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A Generalization of Some Integral Inequalities for Multiplicatively P-Functions

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Abstract: In this paper, using identity for differentiable functions we can derive a general inequality containing all of the midpoint, trapezoid and Simpson inequalities for functions whose derivatives in absolute value at certain power are multiplicatively P-functions. Some applications to special means of real numbers are also given.

Key Words: Convex function, multiplicatively P-functions, Simpson's inequality, Hermite-Hadamard's inequality, midpoint inequality, trapezoid ineqaulity.

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§1. Introduction

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then the function f is said to be concave on interval $I \neq \emptyset$.

This definition is well known in the literature. It is well known that theory of convex sets and convex functions play an important role in mathematics and the other pure and applied sciences. In recent years, the concept of convexity has been extended and generalized in various directions using novel and innovative techniques.

Theorem 1.1 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions.

The classical Hermite-Hadamard integral inequality provides estimates of the mean value of a continuous convex or concave function. See [2-4, 7, 9], for the results of the generalization, improvement and extension of the famous integral inequality (1.1).

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The following inequality is well known in the literature as Simpson's inequality:

Let $f : [a,b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $\left\| f^{(4)} \right\|_\infty = \sup_{x\in(a,b)} \left| f^{(4)}(x) \right| < \infty$. Then the following inequality holds:

$$\left| \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx \right| \le \frac{1}{2880}\left\| f^{(4)} \right\|_\infty (b-a)^2 .$$

In recent years many researchers have studied error estimations for Simpson's inequality; for refinements, counterparts, generalizations and new Simpson's type inequalities, see [1,10-12].

In this paper, in order to provide a unified approach to midpoint inequality, trapezoid inequality and Simpson's inequality for functions whose derivatives in absolute value at certain power are multiplicatively P-functions, we derive a general integral identity for differentiable functions. Finally some applications for special means of real numbers are provided.

Definition 1.2 *Let $I \neq \emptyset$ be an interval in $\mathbb{R}$. The function $f : I \to [0,\infty)$ is said to be multiplicatively P-function (or* log*-P-function), if the inequality*

$$f(tx+(1-t)y) \le f(x)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

In [8], some inequalities of Hermite-Hadamard type for differentiable multiplicatively P-functions were presented as follows.

Theorem 1.3 *Let the function $f : I \to [1,\infty)$,be a multiplicatively P-function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$, then the following inequalities hold:*

(i) $f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)f(a+b-x)\,dx \le [f(a)f(b)]^2$;

(ii) $f\left(\frac{a+b}{2}\right) \le f(a)f(b)\frac{1}{b-a}\int_a^b f(x)dx \le [f(a)f(b)]^2$.

In [5], İşcan obtained inequalities for differentiable convex functions using following lemma.

Lemma 1.4 *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$ and $\alpha, \lambda \in [0,1]$. Then the following equality holds:*

$$\lambda(\alpha f(a) + (1-\alpha) f(b)) + (1-\lambda) f(\alpha a + (1-\alpha) b) - \frac{1}{b-a}\int_a^b f(x)dx \tag{1.2}$$
$$= (b-a)\left[\int_0^{1-\alpha} (t-\alpha\lambda) f'(tb+(1-t)a)\,dt + \int_{1-\alpha}^1 (t-1+\lambda(1-\alpha)) f'(tb+(1-t)a)\,dt \right].$$

§2. Main Results

In this section, in order to prove our main theorems, we shall use the identity (1.4).

Theorem 2.1 *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$ and $\alpha, \lambda \in [0,1]$. If $|f'|^q$ is multiplicatively P-function on $[a,b]$, $q \geq 1$, then the following inequality holds:*

$$\left|\lambda\left(\alpha f(a)+(1-\alpha) f(b)\right)+(1-\lambda) f(\alpha a+(1-\alpha) b)-\frac{1}{b-a}\int_a^b f(x)dx\right| \tag{2.1}$$
$$\leq \begin{cases} (b-a)\left|f'(a)\right|\left|f'(b)\right|\left[\gamma_2+\upsilon_2\right], & \alpha\lambda \leq 1-\alpha \leq 1-\lambda(1-\alpha) \\ (b-a)\left|f'(a)\right|\left|f'(b)\right|\left[\gamma_2+\upsilon_1\right] & \alpha\lambda \leq 1-\lambda(1-\alpha) \leq 1-\alpha \\ (b-a)\left|f'(a)\right|\left|f'(b)\right|\left[\gamma_1+\upsilon_2\right] & 1-\alpha \leq \alpha\lambda \leq 1-\lambda(1-\alpha) \end{cases}$$

where

$$\gamma_1 = (1-\alpha)\left[\alpha\lambda-\frac{(1-\alpha)}{2}\right], \ \gamma_2=(\alpha\lambda)^2-\gamma_1 , \tag{2.2}$$
$$\upsilon_1 = \frac{1-(1-\alpha)^2}{2}-\alpha\left[1-\lambda(1-\alpha)\right], \tag{2.3}$$
$$\upsilon_2 = \frac{1+(1-\alpha)^2}{2}-(\lambda+1)(1-\alpha)\left[1-\lambda(1-\alpha)\right],$$

Proof Suppose that $q \geq 1$. From Lemma 1.4, the well known power mean inequality and property of multiplicatively P-function of $|f'|^q$ on $[a,b]$, that is

$$\left|f'(tb+(1-t)a)\right|^q \leq \left|f'(b)\right|^q\left|f'(a)\right|^q, \quad t\in[0,1],$$

we have

$$\left|\lambda\left(\alpha f(a)+(1-\alpha) f(b)\right)+(1-\lambda) f(\alpha a+(1-\alpha) b)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq (b-a)\left[\int_0^{1-\alpha}|t-\alpha\lambda|\left|f'(tb+(1-t)a)\right|dt+\int_{1-\alpha}^1|t-1+\lambda(1-\alpha)|\left|f'(tb+(1-t)a)\right|dt\right]$$
$$\leq (b-a)\left\{\left(\int_0^{1-\alpha}|t-\alpha\lambda|dt\right)^{1-\frac{1}{q}}\left(\int_0^{1-\alpha}|t-\alpha\lambda|\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{1-\alpha}^1|t-1+\lambda(1-\alpha)|dt\right)^{1-\frac{1}{q}}\left(\int_{1-\alpha}^1|t-1+\lambda(1-\alpha)|\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right\}$$
$$\leq (b-a)\left|f'(a)\right|\left|f'(b)\right|\left\{\int_0^{1-\alpha}|t-\alpha\lambda|dt+\int_{1-\alpha}^1|t-1+\lambda(1-\alpha)|dt\right\}. \tag{2.4}$$

Hence, by simple computation

$$\int_0^{1-\alpha}|t-\alpha\lambda|dt=\begin{cases}\gamma_2, & \alpha\lambda\leq 1-\alpha\\ \gamma_1, & \alpha\lambda\geq 1-\alpha\end{cases}, \tag{2.5}$$

$$\gamma_1=(1-\alpha)\left[\alpha\lambda-\frac{(1-\alpha)}{2}\right], \ \gamma_2=(\alpha\lambda)^2-\gamma_1 ,$$

$$\int_{1-\alpha}^{1} |t-1+\lambda(1-\alpha)|\,dt = \begin{cases} \upsilon_1, & 1-\lambda(1-\alpha) \leq 1-\alpha \\ \upsilon_2, & 1-\lambda(1-\alpha) \geq 1-\alpha \end{cases}, \tag{2.6}$$

$$\begin{aligned} \upsilon_1 &= \frac{1-(1-\alpha)^2}{2} - \alpha\left[1-\lambda(1-\alpha)\right], \\ \upsilon_2 &= \frac{1+(1-\alpha)^2}{2} - (\lambda+1)(1-\alpha)\left[1-\lambda(1-\alpha)\right], \end{aligned}$$

$$\begin{aligned} \int_{0}^{1-\alpha} |t-\alpha\lambda| \left|f'(tb+(1-t)a)\right|^q dt &\leq \int_{0}^{1-\alpha} |t-\alpha\lambda| \left|f'(b)\right|^q \left|f'(a)\right|^q dt \\ &= \left|f'(b)\right|^q \left|f'(a)\right|^q \int_{0}^{1-\alpha} |t-\alpha\lambda|\,dt \\ &= \begin{cases} \gamma_2 \left|f'(a)\right|^q \left|f'(b)\right|^q, & \alpha\lambda \leq 1-\alpha \\ \gamma_1 \left|f'(a)\right|^q \left|f'(b)\right|^q, & \alpha\lambda \geq 1-\alpha \end{cases}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\int_{1-\alpha}^{1} |t-1+\lambda(1-\alpha)| \left|f'(tb+(1-t)a)\right|^q dt \\ &\leq \int_{1-\alpha}^{1} |t-1+\lambda(1-\alpha)| \left|f'(b)\right|^q \left|f'(a)\right|^q dt \\ &= \left|f'(b)\right|^q \left|f'(a)\right|^q \int_{1-\alpha}^{1} |t-1+\lambda(1-\alpha)|\,dt \\ &= \begin{cases} \upsilon_1 \left|f'(b)\right|^q \left|f'(a)\right|^q, & 1-\lambda(1-\alpha) \leq 1-\alpha \\ \upsilon_2 \left|f'(b)\right|^q \left|f'(a)\right|^q, & 1-\lambda(1-\alpha) \geq 1-\alpha \end{cases} \end{aligned} \tag{2.8}$$

Thus, using (2.5)-(2.8) in (2.4), we obtain the inequality (2.1). This completes the proof. □

Corollary 2.2 *Let the assumptions of Theorem* 2.1 *hold. Then for* $\alpha = \frac{1}{2}$ *and* $\lambda = \frac{1}{3}$, *from the inequality* (2.1) *we get the following Simpson type inequality*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{5}{36}(b-a)\left|f'(a)\right|\left|f'(b)\right|. \tag{2.9}$$

Corollary 2.3 *Let the assumptions of Theorem* 2.1 *hold. Then for* $\alpha = \frac{1}{2}$ *and* $\lambda = 0$, *from the inequality* (2.1) *we get the following midpoint inequality*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left|f'(a)\right|\left|f'(b)\right| \tag{2.10}$$

Remark 2.4 We note that the result obtained in Corollary 2.3 coincides with the result in [8].

Corollary 2.5 *Let the assumptions of Theorem 2.1 hold. Then for $\alpha = \frac{1}{2}$ and $\lambda = 1$, from the inequality (2.1) we get the following trapezoid inequality*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left|f'(a)\right|\left|f'(b)\right|$$

Remark 2.6 We note that the result obtained in Corollary 2.5 coincides with the result in [8].

Using Lemma 1.4 we shall give another result for multiplicatively P-functions as follows.

Theorem 2.7 *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$ and $\alpha, \lambda \in [0,1]$. If $|f'|^q$ is multiplicatively P-function on $[a,b]$, $q > 1$, then the following inequality holds:*

$$\left|\lambda\left(\alpha f(a) + (1-\alpha) f(b)\right) + (1-\lambda) f(\alpha a + (1-\alpha) b) - \frac{1}{b-a}\int_a^b f(x)dx\right| \tag{2.11}$$

$$\leq (b-a)\left|f'(a)\right|\left|f'(b)\right|\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \begin{cases} (1-\alpha)^{\frac{1}{q}}\theta_1^{\frac{1}{p}} + \alpha^{\frac{1}{q}}\theta_2^{\frac{1}{p}}, & \alpha\lambda \leq 1-\alpha \leq 1-\lambda(1-\alpha) \\ (1-\alpha)^{\frac{1}{q}}\theta_1^{\frac{1}{p}} + \alpha^{\frac{1}{q}}\theta_4^{\frac{1}{p}}, & \alpha\lambda \leq 1-\lambda(1-\alpha) \leq 1-\alpha \\ (1-\alpha)^{\frac{1}{q}}\theta_3^{\frac{1}{p}} + \alpha^{\frac{1}{q}}\theta_2^{\frac{1}{p}}, & 1-\alpha \leq \alpha\lambda \leq 1-\lambda(1-\alpha) \end{cases}$$

where

$$\begin{aligned} \theta_1 &= (\alpha\lambda)^{p+1} + (1-\alpha-\alpha\lambda)^{p+1}, \quad & \theta_2 &= [\lambda(1-\alpha)]^{p+1} + [\alpha - \lambda(1-\alpha)]^{p+1} \\ \theta_3 &= (\alpha\lambda)^{p+1} - (1-\alpha-\alpha\lambda)^{p+1}, \quad & \theta_4 &= [\lambda(1-\alpha)]^{p+1} - [\alpha - \lambda(1-\alpha)]^{p+1} \end{aligned} \tag{2.12}$$

and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof From Lemma 1.4 and by Hölder's integral inequality, we have

$$\begin{aligned} &\left|\lambda\left(\alpha f(a) + (1-\alpha) f(b)\right) + (1-\lambda) f(\alpha a + (1-\alpha) b) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\leq (b-a)\left[\int_0^{1-\alpha} |t-\alpha\lambda|\left|f'(tb+(1-t)a)\right|dt + \int_{1-\alpha}^1 |t-1+\lambda(1-\alpha)|\left|f'(tb+(1-t)a)\right|dt\right] \\ &\leq (b-a)\left\{\left(\int_0^{1-\alpha} |t-\alpha\lambda|^p dt\right)^{\frac{1}{p}}\left(\int_0^{1-\alpha}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right. \\ &\quad \left. + \left(\int_{1-\alpha}^1 |t-1+\lambda(1-\alpha)|^p dt\right)^{\frac{1}{p}}\left(\int_{1-\alpha}^1\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right\} \\ &\leq (b-a)\left|f'(a)\right|\left|f'(b)\right|\left[(1-\alpha)^{\frac{1}{q}}\left(\int_0^{1-\alpha}|t-\alpha\lambda|^p dt\right)^{\frac{1}{p}}\right. \end{aligned}$$

$$+\alpha^{\frac{1}{q}}\left(\int_{1-\alpha}^{1}|t-1+\lambda(1-\alpha)|^{p}\,dt\right)^{\frac{1}{p}}\Bigg]. \tag{2.13}$$

By simple computation

$$\int_{0}^{1-\alpha}|t-\alpha\lambda|^{p}\,dt=\begin{cases}\frac{(\alpha\lambda)^{p+1}+(1-\alpha-\alpha\lambda)^{p+1}}{p+1}, & \alpha\lambda\leq 1-\alpha\\ \frac{(\alpha\lambda)^{p+1}-(\alpha\lambda-1+\alpha)^{p+1}}{p+1}, & \alpha\lambda\geq 1-\alpha\end{cases}, \tag{2.14}$$

and

$$\int_{1-\alpha}^{1}|t-1+\lambda(1-\alpha)|^{p}\,dt=\begin{cases}\frac{[\lambda(1-\alpha)]^{p+1}+[\alpha-\lambda(1-\alpha)]^{p+1}}{p+1}, & 1-\alpha\leq 1-\lambda(1-\alpha)\\ \frac{[\lambda(1-\alpha)]^{p+1}-[\lambda(1-\alpha)-\alpha]^{p+1}}{p+1}, & 1-\alpha\geq 1-\lambda(1-\alpha)\end{cases}. \tag{2.15}$$

Thus, using (2.15) in (2.13), we obtain the inequality (2.11). This completes the proof. □

Corollary 2.8 *Let the assumptions of Theorem* 2.7 *hold. Then for* $\alpha=\frac{1}{2}$ *and* $\lambda=\frac{1}{3}$, *from the inequality* (2.11) *we get the following Simpson type inequality*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|\leq\frac{b-a}{6}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left|f'(a)\right|\left|f'(b)\right|. \tag{2.16}$$

Corollary 2.9 *Let the assumptions of Theorem* 2.7 *hold. Then for* $\alpha=\frac{1}{2}$ *and* $\lambda=0$, *from the inequality* (2.11) *we get the following midpoint inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|\leq\frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left|f'(a)\right|\left|f'(b)\right|$$

Remark 2.10 Notice that the result obtained in Corollary 2.9 coincides with the result in [8].

Corollary 2.11 *Let the assumptions of Theorem* 2.7 *hold. Then for* $\alpha=\frac{1}{2}$ *and* $\lambda=1$, *from the inequality* (2.11) *we get the following trapezoid inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|\leq\frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left|f'(a)\right|\left|f'(b)\right|.$$

Remark 2.12 Notice that the result obtained in Corollary 2.11 coincides with the result in [8].

§3. Some Applications for Special Means

Let us recall the following special means of the two nonnegative numbers a and b with $\alpha\in[0,1]$:

(1) The weighted arithmetic mean

$$A_{\alpha}=A_{\alpha}(a,b):=\alpha a+(1-\alpha)b,\ \ a,b\geq 0.$$

(2) The unweighted arithmetic mean

$$A = A(a,b) := \frac{a+b}{2},\ a,b \geq 0.$$

(3) The weighted geometric mean

$$G_\alpha = G_\alpha(a,b) := a^\alpha b^{1-\alpha},\ a,b > 0.$$

(4) The unweighted geometric mean

$$G = G(a,b) := \sqrt{ab},\ a,b > 0.$$

(5) The Logarithmic mean

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a},\quad a \neq b,\ a,b > 0.$$

(6) Then n-Logarithmic mean

$$L_n = L_n(a,b) := \left(\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right)^{\frac{1}{n}},\ n \in \mathbb{Z}\setminus\{-1,0\},\ a,b > 0,\ a \neq b.$$

Proposition 3.1 *Let $a,b \in \mathbb{R}$ with $0 < a < b$ and $n \in \mathbb{Z}^+ \cup \{0\}$. Then, for $\alpha, \lambda \in [0,1]$ and $q \geq 1$, we have the following inequality:*

$$\begin{aligned} &|\lambda A_\alpha(a^n,b^n) + (1-\lambda)A_\alpha^n - L_n^n| \\ &\leq \begin{cases} (b-a)n^2(ab)^{n-1}[\gamma_2+\upsilon_2] & \alpha\lambda \leq 1-\alpha \leq 1-\lambda(1-\alpha) \\ (b-a)n^2(ab)^{n-1}[\gamma_2+\upsilon_1] & \alpha\lambda \leq 1-\lambda(1-\alpha) \leq 1-\alpha \\ (b-a)n^2(ab)^{n-1}[\gamma_1+\upsilon_2] & 1-\alpha \leq \alpha\lambda \leq 1-\lambda(1-\alpha) \end{cases}, \end{aligned}$$

where $\gamma_1,\ \gamma_2,\ \upsilon_1,\ \ \upsilon_2$, numbers are defined as in $(2.2)-(2.3)$.

Proof This assertion immediately follows from Theorem 2.1 in the case of $f(x) = x^n$, $x \in [1,\infty)$, $n \in \mathbb{Z}^+ \cup \{0\}$. □

Proposition 3.2 *Let $a,b \in \mathbb{R}$ with $0 < a < b$ and $n \in \mathbb{Z}^+ \cup \{0\}$. Then, for $\alpha, \lambda \in [0,1]$ and $q > 1$, we have the following inequality:*

$$|\lambda A_\alpha(a^n,b^n) + (1-\lambda)A_\alpha^n - L_n^n| \leq (b-a)n^2G^{2n-2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$

$$\times \begin{cases} \left[(1-\alpha)^{\frac{1}{q}}\theta_1^{\frac{1}{p}} + \alpha^{\frac{1}{q}}\theta_2^{\frac{1}{p}}\right], & \alpha\lambda \leq 1-\alpha \leq 1-\lambda(1-\alpha) \\ \left[(1-\alpha)^{\frac{1}{q}}\theta_1^{\frac{1}{p}} + \alpha^{\frac{1}{q}}\theta_4^{\frac{1}{p}}\right], & \alpha\lambda \leq 1-\lambda(1-\alpha) \leq 1-\alpha \\ \left[(1-\alpha)^{\frac{1}{q}}\theta_3^{\frac{1}{p}} + \alpha^{\frac{1}{q}}\theta_2^{\frac{1}{p}}\right], & 1-\alpha \leq \alpha\lambda \leq 1-\lambda(1-\alpha) \end{cases},$$

where $\theta_1,\theta_2,\theta_3,\theta_4$ numbers are defined as in (2.12).

Proof This assertion immediately follows from Theorem 2.7 in the case of $f(x) = x^n$, $x \in [1,\infty)$, $n \in \mathbb{Z}^+ \cup \{0\}$. □

Proposition 3.3 *Let $a,b\in\mathbb{R}$ with $0<a<b$. Then, for $\alpha,\lambda\in[0,1]$ and $q\geq 1$, we have the following inequality:*

$$\left|A_\lambda\left(A_\alpha\left(e^a,e^b\right),G_\alpha\left(e^a,e^b\right)\right)-L\left(e^a,e^b\right)\right|$$
$$\leq\begin{cases}(b-a)\,e^{2A}\left[\gamma_2+\upsilon_2\right] & \alpha\lambda\leq 1-\alpha\leq 1-\lambda(1-\alpha)\\ (b-a)\,e^{2A}\left[\gamma_2+\upsilon_1\right] & \alpha\lambda\leq 1-\lambda(1-\alpha)\leq 1-\alpha\\ (b-a)\,e^{2A}\left[\gamma_1+\upsilon_2\right] & 1-\alpha\leq\alpha\lambda\leq 1-\lambda(1-\alpha)\end{cases},$$

where $\gamma_1,\ \gamma_2,\ \upsilon_1,\ \ \upsilon_2,$ numbers are defined as in (2.2) − (2.3).

Proof The assertion follows from Theorem 2.1 in the case of $f(x)=e^x,\ x\in[0,\infty)$. □

Proposition 3.4 *Let $a,b\in\mathbb{R}$ with $0<a<b$. Then, for $\alpha,\lambda\in[0,1]$ and $q>1$, we have the following inequality:*

$$\left|A_\lambda\left(A_\alpha\left(e^a,e^b\right),G_\alpha\left(e^a,e^b\right)\right)-L\left(e^a,e^b\right)\right|\leq(b-a)\,e^{2A}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$
$$\times\begin{cases}\left[(1-\alpha)^{\frac{1}{q}}\,\theta_1^{\frac{1}{p}}+\alpha^{\frac{1}{q}}\theta_2^{\frac{1}{p}}\right], & \alpha\lambda\leq 1-\alpha\leq 1-\lambda(1-\alpha)\\ \left[(1-\alpha)^{\frac{1}{q}}\,\theta_1^{\frac{1}{p}}+\alpha^{\frac{1}{q}}\theta_4^{\frac{1}{p}}\right], & \alpha\lambda\leq 1-\lambda(1-\alpha)\leq 1-\alpha\\ \left[(1-\alpha)^{\frac{1}{q}}\,\theta_3^{\frac{1}{p}}+\alpha^{\frac{1}{q}}\theta_2^{\frac{1}{p}}\right], & 1-\alpha\leq\alpha\lambda\leq 1-\lambda(1-\alpha)\end{cases},$$

where $\theta_1,\theta_2,\theta_3,\theta_4$ numbers are defined as in (2.12).

Proof The assertion follows from Theorem 2.7 in the case of $f(x)=e^x,\ x\in[0,\infty)$. □

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