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A Note on Detour Radial Signed Graphs

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Abstract: In this paper we introduced a new notion detour radial signed graph of a signed graph and its properties are obtained. Also, we obtained the structural characterization of detour radial signed graphs. Further, we presented some switching equivalent characterizations.

Key Words: Signed graphs, balance, switching, detour radial signed graph, radial signed graph.

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§1. Introduction

For standard terminology and notion in graph theory, we refer the reader to the text-book of Harary [2]. The non-standard will be given in this paper as and when required.

Let $G = (V, E)$ be a connected graph.For any two vertices $u, v \in V(G)$, the detour distance $D(u, v)$ is the length of the longest $u - v$ path in G. The eccentricity $e(u)$ of a vertex u is the distance to a vertex farthest from u. The radius $r(G)$ of G is defined by

$$r(G) = \min\{e(u) : u \in G\}.$$

For any vertex u in G, the detour eccentricity $D_e(u)$ of u is the detour distance to a vertex farthest from u. The detour radius $D_r(G)$ of G is defined by $D_r(G) = \min\{D_e(u) : u \in G\}$. The diameter $d(G)$ of G is defined by $d(G) = \max\{e(u) : u \in G\}$ and the detour diameter $D_d(G)$ of G is $\max\{D_e(u) : u \in G\}$.

The detour radial graph $\mathcal{DR}(G)$ of $G = (V, E)$ is a graph with $V(\mathcal{DR}(G)) = V(G)$ and any two vertices u and v in $\mathcal{DR}(G)$ are joined by an edge if and only if $D(u, v) = D_r(G)$. This concept were introduced by Ganeshwari and Pethanachi Selvam [1].

To model individuals' preferences towards each other in a group, Harary [3] introduced the concept of signed graphs in 1953. A signed graph $S = (G, \sigma)$ is a graph $G = (V, E)$ whose edges are labeled with positive and negative signs (i.e., $\sigma : E(G) \to \{+, -\}$). The vertexes of a graph represent people and an edge connecting two nodes signifies a relationship between individuals. The signed graph captures the attitudes between people, where a positive (negative edge) represents liking (disliking). An unsigned graph is a signed graph with the signs removed. Similar to an unsigned graph, there are many active

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areas of research for signed graphs. For more new notions on signed graphs refer the papers.

The sign of a cycle (this is the edge set of a simple cycle) is defined to be the product of the signs of its edges; in other words, a cycle is positive if it contains an even number of negative edges and negative if it contains an odd number of negative edges. A signed graph S is said to be balanced if every cycle in it is positive. A signed graph S is called totally unbalanced if every cycle in S is negative. A chord is an edge joining two non adjacent vertices in a cycle.

A *marking* of S is a function $\zeta : V(G) \to \{+, -\}$. Given a signed graph S one can easily define a marking ζ of S as follows:

For any vertex $v \in V(S)$,

$$\zeta(v) = \prod_{uv \in E(S)} \sigma(uv),$$

the marking ζ of S is called canonical marking *of S.*

The following are the fundamental results about balance, the second being a more advanced form of the first. Note that in a bipartition of a set, $V = V_1 \cup V_2$, the disjoint subsets may be empty.

Theorem 1.1 *A signed graph S is balanced if and only if either of the following equivalent conditions is satisfied:*

(i) Its vertex set has a bipartition $V = V_1 \cup V_2$ such that every positive edge joins vertices in V_1 or in V_2, and every negative edge joins a vertex in V_1 and a vertex in V_2; (Harary [3])

(ii) There exists a marking μ of its vertices such that each edge uv in Γ satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. (Sampathkumar [4])

Switching S with respect to a marking ζ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs. The resulting signed graph $S_\zeta(S)$ is said switched signed graph. A signed graph S is called to switch to another signed graph S' written $S \sim S'$, whenever their exists a marking ζ such that $S_\zeta(S) \cong S'$, where $\cong$ denotes the usual equivalence relation of isomorphism in the class of signed graphs. Hence, if $S \sim S'$, we shall say that S and S' are switching equivalent. Two signed graphs S_1 and S_2 are signed isomorphic (written $S_1 \cong S_2$) if there is a one-to-one correspondence between their vertex sets which preserve adjacency as well as sign.

Two signed graphs $S_1 = (G_1, \sigma_1)$ and $S_2 = (G_2, \sigma_2)$ are said to be *weakly isomorphic* (see [21]) or *cycle isomorphic* (see [22]) if there exists an isomorphism $\phi : G_1 \to G_2$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. More results on signed graphs can be found in references [4-22]. For example, the following result is well known.

Theorem 1.2 (T. Zaslavsky, [22]) *Given a graph G, any two signed graphs in $\psi(G)$, where $\psi(G)$ denotes the set of all the signed graphs possible for a graph G, are switching equivalent if and only if they are cycle isomorphic.*

§2. Detour Radial Signed Graphs

Motivated by the existing definition of complement of a signed graph, we now extend the notion of detour radial graphs to signed graphs as follows: The *detour radial signed graph* $\mathcal{DR}(S)$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $\mathcal{DR}(G)$ and sign of any edge uv is $\mathcal{DR}(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S. Further, a signed graph $S = (G, \sigma)$ is called detour radial signed graph, if $S \cong \mathcal{DR}(S')$ for some signed graph S'. The following result restricts the class of detour radial graphs.

Theorem 2.1 *For any signed graph $S = (G, \sigma)$, its detour radial signed graph $\mathcal{DR}(S)$ is balanced.*

Proof Since sign of any edge $e = uv$ in $\mathcal{DR}(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S, by Theorem 1.1, $\mathcal{DR}(S)$ is balanced. □

For any positive integer k, the k^{th} iterated detour radial signed graph, $\mathcal{DK}^k(S)$ of S is defined as follows:

$$\mathcal{DR}^0(S) = S,\ \mathcal{DR}^k(S) = \mathcal{DR}(\mathcal{DR}^{k-1}(S)).$$

Corollary 2.2 *For any signed graph $S = (G, \sigma)$ and for any positive integer k, $\mathcal{DR}^k(S)$ is balanced.*

The following result characterize signed graphs which are detour radial signed graphs.

Theorem 2.3 *A signed graph $S = (G, \sigma)$ is a detour radial signed graph if, and only if, S is balanced signed graph and its underlying graph G is a detour radial graph.*

Proof Suppose that S is balanced and G is a detour radial graph. Then there exists a graph G' such that $\mathcal{DR}(G') \cong G$. Since S is balanced, by Theorem 1.1, there exists a marking ζ of G such that each edge uv in S satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. Now consider the signed graph $S' = (G', \sigma')$, where for any edge e in G', $\sigma'(e)$ is the marking of the corresponding vertex in G. Then clearly, $\mathcal{DR}(S') \cong S$. Hence S is a detour radial signed graph.

Conversely, suppose that $S = (G, \sigma)$ is a detour radial signed graph. Then there exists a signed graph $S' = (G', \sigma')$ such that $\mathcal{DR}(S') \cong S$. Hence, G is the detour radial graph of G' and by Theorem 2.1, S is balanced. □

In [1], the authors characterizes the graphs $G = (V, E)$ such that $G \cong \mathcal{DR}(G)$.

Theorem 2.4 *Let $G = (V, E)$ be a graph with atleast one cycle which covers all the vertices of G. Then G and the detour radial graph $\mathcal{DR}(G)$ are isomorphic if and only if G is isomorphic to either K_n or C_n or $K_{m,n}$ with $m = n$.*

In view of the above result, we now characterize the signed graphs such that the detour radial signed graph and its corresponding signed graph are switching equivalent.

Theorem 2.5 *For any signed graph $S = (G, \sigma)$ and its underlying graph G contains atleast one cycle which covers all the vertices. Then S and the detour radial signed graph $\mathcal{DR}(S)$ are cycle isomorphic if and only if the underlying of S satisfies the conditions of Theorem 2.4 and S is balanced.*

Proof Suppose $\mathcal{RD}(S) \sim S$. This implies, $\mathcal{DR}(G) \cong G$ and hence by Theorem 2.4, we see that the graph G satisfies the conditions in Theorem 2.4. Now, if S is any signed graph with underlying graph contains at least one Hamilton cycle and satisfies the conditions of Theorem 2.4. Then $\mathcal{DR}(S)$ is balanced and hence if S is unbalanced and its detour radial signed graph $\mathcal{DR}(S)$ being balanced can not be switching equivalent to S in accordance with Theorem 1.2. Therefore, S must be balanced.

Conversely, suppose that S balanced signed graph with the underlying graph G satisfies the conditions of Theorem 2.4. Then, since $\mathcal{DR}(S)$ is balanced as per Theorem 2.1 and since $\mathcal{DR}(G) \cong G$ by Theorem 2.4, the result follows from Theorem 1.2 again. □

In [5], P.S.K.Reddy introduced the notion radial signed graph of a signed graph and proved some results.

Theorem 2.6 *For any signed graph $S = (G, \sigma)$, its radial signed graph $\mathcal{R}(S)$ is balanced.*

In [1], the authors remarked that $\mathcal{DR}(G)$ and $\mathcal{R}(G)$ are isomorphic, if G is any cycle of odd length. We now characterize the signed graphs S such that $\mathcal{DR}(S) \sim \mathcal{R}(S)$.

Theorem 2.7 *For any signed graph $S = (G, \sigma)$, $\mathcal{DR}(S) \sim \mathcal{R}(S)$ if, and only if, $G \cong C_n$, where n is odd.*

Proof Suppose that $\mathcal{DR}(S) \sim \mathcal{R}(S)$. Then clearly, $\mathcal{DR}(G) \sim \mathcal{R}(G)$. Hence, G is any cycle of odd length.

Conversely, suppose that S is a signed graph whose underlying graph G is C_n, where n is odd. Then, $\mathcal{DR}(G) \cong \mathcal{R}(G)$. Since for any signed graph S, both $\mathcal{DR}(S)$ and $\mathcal{R}(S)$ are balanced, the result follows by Theorem 1.2. □

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