2i - j - 5, & \left\lfloor \frac{p_j}{2} \right\rfloor + 3 \leq i \leq p_j \end{cases}$$

and for $3 \leq j \leq n$,

$$f(u_i^{(j-1)}) = \sum_{k=1}^{j-2}(p_k + m_k) + p_{j-1} + i + 2 - j, \text{ for } 2 \leq i \leq m_{j-1}.$$

Then the induced edge labeling is obtained as follows:

If p_1 is odd and $p_1 \neq 3$,

$$f^*(v_i^{(1)}v_{i+1}^{(1)}) = \begin{cases} 2i, & 1 \leq i \leq 2, \\ 2i-1, & 3 \leq i \leq \lfloor \frac{p_1}{2} \rfloor + 1, \\ 2p_1+4-2i, & \lfloor \frac{p_1}{2} \rfloor + 2 \leq i \leq p_1 - 1, \end{cases}$$
$$f^*(v_{p_1}^{(1)}v_1^{(1)}) = 3.$$

and if p_1 is even,

$$f^*(v_i^{(1)}v_{i+1}^{(1)}) = \begin{cases} 4, & j = 1, \\ 2i+1, & 2 \leq i \leq \lfloor \frac{p_1}{2} \rfloor, \\ 2p_1+2-2i, & \lfloor \frac{p_1}{2} \rfloor + 1 \leq i \leq p_1 - 2, \end{cases}$$
$$f^*(v_{p_1-1}^{(1)}v_{p_1}^{(1)}) = 3,$$
$$f^*(v_{p_1}^{(1)}v_1^{(1)}) = 2,$$
$$f^*(u_i^{(1)}u_{i+1}^{(1)}) = p_1 + i, \quad 1 \leq i \leq m_1 - 1$$

and for $2 \leq j \leq n$,

$$f^*(v_i^{(j)}v_{i+1}^{(j)}) = \begin{cases} \sum\limits_{k=1}^{j-1}(p_k+m_k)+2i-j+1, & 1 \leq i \leq \lfloor \frac{p_j}{2} \rfloor, \\ \sum\limits_{k=1}^{j-1}(p_k+m_k)+2i-j+1, & i = \lfloor \frac{p_j}{2} \rfloor + 1 \text{ and } p_j \text{ is odd}, \\ \sum\limits_{k=1}^{j-1}(p_k+m_k)+2p_j-2i-j+4, & i = \lfloor \frac{p_j}{2} \rfloor + 1 \text{ and } p_j \text{ is even}, \\ \sum\limits_{k=1}^{j-1}(p_k+m_k)+2p_j-2i-j+4, & \lfloor \frac{p_j}{2} \rfloor + 2 \leq i \leq p_j - 1, \end{cases}$$
$$f^*(v_{p_j}^{(j)}v_1^{(j)}) = \sum_{k=1}^{j-1}(p_k+m_k) - j + 4$$

and

$$f^*(u_i^{(j-1)}u_{i+1}^{(j-1)}) = \sum_{k=1}^{j-2}(p_k+mk) + p_{j-1} + i + 3 - j, \text{ for } 1 \leq i \leq m_{j-1} - 1 \text{ and } 3 \leq j \leq n.$$

Hence, f is a C-Geometric mean labeling of $\widehat{G}(p_1, m_1, p_2, m_2 \cdots, m_{n-1}, p_n)$. Thus the graph $\widehat{G}(p_1, m_1, p_2, m_2 \cdots, m_{n-1}, p_n)$ is a C-Geometric mean graph, for $p_1 \neq 3$. □

Corollary 2.7 *A graph $G^*(p_1, p_2, \cdots, p_n)$ is a C-Geometric mean graph if $p_1 \neq 3$.*

Theorem 2.8 *A graph $G'(p_1, p_2, \cdots, p_n)$ is a C-Geometric mean graph if all p_j's are odd and $p_1 \neq 3$ or all p_j's $1 \leq j \leq n$ are even.*

Proof Let $\{v_i^{(j)}; 1 \leq j \leq n, 1 \leq i \leq p_j\}$ be the vertices of the n number of cycles.

Case 1. p_j is odd and $p_1 \neq 3$ for $1 \leq j \leq n$.

For $1 \leq j \leq n-1$, the j^{th} and $(j+1)^{th}$ cycles are identified by the edges $v_{\frac{p_j+1}{2}}^{(j)}v_{\frac{p_j+3}{2}}^{(j)}$ and $v_1^{(j+1)}v_{p_{j+1}}^{(j+1)}$ while j is odd and $v_{\frac{p_j-1}{2}}^{(j)}v_{\frac{p_j+1}{2}}^{(j)}$ and $v_1^{(j+1)}v_{p_{j+1}}^{(j+1)}$ while j is even.

Define $f : V(G'(p_1, p_2, \cdots, p_n)) \to \left\{1, 2, 3, \cdots, \sum\limits_{j=1}^{n} p_j - n + 2\right\}$ as follows:

$$f(v_i^{(1)}) = \begin{cases} 3, & i = 1, \\ 2i, & 2 \le i \le \left\lfloor \frac{p_1}{2} \right\rfloor + 1, \\ 2p_1 + 3 - 2i, & \left\lfloor \frac{p_1}{2} \right\rfloor + 2 \le i \le p_1 - 1, \end{cases}$$

$$f(v_{p_1}^{(1)}) = 1$$

and for $2 \le j \le n$,

$$f(v_i^{(j)}) = \begin{cases} \sum\limits_{k=1}^{j-1} p_k - j + 2i + 2, & 2 \le i \le \left\lfloor \frac{p_j}{2} \right\rfloor \text{ and } j \text{ is even}, \\ \sum\limits_{k=1}^{j-1} p_k + 2p_j + 3 - j - 2i, & \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \le i \le p_j - 1 \text{ and } j \text{ even}, \\ \sum\limits_{k=1}^{j-1} p_k - j + 2i + 1, & 2 \le i \le \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \text{ and } j \text{ is odd}, \\ \sum\limits_{k=1}^{j-1} p_k + 2p_j + 4 - j - 2i, & \left\lfloor \frac{p_j}{2} \right\rfloor + 2 \le i \le p_j - 1 \text{ and } j \text{ odd}. \end{cases}$$

The induced edge labeling is obtained as follows:

$$f^*(v_i^{(1)} v_{i+1}^{(1)}) = \begin{cases} 4, & i = 1, \\ 2i + 1, & 2 \le i \le \left\lfloor \frac{p_1}{2} \right\rfloor, \\ 2p_1 + 2 - 2i & \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \le i \le p_1 - 2, \end{cases}$$

$$f^*(v_{p_1-1}^{(1)} v_{p_1}^{(1)}) = 3, \quad f^*(v_{p_1}^{(1)} v_1^{(1)}) = 2$$

and for $2 \le j \le n$,

$$f^*(v_i^{(j)} v_{i+1}^{(j)}) = \begin{cases} \sum\limits_{k=1}^{j-1} p_k - j + 2i + 3, & 1 \le i \le \left\lfloor \frac{p_j}{2} \right\rfloor \text{ and } j \text{ is even}, \\ \sum\limits_{k=1}^{j-1} p_k + 2p_j + 2 - j - 2i, & \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \le i \le p_j - 1 \text{ and } j \text{ even}, \\ \sum\limits_{k=1}^{j-1} p_k - j + 2i + 2, & 1 \le i \le \left\lfloor \frac{p_j}{2} \right\rfloor \text{ and } j \text{ is odd}, \\ \sum\limits_{k=1}^{j-1} p_k + 2p_j + 3 - j - 2i, & \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \le i \le p_j - 1 \text{ and } j \text{ odd}. \end{cases}$$

Case 2. p_j is even for $1 \le j \le n$.

For $1 \le j \le n-1$, the j^{th} and $(j+1)^{th}$ cycles are identified by the edges $v_{\frac{p_j}{2}}^{(j)} v_{\frac{p_j+2}{2}}^{(j)}$ and $v_1^{(j+1)} v_{p_{j+1}}^{(j+1)}$.

Define $f : V(G'(p_1, p_2, \cdots, p_n)) \to \left\{1, 2, 3, \cdots, \sum\limits_{j=1}^{n} p_j - n + 2\right\}$ as follows:

$$f(v_i^{(1)}) = \begin{cases} 3, & i = 1, \\ 2i, & 2 \le i \le \left\lfloor \frac{p_1}{2} \right\rfloor, \\ 2p_1 + 3 - 2i, & \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \le i \le p_1 - 1, \end{cases}$$

$$f(v_{p_1}^{(1)}) = 1$$

and for $2 \leq j \leq n$,

$$f(v_i^{(j)}) = \begin{cases} \sum\limits_{k=1}^{j-1} p_k - j + 2i + 1, & 2 \leq i \leq \lfloor \frac{p_j}{2} \rfloor, \\ \sum\limits_{k=1}^{j-1} p_k + 2p_j + 4 - j - 2i, & \lfloor \frac{p_j}{2} \rfloor + 1 \leq i \leq p_j - 1. \end{cases}$$

The induced edge labeling is obtained as follows:

$$f^*(v_i^{(1)} v_{i+1}^{(1)}) = \begin{cases} 4, & i = 1, \\ 2i + 1, & 2 \leq i \leq \lfloor \frac{p_1}{2} \rfloor, \\ 2p_1 + 2 - 2i, & \lfloor \frac{p_1}{2} \rfloor + 1 \leq i \leq p_1 - 2, \end{cases}$$

$$f^*(v_{p_1-1}^{(1)} v_{p_1}^{(1)}) = 3,$$

$$f^*(v_{p_1}^{(1)} v_1^{(1)}) = 2$$

and for $2 \leq j \leq n$,

$$f^*(v_i^{(j)} v_{i+1}^{(j)}) = \begin{cases} \sum\limits_{k=1}^{j-1} p_k - j + 2i + 2, & 1 \leq i \leq \lfloor \frac{p_j}{2} \rfloor, \\ \sum\limits_{k=1}^{j-1} p_k + 2p_j + 3 - j - 2i, & \lfloor \frac{p_j}{2} \rfloor + 1 \leq i \leq p_j - 1. \end{cases}$$

Hence, f is a C-Geometric mean labeling of $G'(p_1, p_2, \ldots, p_n)$. Thus the graph $G'(p_1, p_2, \ldots, p_n)$ is a C-Geometric mean graph, for $p_1 \neq 3$. □

Theorem 2.9 *A graph $P_{a,b}$ is a C-Geometric mean graph if $b \leq 4$ and $a \geq 2$.*

Proof Let $v_0^{(i)}, v_1^{(i)}, v_2^{(i)}, \cdots, v_a^{(i)}$ be the vertices of the i^{th} copy of the path of length 'a' where $i = 1, 2, \cdots, b$, $v_0^{(i)} = u$ and $v_a^{(i)} = v$, for all i. Clearly, $|V(P_{a,b})| = ab - b + 2$ and $|E(P_{a,b})| = ab$. Consider a graph $P_{a,b}$ with $a \geq 2$.

Case 1. $b = 2$.

Notice that $P_{a,2}$ is a cycle of length more than 3. By Theorem 2.1, it admits a C-Geometric mean labeling.

Case 2. $b = 3$.

Define $f : V(P_{a,3}) \to \{1, 2, 3, \cdots, 3a + 1\}$ as follows:

$$f(u) = a + 1,$$

$$f(v) = 3a + 1,$$

$$f(v_{a-j}^{(1)}) = \begin{cases} j, & 1 \leq j \leq \lceil \sqrt{3a+1} \rceil - 2, \\ j + 1, & \lceil \sqrt{3a+1} \rceil - 1 \leq j \leq a - 1, \end{cases}$$

$$f(v_j^{(i)}) = a + i - 1 + 2j, \text{ for } 2 \leq i \leq 3,\ 1 \leq j \leq a - 1.$$

Then, the induced edge labeling is obtained as follows:

$$f^*(v_{a-j}^{(1)}v_{a-j-1}^{(1)}) = \begin{cases} j+1, & 1 \leq j \leq \lceil\sqrt{3a+1}\rceil - 2, \\ j+2, & \lceil\sqrt{3a+1}\rceil - 1 \leq j \leq a-2, \end{cases}$$

$$\begin{aligned} f^*(vv_{a-1}^{(1)}) &= \lceil\sqrt{3a+1}\rceil, \\ f^*(uv_1^{(1)}) &= a+1, \end{aligned}$$

$$\begin{aligned} f^*(vv_{a-1}^{(i)}) &= 3a-2+i, \quad \text{for } 2 \leq i \leq 3, \\ f^*(uv_1^{(i)}) &= a+i, \quad \text{for } 2 \leq i \leq 3, \\ f^*(v_j^{(i)}v_{j+1}^{(i)}) &= a+i+2j, \quad \text{for } 2 \leq i \leq 3 \text{ and } 1 \leq j \leq a-2. \end{aligned}$$

Case 3. $b = 4$.

Consider a graph $P_{a,b}$ with $a \geq 3$. Define $f : V(P_{a,4}) \to \{1, 2, 3, \cdots, 4a+1\}$ as follows:

$$\begin{aligned} f(u) &= a+1, \\ f(v) &= 4a+1, \end{aligned}$$

$$f(v_{a-j}^{(1)}) = \begin{cases} j, & 1 \leq j \leq \lceil\sqrt{4a+1}\rceil - 2, \\ j+1, & \lceil\sqrt{4a+1}\rceil - 1 \leq j \leq a-1, \end{cases}$$

$$f(v_j^{(2)}) = \begin{cases} a+3j-1, & 1 \leq j \leq a-1 \text{ and } j \text{ is odd}, \\ a+3j+1, & 1 \leq j \leq a-1 \text{ and } j \text{ is even}, \end{cases}$$

$$f(v_j^{(3)}) = \begin{cases} a+1+3j, & 1 \leq j \leq a-1 \text{ and } j \text{ is odd}, \\ a+3+3j, & 1 \leq j \leq a-1 \text{ and } j \text{ is even} \end{cases}$$

$$f(v_j^{(4)}) = \begin{cases} a+3+3j, & 1 \leq j \leq a-1 \text{ and } j \text{ is odd}, \\ a-1+3j, & 1 \leq j \leq a-1 \text{ and } j \text{ is even}. \end{cases}$$

Then, the induced edge labeling is obtained as follows:

$$f^*(v_{a-j}^{(1)}v_{a-j-1}^{(1)}) = \begin{cases} j+1, & 1 \leq j \leq \lceil\sqrt{4a+1}\rceil - 2, \\ j+2, & \lceil\sqrt{4a+1}\rceil - 1 \leq j \leq a-2, \end{cases}$$

$$\begin{aligned} f^*(v_{a-1}^{(1)}v) &= \lceil\sqrt{4a+1}\rceil, \\ f^*(uv_1^{(1)}) &= a+1, \\ f^*(uv_1^{(i)}) &= a+i, \text{ for } 2 \leq i \leq 4 \\ f^*(v_{a-1}^{(i)}v) &= 4a-3+i, \text{ for } 2 \leq i \leq 4 \text{ and} \end{aligned}$$

$$f^*(v_j^{(i)}v_{j+1}^{(i)}) = \begin{cases} a+2+3j, & i = 2 \text{ and } 1 \leq j \leq a-2, \\ a+7-i+3j & 3 \leq i \leq 4 \text{ and } 1 \leq j \leq a-2. \end{cases}$$

For $a = 2$, a C-Geometric mean labeling of $P_{2,4}$ is as shown in Figure 5.

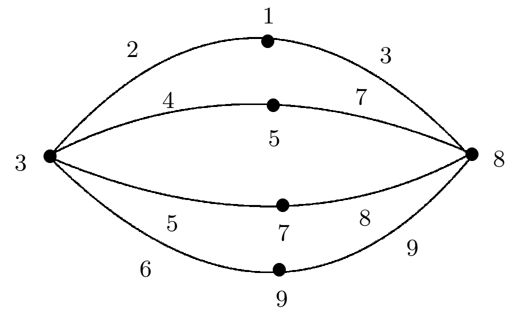

Figure 5 A C-Geometric mean labeling of $P_{2,4}$

Hence, the graph $P_{a,b}$ for $b \leq 4$ admits a C-Geometric mean labeling. Thus the graph $P_{a,b}$ for $b \leq 4$ is a C-Geometric mean graph. □

Theorem 2.10 *A graph P_a^b is a C-Geometric mean graph if $b \leq 3$.*

Proof Let $y_i, x_{ij1}, x_{ij2}, \cdots, x_{iji}, y_{i+1}$ be the vertices of the j^{th} path of i^{th} block of P_a^b, where $1 \leq i \leq a-1$ and $1 \leq j \leq b$. Obviously,

$$V(P_a^b) = \{y_i; 1 \leq i \leq a\} \bigcup \left(\bigcup_{i=1}^{a-1} \bigcup_{j=1}^{b} \{x_{ijk}; 1 \leq k \leq i\} \right)$$

$$E(P_a^b) = \bigcup_{i=1}^{a-1} \{y_i x_{ij1} : 1 \leq j \leq b\} \bigcup \left(\bigcup_{i=1}^{a-1} \bigcup_{j=1}^{b} \{x_{ijk} x_{ij(k+1)}; 1 \leq k \leq i-1\} \right) \bigcup \left(\bigcup_{i=1}^{a-1} \{x_{iji} y_{i+1}; 1 \leq j \leq b\} \right)$$

Hence, $|V(P_a^b)| = \frac{ab(a-1)}{2} + a$ and $|E(P_a^b)| = \frac{b(a-1)(a+2)}{2}$.

Case 1. $b = 2$.

Notice that the graph P_a^2 is $G^*(p_1, p_2, \cdots, p_n)$. Applying Corollary 2.9, P_a^2 is a C-Geometric mean graph for $p_1 \neq 3$.

Case 2. $b = 3$.

Define $f : V(P_a^3) \to \left\{1, 2, 3, \cdots, \frac{3(a-1)(a+2)}{2} + 1\right\}$ as follows:

$$f(y_1) = 3,$$
$$f(y_i) = \frac{3(i-1)(i+2)}{2} + 1, \text{ for } 2 \leq i \leq a,$$
$$f(x_{111}) = 1,$$
$$f(x_{1j1}) = j + 3, \text{ for } 2 \leq j \leq 3,$$
$$f(x_{21k}) = 4k + 5, \text{ for } 1 \leq k \leq 2,$$
$$f(x_{22k}) = 5k + 5, \text{ for } 1 \leq k \leq 2,$$
$$f(x_{23k}) = 13 - k, \text{ for } 1 \leq k \leq 2$$

and for $3 \leq i \leq a-1$,

$$f(x_{ij1}) = \begin{cases} \frac{3(i-1)(i+2)}{2} + 2 + j, & 1 \leq j \leq 2, \\ \frac{3(i-1)(i+2)}{2} + 2j, & j = 3, \end{cases}$$

$$f(x_{ijk}) = \begin{cases} \frac{3(i-1)(i+2)}{2} + 2j + 3k - 1 & 1 \leq j \leq 2, 2 \leq k \leq i-1 \text{ and } k \text{ is even}, \\ \frac{3(i-1)(i+2)}{2} + 3k - 1, & j = 3, 2 \leq k \leq i-1 \text{ and } k \text{ is even}, \\ \frac{3(i-1)(i+2)}{2} + 2j + 3k - 3, & 1 \leq j \leq 3, 2 \leq k \leq i-1 \text{ and } k \text{ is odd}, \\ \frac{3(i-1)(i+2)}{2} + 3k - 1, & j = 1, k = i \text{ and } k \text{ is odd}, \\ \frac{3(i-1)(i+2)}{2} + 3k + j, & 2 \leq j \leq 3, k = i \text{ and } k \text{ is odd}, \\ \frac{3(i-1)(i+2)}{2} + 3k + j + 1, & 1 \leq j \leq 2, k = i \text{ and } k \text{ is even}, \\ \frac{3(i-1)(i+2)}{2} + 3k - 1, & j = 3, k = i \text{ and } k \text{ is even}. \end{cases}$$

Then, the induced edge labeling is as follows:

$$f^*(y_i x_{ij1}) = \frac{3(i-1)(i+2)}{2} + j + 1, \text{ for } 1 \leq j \leq 3 \text{ and } 2 \leq i \leq a-1,$$

$$f^*(y_1 x_{1j1}) = \begin{cases} 2, & j = 1, \\ j + 2, & 2 \leq j \leq 3, \end{cases}$$

$$f^*(x_{1j1} y_2) = \begin{cases} 3, & j = 1, \\ j + 4, & 2 \leq j \leq 3, \end{cases}$$

$$f^*(x_{2j1} x_{2j2}) = \begin{cases} 2j + 9, & 1 \leq j \leq 2, \\ 12, & j = 3, \end{cases}$$

$$f^*(x_{2j2} y_3) = \begin{cases} j + 14, & 1 \leq j \leq 2, \\ 14, & j = 3 \end{cases}$$

and for $3 \leq i \leq a-1$,

$$f^*(x_{ijk} x_{ijk+1}) = \begin{cases} \frac{3(i-1)(i+2)}{2} + 3k + 2j - 1, & 1 \leq k \leq i-1, \\ & \text{and } 1 \leq j \leq 2, \\ \frac{3(i-1)(i+2)}{2} + 3k + 2, & 1 \leq k \leq i-1, \\ & \text{and } j = 3, \end{cases}$$

$$\text{and } f^*(x_{iji} y_{i+1}) = \begin{cases} \frac{3i(i+3)}{2} + j - 2, & 1 \leq j \leq 3 \text{ and } i \text{ is odd}, \\ \frac{3i(i+3)}{2} + j - 1, & 1 \leq j \leq 2 \text{ and } i \text{ is even}, \\ \frac{3i(i+3)}{2} - 1, & j = 3 \text{ and } i \text{ is even}. \end{cases}$$

Hence, f is a C-Geometric mean labeling of P_a^b, for $b \leq 3$. Thus the graph P_a^b for $b \leq 3$ is a C-Geometric mean graph. □

Theorem 2.11 *Let G be a graph obtained from a path by identifying any of its edges by an edge of a cycle and none of the pendent edges is identified by an edge of a cycle of length 3. Then, G is a C-Geometric mean graph.*

Proof Let $v_1, v_2, \cdots, v_p$ be the vertices of the path on p vertices. Let m be the number of cycles are placed in a path in order to get G and the edges of the j^{th} cycle be identified with the edge (v_{i_j}, v_{i_j+1}) of the path having the length n_j and $n_1 \neq 3$ when $i_1 = 1$. For $1 \leq j \leq m$, the vertices of the j^{th} cycle be $v_{i_j,l}, 1 \leq l \leq n_j$ where $v_{i_j,1} = v_{i_j}$ and $v_{i_j,n_j} = v_{i_j+1}$.

Define $f : V(G) \to \left\{1, 2, 3, \cdots, \sum\limits_{j=1}^{m} n_j + p - m\right\}$ as follows:

$$\begin{aligned}
f(v_k) &= k, \text{ for } 1 \leq k \leq i_1,\\
f(v_{i_j}) &= i_j + \sum_{k=1}^{j-1}(n_k - 2) + j - 1, \text{ for } 1 \leq j \leq m,\\
f(v_{i_j+1}) &= f(v_{i_j}) + n_j, \text{ for } 1 \leq j \leq m,\\
f(v_{i_j+k}) &= f(v_{i_j+1}) + k - 1, \text{ for } 2 \leq k \leq i_{j+1} - i_j - 1 \text{ and } 1 \leq j \leq m-1,\\
f(v_{i_m+k+1}) &= f(v_{i_m+k}) + k - 1, \text{ for } 2 \leq k \leq p - i_m
\end{aligned}$$

and for $1 \leq j \leq m$,

$$f(v_{i_j,l}) = \begin{cases} f(v_{i_j}) + l - 1, & 2 \leq l \leq \left\lceil \sqrt{f(v_{i_j})f(v_{i_j+1})} \right\rceil - f(v_{i_j}) - 1,\\ f(v_{i_j}) + l, & \left\lceil \sqrt{f(v_{i_j})f(v_{i_j+1})} \right\rceil - f(v_{i_j}) \leq l \leq n_j - 1. \end{cases}$$

Then, the induced edge labeling is obtained as follows:

$$\begin{aligned}
f^*(v_k v_{k+1}) &= k + 1, \text{ for } 1 \leq k \leq i_1 - 1,\\
f^*(v_{i_j+k} v_{i_j+k+1}) &= v_{i_j+k} + 1, \text{ for } 1 \leq k \leq i_{j+1} - i_j - 1 \text{ and } 1 \leq j \leq m-1,\\
f^*(v_{i_m+k} v_{i_m+k+1}) &= f(v_{i_m+k}) + 1, \text{ for } 1 \leq k \leq p - i_m - 1
\end{aligned}$$

and for $1 \leq j \leq m$,

$$\begin{aligned}
f^*(v_{i_j,l} v_{i_j,l+1}) &= \begin{cases} f(v_{i_j}) + l, & 1 \leq l \leq \left\lceil \sqrt{f(v_{i_j})f(v_{i_j+1})} \right\rceil - f(v_{i_j}) - 1,\\ f(v_{i_j}) + l + 1, & \left\lceil \sqrt{f(v_{i_j})f(v_{i_j+1})} \right\rceil - f(v_{i_j}) \leq l \leq n_j - 1, \end{cases}\\
f^*(v_{i_j} v_{i_{j+1}}) &= \left\lceil \sqrt{f(v_{i_j})f(v_{i_{j+1}})} \right\rceil, \text{ for } 1 \leq j \leq m.
\end{aligned}$$

Hence, the graph G admits a C-Geometric mean labeling. Thus the graph G is obtained from a path by identifying any of its edges by an edge of a cycle and none of the pendent edges is identified by an edge of a cycle of length 3, is a C-Geometric mean graph. □

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