INTERNATIONAL JOURNAL OF

MATHEMATICAL COMBINATORICS

International Journal of

Mathematical Combinatorics

(www.mathcombin.com)

Edited By

The Madis of Chinese Academy of Sciences and

Academy of Mathematical Combinatorics & Applications, USA

June,   2017

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Famous Words:

I want to bring out the secrets of nature and apply them for the happiness of man. I don't know of any better service to offer for the short time we are in the world.

By Thomas Edison, an American inventor.

A New Approach on the Striction Curves Belonging to

Bertrandian Frenet Ruled Surfaces

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Abstract: In this study, we think six special ruled surfaces associated to the Bertrand curves pair $\{\alpha, \alpha^*\}$. We describe Bertrandian Frenet ruled surfaces and striction curves of these surfaces are expressed, as depending on the angle between the tangent vectors of the Bertrand curves pair $\{\alpha, \alpha^*\}$. Also, we examined the situation of the tangent vectors belonging to Striction curves of Frenet and Bertrandian Frenet ruled surfaces.

Key Words: Striction curves, Bertrand curves pair, ruled surfaces, Bertrandian Frenet ruled surface, Frenet ruled surface.

AMS(2010): 53A04, 53A05.

§1. Introduction and Preliminaries

A surface is told to be ruled if it is generated by moving a direct line continuously in Euclidean space [8]. Ruled surfaces are one of the simplest objects in geometric modeling. The basis notions on ruled surfaces in are given in [4]. A ruled surface is a surface swept out by a straight line L moving along a curve $\alpha(u) \in \mathbb{R}^3$. The various positions of the generating line L are called the rulings of the surface. Such a surface always has a ruled parametrization

$$\varphi(u, v) = \alpha(u) + vX(u), \ \ u \in I, \ \ v \in \mathbb{R} \tag{1.1}$$

We call α base curve and X the director curve, although X is usually pictured as a vector field on α pointing along the line L. The striction point on a ruled surface is the foot of the common normal between two consecutive generators. The set of striction points defines the striction curve given by ([2])

$$c(s) = \alpha(s) - \frac{\langle \alpha', X' \rangle}{\langle X', X' \rangle} X. \tag{1.2}$$

Bertrand curves discovered by J. Bertrand in 1850 are one of the important and interesting

---

topics of classical special curve theory. These well known properties of Bertrand curves in Euclidean 3-space was extended by L. R. Pears in [9]. The set, whose elements are frame vectors and curvatures of a curve α, is called Frenet-Serret apparatus of the curves. Let Frenet vector fields be $V_1(s), V_2(s), V_3(s)$ of α and let the first and second curvatures of the curve $\alpha(s)$ be $k_1(s)$ and $k_2(s)$, respectively. The quantities $\{V_1, V_2, V_3, k_1, k_2\}$ are collectively Frenet-Serret apparatus of the curves, ([3], [6]). The Frenet formulae are known as

$$
\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.
$$

Let $\alpha : I \to \mathbb{E}^3$ and $\alpha^* : I \to \mathbb{E}^3$ be the C^2- class differentiable two curves and let $V_1(s), V_2(s), V_3(s)$ and $V_1^*(s), V_2^*(s), V_3^*(s)$ be the Frenet frames of the curves α and α^*, respectively. If the principal normal vector V_2 of the curve α is linearly dependent on the principal normal vector V_2^* of the curve α^*, then the pair $\{\alpha, \alpha^*\}$ are called Bertrand curves pair, [4], [7]. Also α^* is called Bertrand mate. If the curve α^* is Bertrand mate of α, then we may write that

$$
\alpha^*(s) = \alpha(s) + \lambda V_2(s). \tag{1.3}
$$

If the curve α^* is Bertrand mate $\alpha(s)$, then we have that $\langle V_1^*(s), V_1(s) \rangle = \cos\theta = constant$. The distance between corresponding points of the Bertrand curves pair in $\mathbb{E}^3$ is constant. $\alpha(s)$ is a Bertrand curve if and only if there exist nonzero real numbers λ and β such that constant $\lambda k_1 + \beta k_2 = 1$ for any $s \in I$, [4], [7]. The relationship between Frenet apparatus belonging to α and the Bertrand mate α^* is as follows:

$$
\begin{cases} V_1^* = \cos\theta V_1 + \sin\theta V_3 \\ V_2^* = V_2 \\ V_3^* = -\sin\theta V_1 + \cos\theta V_3. \end{cases} \tag{1.4}
$$

$$
k_1^* = \frac{\lambda k_1 - \sin^2\theta}{\lambda(1-\lambda k_1)}, \ \ k_2^* = \frac{\sin^2\theta}{\lambda^2 k_2}, \tag{1.5}
$$

where $\angle(V_1, V_1^*) = \theta$, [10]. Due to this equations, we can write

$$
\frac{k_1^*}{\eta^*} = \frac{\lambda k_1 - \sin^2\theta}{\lambda(1-\lambda k_1)\left(\dfrac{(\lambda k_1 - \sin^2\theta)^2}{\lambda^2(1-\lambda k_1)^2} + \dfrac{\sin^4\theta}{\lambda^4 k_2^2}\right)},
$$

$$
\left(\frac{k_1^*}{\eta^*}\right)' = \left[\frac{(\lambda k_1 - \sin^2\theta)\lambda^3 k_2^2(1-\lambda k_1)}{\left((\lambda k_1 - \sin^2\theta)^2\lambda^2 k_2^2 + \sin^4\theta(1-\lambda k_1)^2\right)}\right]' \frac{1}{k_2\sqrt{\lambda^2+\beta^2}},
$$

$$
\frac{ds}{ds^*} = \frac{1}{k_2\sqrt{\lambda^2+\beta^2}},
$$

where $\eta^* = k_1^{*2} + k_2^{*2}$. Frenet ruled surface can be generated by the motion of a Frenet vector of

curve in $\mathbb{E}^3$. Tangent, Normal and Binormal ruled surfaces are collectively named Frenet ruled surfaces, [5].

Theorem 1.1([5]) *Striction curves belonging to Frenet ruled surfaces are given in equations*

$$
\begin{aligned}
c_1(s) &= c_3(s) = \alpha(s) \\
c_2(s) &= \alpha(s) + \frac{k_1(s)}{k_2^2(s) + k_2^2(s)} V_2(s)
\end{aligned}
\tag{0.1}
$$

Theorem 1.2([5]) *Tangent vector fields T_1, T_2 and T_3 of striction curves belonging to Frenet ruled surface are given by*

$$
\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{k_2^2}{\eta \|c_2'(s)\|} & \frac{\left(\frac{k_1}{\eta}\right)'}{\|c_2'(s)\|} & \frac{k_1 k_2}{\eta \|c_2'(s)\|} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}
\tag{1.7}
$$

where $\eta = k_1^2 + k_2^2$.

§2. **A New Approach on the Striction Curves Belonging to Bertrandian**

   Frenet Ruled Surfaces

In this section first, we give Bertrandian tangent, Bertrandian normal and Bertrandian binormal Frenet ruled surfaces depending on the angle between the tangent vectors of the Bertrand curves pair. The later, we examined the situation of the tangent vectors belonging to Striction curves of Frenet and Bertrandian Frenet ruled surfaces.

Definition 2.1 *Let Bertrand curves pair $\{\alpha, \alpha^*\}$. Bertrandian Frenet ruled surfaces which are depended on the angle between the tangent vectors of Bertrand curves pair are defined as follow, respectively*

$$
\begin{aligned}
\varphi_1^*(s, w_1) &= \alpha^*(s) + w_1 V_1^*(s) = \alpha + \lambda V_2 + w_1(\cos\theta V_1 + \sin\theta V_3), \\
\varphi_2^*(s, w_2) &= \alpha^*(s) + w_2 V_2^*(s) = \alpha + (\lambda + w_2) V_2, \qquad (2.1) \\
\varphi_3^*(s, w_3) &= \alpha^*(s) + w_3 V_3^*(s) = \alpha + \lambda V_2 + w_3(-\sin\theta V_1 + \cos\theta V_3).
\end{aligned}
$$

 Theorem 2.2([11]) *Striction curves belonging to Bertrandian Frenet ruled surfaces depending on the angle between the tangent vectors of Bertrand curves pair can be expressed as follows,*

$$
\begin{aligned}
c_1^* &= \alpha + \lambda V_2, \\
c_2^* &= \alpha + \lambda V_2 + \frac{(\lambda k_1 - \sin^2\theta)V_2}{\lambda(1-\lambda k_2)\left(\frac{(\lambda k_1 - \sin^2\theta)^2}{\lambda^2(1-\lambda k_2)^2} + \frac{\sin^4\theta}{\lambda^4 k_2{}^2}\right)}, \\
c_3^* &= \alpha + \lambda V_2.
\end{aligned}
$$

Theorem 2.3([11])   *Tangent vector fields* T_1^*, T_2^*, T_3^*, *and* T_4^* *of striction curves belonging to Bertarndian Frenet ruled surfaces are given by*

$$
\begin{bmatrix} T_1^* \\ T_2^* \\ T_3^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a^* & b^* & c^* \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \end{bmatrix} \tag{2.2}
$$

where

$$
a^* = \frac{k_2^{*2}}{\eta^* \left\| c_2^{*\prime}(s) \right\|} = \frac{\sin^4\theta}{\lambda^4 k_2{}^2 \left(\frac{(\lambda k_1 - \sin^2\theta)^2}{\lambda^2(1-\lambda k_1)^2} + \frac{\sin^4\theta}{\lambda^4 k_2{}^2}\right)\sqrt{\psi_1{}^2 + \psi_2{}^2 + \psi_3{}^2}},
$$

$$
b^* = \frac{\left(\frac{k_1^*}{\eta^*}\right)'}{\left\| c_2^{*\prime}(s) \right\|} = \frac{\left[(\lambda k_1 - \sin^2\theta)\lambda^{-1}\left(1-\lambda k_1\right)^{-1}\left(\frac{\left(\lambda k_1 - \sin^2\theta\right)^2}{\lambda^2\left(1-\lambda k_1\right)^2} + \frac{\sin^4\theta}{\lambda^4 k_2{}^2}\right)^{-1}\right]'}{k_2\sqrt{(\lambda^2+\beta^2)(\psi_1{}^2 + \psi_2{}^2 + \psi_3{}^2)}},
$$

$$
c^* = \frac{k_1^* k_2^*}{\eta^* \left\| c_2^{*\prime}(s) \right\|} = \frac{(\lambda k_1 - \sin^2\theta)\sin^2\theta}{\lambda^3 k_2(1-\lambda k_1)\left(\frac{(\lambda k_1 - \sin^2\theta)^2}{\lambda^2(1-\lambda k_1)^2} + \frac{\sin^4\theta}{\lambda^4 k_2{}^2}\right)\sqrt{\psi_1{}^2 + \psi_2{}^2 + \psi_3{}^2}}.
$$

$$
\psi_1 = \left[\frac{\sin^4\theta\cos\theta}{\lambda^4 k_2{}^2\left(\frac{(\lambda k_1 - \sin^2\theta)^2}{\lambda^2(1-\lambda k_1)^2} + \frac{\sin^4\theta}{\lambda^4 k_2{}^2}\right)} - \frac{(\lambda k_1 - \sin^2\theta)\sin^3\theta}{\lambda^3 k_2(1-\lambda k_1)\left(\frac{(\lambda k_1 - \sin^2\theta)^2}{\lambda^2(1-\lambda k_1)^2} + \frac{\sin^4\theta}{\lambda^4 k_2{}^2}\right)}\right],
$$

$$
\psi_2 = \left[\frac{\lambda k_1 - \sin^2\theta}{\lambda\left(1-\lambda k_1\right)\left(\frac{\left(\lambda k_1 - \sin^2\theta\right)^2}{\lambda^2\left(1-\lambda k_1\right)^2} + \frac{\sin^4\theta}{\lambda^4 k_2{}^2}\right)}\right]' \frac{1}{k_2\sqrt{\lambda+\beta}},
$$

$$
\psi_3 = \left[\frac{\sin^5\theta}{\lambda^4 k_2{}^2\left(\frac{(\lambda k_1 - \sin^2\theta)^2}{\lambda^2(1-\lambda k_1)^2} + \frac{\sin^4\theta}{\lambda^4 k_2{}^2}\right)} + \frac{(\lambda k_1 - \sin^2\theta)\sin^2\theta\cos\theta}{\lambda^3 k_2(1-\lambda k_1)\left(\frac{(\lambda k_1 - \sin^2\theta)^2}{\lambda^2(1-\lambda k_1)^2} + \frac{\sin^4\theta}{\lambda^4 k_2{}^2}\right)}\right].
$$

Theorem 2.4   *The matrix of tangent vector fields on striction curves belonging to Frenet and*

Bertrandian Frenet ruled surfaces are given by

$$\begin{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \begin{bmatrix} T_1^* \\ T_2^* \\ T_3^* \end{bmatrix} \end{bmatrix}^{\mathbf{T}} = \begin{bmatrix} \cos\theta & a^*\cos\theta - c^*\sin\theta & \cos\theta \\ X & a^*X + \frac{\left(\frac{k_1}{\eta}\right)'}{\|c_2'(s)\|}b^* + c^*Y & X \\ \cos\theta & a^*\cos\theta - c^*\sin\theta & \cos\theta \end{bmatrix}$$

where $X = \frac{k_2^2}{\eta\|c_2'(s)\|}\cos\theta + \frac{k_1 k_2}{\eta\|c_2'(s)\|}\sin\theta,\ Y = -\frac{k_2^2}{\eta\|c_2'(s)\|}\sin\theta + \frac{k_1 k_2}{\eta\|c_2'(s)\|}\cos\theta.$

Proof Let be $[T] = [A][V]$ and $[T^*] = [A^*][V^*]$. By using the properties of the matrix, the following result is obtained.

$$\begin{aligned} [T][T^*]^{\mathbf{T}} &= [A][V]([A^*][V^*])^{\mathbf{T}} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ \frac{k_2^2}{\eta\|c_2'(s)\|} & \frac{\left(\frac{k_1}{\eta}\right)'}{\|c_2'(s)\|} & \frac{k_1 k_2}{\eta\|c_2'(s)\|} \\ 1 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}\begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \end{bmatrix}\right)^{\mathbf{T}} \begin{bmatrix} 1 & 0 & 0 \\ a^* & b^* & c^* \\ 1 & 0 & 0 \end{bmatrix}^{\mathbf{T}} \\ &= \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ X & \frac{\left(\frac{k_1}{\eta}\right)'}{\|c_2'(s)\|} & Y \\ \cos\theta & 0 & -\sin\theta \end{bmatrix}\begin{bmatrix} 1 & a^* & 1 \\ 0 & b^* & 0 \\ 0 & c^* & 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & a^*\cos\theta - c^*\sin\theta & \cos\theta \\ X & a^*X + \frac{\left(\frac{k_1}{\eta}\right)'}{\|c_2'(s)\|}b^* + c^*Y & X \\ \cos\theta & a^*\cos\theta - c^*\sin\theta & \cos\theta \end{bmatrix} \qquad \Box \end{aligned}$$

Corollary 2.1 (*i*) *If the Bertrand mate is helix curve, then tangent vector fields of striction curves belonging to Tangent and Bertrandian normal ruled surfaces can be orthogonal;*

(*ii*) *If the Bertrand mate is helix curve, then tangent vector fields of striction curves belonging to Binormal and Bertrandian normal ruled surfaces can be orthogonal.*

Proof i) If inner product of T_1 and T_2^* is zero, then T_1 and T_2^* are orthogonal vectors.

$$\begin{aligned} \langle T_1, T_2^* \rangle = 0 \Rightarrow \langle T_1, T_2^* \rangle &= a^*\cos\theta - c^*\sin\theta = 0 \Longrightarrow \frac{a^*}{c^*} = \frac{\sin\theta}{\cos\theta} = \tan\theta \\ \frac{k_2^*}{k_1^*} &= \tan\theta = constant \end{aligned}$$

this completes the proof.

For (*ii*), since T_1 and T_3 are equivalent vectors, then the proof is clear. $\qquad\Box$

Corollary 2.2 (*i*) *If Bertrand curve is helix curve, then tangent vector fields of striction curves belonging to Normal and Bertrandian tangent ruled surfaces can be orthogonal;*

(*ii*) *If the Bertrand curve is helix curve, then tangent vector fields of striction curves belonging to Normal and Bertrandian binormal ruled surfaces can be orthogonal.*

Proof (i) If inner product of T_2 and T_1^* is zero, then T_2 and T_1^* are orthogonal vectors.

$$\langle T_2, T_1^* \rangle = 0 \Rightarrow \langle T_2, T_1^* \rangle = X = \frac{k_2^2}{\eta \, \|c_2'(s)\|} \cos\theta + \frac{k_1 k_2}{\eta \, \|c_2'(s)\|} \sin\theta = 0$$

$$\Rightarrow \frac{k_2}{k_1} = -\tan\theta = constant.$$

This completes the proof.

For (ii), since T_1^* and T_3^* are equivalent vectors, the proof is clear. $\qquad\square$

Corollary 2.3 *Tangent vector fields of striction curves belonging to Normal and Bertrandian normal ruled surfaces have orthogonal under the condition*

$$\left[\frac{(\lambda k_1 - \sin^2\theta)\lambda^3 k_2^2 (1 - \lambda k_1)}{\left((\lambda k_1 - \sin^2\theta)^2 \lambda^2 k_2^2 + \sin^4\theta (1 - \lambda k_1)^2 \right)} \right]' = \frac{\frac{\sin^2\theta}{\lambda^2 k_2}\left(\frac{\sin^2\theta}{\lambda^2 k_2} X + \frac{\lambda k_1 - \sin^2\theta}{\lambda(1 - \lambda k_1)} Y \right)}{-\frac{\left(\frac{k_1}{\eta} \right)'}{\|c_2'(s)\|}\left(\frac{(\lambda k_1 - \sin^2\theta)^2}{\lambda^2(1 - \lambda k_1)^2} + \frac{\sin^4\theta}{\lambda^4 k_2^2} \right)}.$$

Proof $\langle T_2, T_2^* \rangle = 0 \Rightarrow \langle T_2, T_2^* \rangle = a^* X + \frac{\left(\frac{k_1}{\eta} \right)'}{\|c_2'(s)\|} b^* + c^* Y = 0$

$$a^* X + \frac{\left(\frac{k_1}{\eta} \right)'}{\|c_2'(s)\|} b^* + c^* Y = 0$$

$$k_2^{*2} X + \frac{\left(\frac{k_1}{\eta} \right)'}{\|c_2'(s)\|} \left(\frac{k_1^*}{\eta^*} \right)' \eta^* + k_1^* k_2^* Y = 0$$

$$k_2^*(k_2^* X + k_1^* Y) = -\frac{\left(\frac{k_1}{\eta} \right)'}{\|c_2'(s)\|} \left(\frac{k_1^*}{\eta^*} \right)' \eta^*$$

$$\left(\frac{k_1^*}{\eta^*} \right)' = \frac{k_2^*(k_2^* X + k_1^* Y)}{-\frac{\left(\frac{k_1}{\eta} \right)'}{\|c_2'(s)\|} \eta^*}$$

$$\left[\frac{(\lambda k_1 - \sin^2\theta)\lambda^3 k_2^2 (1 - \lambda k_1)}{\left((\lambda k_1 - \sin^2\theta)^2 \lambda^2 k_2^2 + \sin^4\theta (1 - \lambda k_1)^2 \right)} \right]' = \frac{\frac{\sin^2\theta}{\lambda^2 k_2}\left(\frac{\sin^2\theta}{\lambda^2 k_2} X + \frac{\lambda k_1 - \sin^2\theta}{\lambda(1 - \lambda k_1)} Y \right)}{-\frac{\left(\frac{k_1}{\eta} \right)'}{\|c_2'(s)\|}\left(\frac{(\lambda k_1 - \sin^2\theta)^2}{\lambda^2(1 - \lambda k_1)^2} + \frac{\sin^4\theta}{\lambda^4 k_2^2} \right)}.$$

This completes the proof. $\qquad\square$

Corollary 2.4 *Four pairs of tangent vectors fields of Frenet ruled surface belonging to Bertrand pair $\{\alpha, \alpha^*\}$ are equal.*

Proof From the equations (1.7) and (2.2)

$$\langle T_1, T_1^* \rangle = \langle T_1, T_3^* \rangle = \langle T_3, T_1^* \rangle = \langle T_3, T_3^* \rangle = \langle V_1, \cos\theta V_1 + \sin\theta V_3 \rangle = \cos\theta. \qquad\square$$

Example 2.1 Let us consider the following Bertrand curve α and Bertrand mate α^*, respec-

tively.

$$\alpha(s) = \left(\frac{9}{208}\sin 16s - \frac{1}{117}\sin 36s, -\frac{9}{208}\cos 16s + \frac{1}{117}\cos 36s, \frac{6}{65}\sin 10s\right),$$

$$\alpha^*(s) = \left(\frac{9}{208}\sin 16s - \frac{1}{117}\sin 36s + \frac{12}{13}\lambda\cos(26s), -\frac{9}{208}\cos 16s + \frac{1}{117}\cos 36s\right.$$

$$\left. + \frac{12}{13}\lambda\sin 26s, \frac{6}{65}\sin 10s - \frac{5}{13}\lambda\right), [?].$$

Striction curves of the ruled surface formed by Frenet vector of this curves is as follows:

$$\begin{cases}
c_1(s) = \left(\frac{9}{208}\sin 16s - \frac{1}{117}\sin 36s, -\frac{9}{208}\cos 16s + \frac{1}{117}\cos 36s, \frac{6}{65}\sin 10s\right), \\[2mm]
c_2(s) = \left(\frac{9}{208}\sin 16s - \frac{1}{117}\sin 36s + \frac{1}{24}\sin 10s\left(\frac{9}{13}\cos 16s - \frac{4}{13}\cos 36s\right), \right. \\[2mm]
\quad -\frac{9}{208}\cos 16s + \frac{1}{117}\cos 36s + \frac{1}{24}\sin 10s\left(\frac{9}{13}\sin 16s - \frac{4}{13}\sin 36s\right), \\[2mm]
\quad \left. \frac{6}{65}\sin 10s + \frac{1}{26}\sin 10s\cos 10s\right), \\[2mm]
c_3(s) = \left(\frac{9}{208}\sin 16s - \frac{1}{117}\sin 36s, -\frac{9}{208}\cos 16s + \frac{1}{117}\cos 36s, \frac{6}{65}\sin 10s\right)
\end{cases}$$

$$\begin{cases}
c_1{}^*(s) = \left(\frac{9}{208}\sin 16s - \frac{1}{117}\sin 36s + \frac{12}{13}\lambda\cos(26s), -\frac{9}{208}\cos 16s + \frac{1}{117}\cos 36s\right. \\[2mm]
\quad \left. +\frac{12}{13}\lambda\sin 26s, \frac{6}{65}\sin 10s - \frac{5}{13}\lambda\right), \\[2mm]
c_2{}^*(s) = \left[\left(\frac{9}{208}\sin 16s - \frac{1}{117}\sin 36s + \frac{12}{13}\lambda\cos 26s\right)\cos^2 10s + \frac{1}{14976}\sin 10s\cos 26s\right. \\[2mm]
\quad -\frac{1}{624}\lambda\cos 26s, \cos^2 10s\left(-\frac{9}{208}\cos 16s + \frac{1}{117}\cos 36s + \frac{12}{13}\lambda\sin 26s\right) \\[2mm]
\quad +\frac{1}{14976}\sin 10s\sin 26s - \frac{1}{624}\lambda\sin 26s, \cos^2 10s\left(\frac{6}{65}\sin 10s - \frac{5}{13}\lambda\right) \\[2mm]
\quad \left. -\frac{5}{179712}\sin 10s + \frac{5}{7488}\lambda\right]\cos^{-2} 10s, \\[2mm]
c_3{}^*(s) = \left(\frac{9}{208}\sin 16s - \frac{1}{117}\sin 36s + \frac{12}{13}\lambda\cos(26s), -\frac{9}{208}\cos 16s + \frac{1}{117}\cos 36s\right. \\[2mm]
\quad \left. +\frac{12}{13}\lambda\sin 26s, \frac{6}{65}\sin 10s - \frac{5}{13}\lambda\right),
\end{cases}$$

Bertrand Curve   Bertrand mate curve
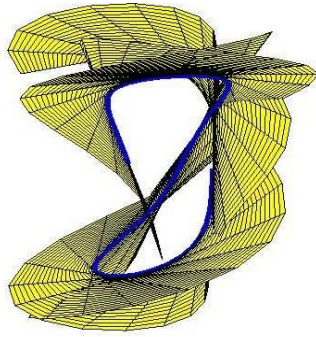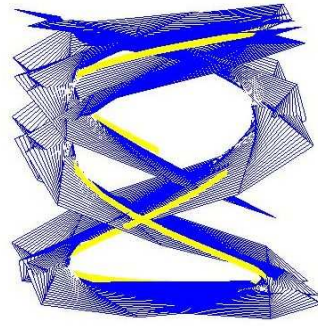
a)   b)

Figure 1. The figures a) and b) show, respectively, Tangent ruled surface and Bertrandian Tangent ruled surface. Blue and yellow colors are striction curves on these surfaces, respectively.
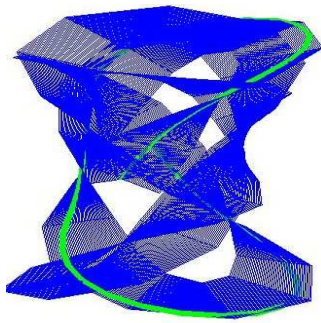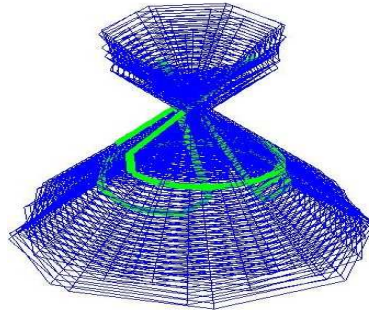
c)   d)

Figure 2. The figures c) and d) show, respectively, Normal ruled surface and Bertrandian Normal ruled surface. Green colors are striction curves on these surfaces, respectively.

e)　　　　　　　　　　　　f)
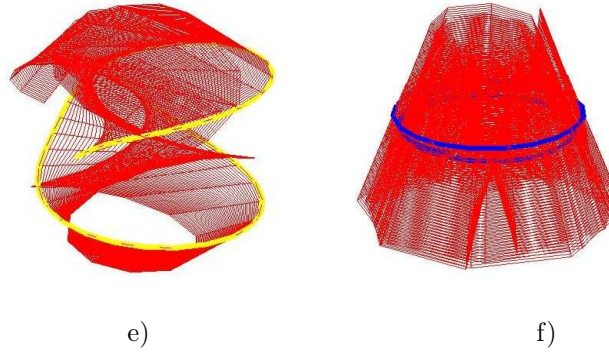
Figure 3.　The figures e) and f) show, respectively, Binormal ruled surface and Bertrandian Binormal ruled surface. Yellow and blue colors are striction curves on these surfaces, respectively.

k)　　　　　　　　　　　　m)
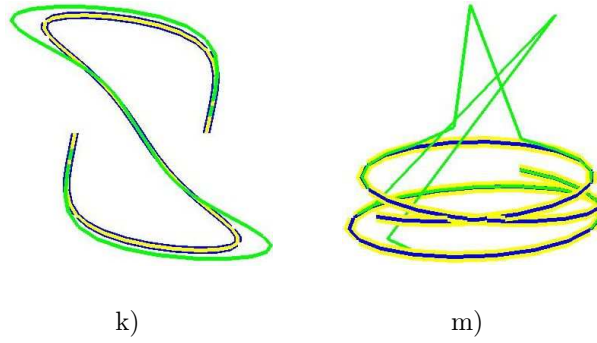
Figure 4.　The figures k) and m) show Striction curves of Frenet ruled surfaces and Bertrandian Frenet ruled surfaces.

These figures are drawn with Mapple program for $\lambda = 1$.

Acknowledgement

References

[1] Ali A.T., Special Smarandache curves in the Euclidean space, *International J.Math. Combin.*, 2, 30-36, 2010.

[2] Do Carmo, M. P., *Differential Geometry of Curves and Surfaces*, Prentice-Hall, ISBN 0-13-212589-7, 1976.

[3] Gray, A. *Modern Differential Geometry of Curves and Surfaces with Mathematica,* 2nd ed. Boca Raton, FL: CRC Press, 205, 1997.

[4] Hacısalihoğlu H.H., *Diferensiyel Geometri*, Cilt 1, İnönü Üniversitesi Yayinlari, Malatya,

1994.

[5] Kılıçoğlu Ş, Şenyurt, S. and Hacısalihoğlu H.H., On the striction curves of Involute and Bertrandian Frenet ruled surfaces in $\mathbb{E}^3$, *Applied Mathematical Sciences*, 9(142), 7081 - 7094, 2015.

[6] Izumiya, S., Takeuchi, N., Special curves and Ruled surfaces, *Beitrage zur Algebra und Geometrie Contributions to Algebra and Geometry*, 44(1), 203-212, 2003.

[7] Lipschutz M.M., *Differential Geometry*, Schaum's Outlines.

[8] O'Neill B., *Elemantary Differential Geometry*, 2nd ed., Academic Press, New York, 1997.

[9] Pears L.R., Bertrand curves in Riemannian space, *J. London Math. Soc.*, 1-10(2), 180-183, 1935.

[10] Sabuncuoğlu A., *Differential Geometry*, Nobel Publications, Ankara, 2006.

[11] Şenyurt, S. and Çalışkan, A., A new approach on the striction curves along Bertrandian Darboux Frenet ruled surface, *AIP Conf. Proc.*, 1726, 020032 1-5, 2016.

Mathematical Combinatorics with Natural Reality

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Abstract: There are 2 contradictory views on our world, i.e., continuous or discrete, which results in that only partially reality of a thing T can be understood by one of continuous or discrete mathematics because of the universality of contradiction and the connection of things in the nature, just as the philosophical meaning in the story of the blind men with an elephant. Holding on the reality of natural things motivates the combination of continuous mathematics with that of discrete, i.e., an envelope theory called *mathematical combinatorics* which extends classical mathematics over topological graphs because a thing is nothing else but a multiverse over a spacial structure of graphs with conservation laws hold on its vertices. Such a mathematical object is said to be an *action flow*. The main purpose of this survey is to introduce the powerful role of action flows, or mathematics over graphs with applications to physics, biology and other sciences, such as those of G-solution of non-solvable algebraic or differential equations, Banach or Hilbert $\overrightarrow{G}$-flow spaces with multiverse, multiverse on equations, $\cdots$ and with applications to complex systems, for examples, the understanding of particles, spacetime and biology. All of these make it clear that holding on the reality of things by classical mathematics is only on the coherent behaviors of things for its homogenous without contradictions, but the mathematics over graphs G is applicable for contradictory systems, i.e., complex systems because contradiction is universal only in eyes of human beings but not the nature of a thing itself.

Key Words: Graph, Banach space, Smarandache multispace, $\overrightarrow{G}$-flow, observation, natural reality, complex system, non-solvable equation, mathematical combinatorics.

AMS(2010): 03A10,05C15,20A05, 34A26,35A01,51A05,51D20,53A35.

§1. Introduction

Generally, the reality of a thing T is its state of existed, exists, or will exist in the world, whether or not they are observable or comprehensible by human beings. However, the recognized reality maybe very different from that of the truth because it depends on the way of the observer and his world view is continuous or discrete, i.e., view the behavior of thing T a continuous function

f, or an infinite or finite sequence $x_1, x_2, \cdots, x_n$ with $n \geq 1$ on time t.

Is our world continuous or discrete? Certainly not because there exist both continuous or discrete things in the eyes of human beings. For example, all apples on a tree is discrete but the moving of a car on the road is continuous, such as those figures (a) and (b) shown in Fig.1.

(a) (b)

Fig.1

And historically, holding on the behavior of things mutually develops the continuous and discrete mathematics, i.e., research a discrete (continuous) question by that of continuous (discrete) mathematical methods. For example, let x, y be the populations in a self-system of cats and rats, such as Tom and Jerry shown in Fig.2,

Fig.2

then they were continuously characterized by Lotka-Volterra with differential equations ([4])

$$\begin{cases} \dot{x} &=& x(\ \lambda - by), \\ \dot{y} &=& y(-\mu - cx). \end{cases} \tag{1.1}$$

Similarly, all numerical calculations by computer for continuous questions are carried out by discrete methods because algorithms language recognized by computer is essentially discrete. Such a typical example is the movies by discrete images for a continuous motion shown in Fig.3. Thus, the reality of things needs the combination of the continuous mathematics with that of the discrete.

Fig.3

Physically, the behavior of things T is usually characterized by differential equation

$$\mathscr{F}\left(t, x_1, x_2, x_3, \psi_t, \psi_{x_1}, \psi_{x_2}, \cdots, \psi_{x_1 x_2}, \cdots\right) = 0 \tag{1.2}$$

established on observed characters of $\mu_1, \mu_2, \cdots, \mu_n$ for its state function $\psi(t, x)$ in reference system $\mathbb{R}^3$ by Newtonian and $\mathbb{R}^4$ by Einstein ([2]).

Fig.4

Usually, these physical phenomenons of a thing is complex, and hybrid with other things. *Is the reality of particle P all solutions of that equation* (1.2) *in general?* Certainly not because the equation (1.2) only characterizes the behavior of P on some characters of $\mu_1, \mu_2, \cdots, \mu_n$ at time t abstractly, not the whole in philosophy. For example, the behavior of a particle is characterized by the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + U\psi \tag{1.3}$$

in quantum mechanics ([24]) but observation shows it in two or more possible states of being, i.e., superposition such as the asking question of Schrödinger for the alive or dead of the cat in the box with poison switch shown in Fig.4. We can not even say which solution of the Schrödinger equation (1.3) is the particle because each solution is only for one determined state.

Furthermore, *can we conclude the equation* (1.2) *is absolutely right for a particle P?* Certainly not also because the dynamic equation (1.2) is always established with an additional

assumption, i.e., the geometry on a particle P is a point in classical mechanics or a field in quantum mechanics and dependent on the observer is out or in the particle. For example, a water molecule H_2O consists of 2 hydrogen atoms and 1 oxygen atom such as those shown in Fig.5. If an observer receives information on the behaviors of hydrogen or oxygen atom but stands out of the water molecule H_2O by viewing it a geometrical point, then he can only receives coherent information on atoms H and O with the water molecule H_2O.
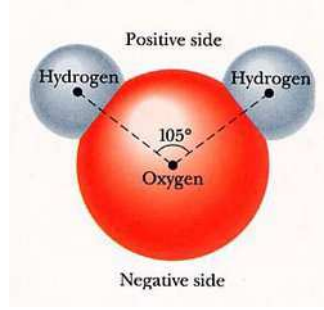
Fig.5

But if he enters the interior of the molecule, he will view a different sceneries for atoms H and O, which are respectively called *out-observation* and *in-observation*, and establishes equation (1.3) on H_2O or 3 dynamic equations

$$\begin{cases} -i\hbar\dfrac{\partial \psi_O}{\partial t} = \dfrac{\hbar^2}{2m_O}\nabla^2\psi_O - V(x)\psi_O \\ -i\hbar\dfrac{\partial \psi_{H_1}}{\partial t} = \dfrac{\hbar^2}{2m_{H_1}}\nabla^2\psi_{H_1} - V(x)\psi_{H_1} \\ -i\hbar\dfrac{\partial \psi_{H_2}}{\partial t} = \dfrac{\hbar^2}{2m_{H_2}}\nabla^2\psi_{H_2} - V(x)\psi_{H_2} \end{cases} \tag{1.4}$$

on atoms H and O. *Which is the right model on H_2O, the* (1.3) *or* (1.4) *dynamic equations?* The answer is not easy because the equation model (1.3) can only characterizes those of coherent behavior of atoms H and O in H_2O, but equations (1.4) have no solutions, i.e., non-solvable in mathematics ([17]).

The main purpose of this survey is to clarify that the reality of a thing T should be a contradictory system in one's eyes, or multiverse with non-solvable systems of equations in geometry, conclude that they essentially describe its nature, which results in mathematical combinatorics, i.e., mathematics over graphs in space, and show its powerful role to mathematics with applications to elementary particles, gravitational field and other sciences, such as those of extended Banach or Hilbert $\overrightarrow{G}$-flow spaces, geometry on non-solvable systems of solvable differential equations, $\cdots$ with applications to the understanding of particles, population biology and other sciences.

For terminologies and notations not mentioned here, we follow references [1] for mechanics, [4] for biological mathematics, [8] for combinatorial geometry, [23]-[24] for elementary particles, and [25] for Smarandache systems and multispaces, and all phenomenons discussed in this paper are assumed to be true in the nature.

§2. **Contradiction, a By-product of Non-complete Recognizing**

A philosophical proposition following clarifies the fundamental relation between the reality and the reality understood by classical mathematics, which is clear but few peoples noted in the past.

Proposition 2.1 *Let $\mathscr{R}$ and $\mathscr{MR}$ be respectively the sets of reality and the reality known by classical mathematics on things. Then*

$$\mathscr{MR} \subset \mathscr{R} \quad and \quad \mathscr{MR} \neq \mathscr{R}. \tag{2.1}$$

Proof Notice that classical mathematical systems are homogenous without contradictions, i.e. a compatible one in logic but contradictions exist everywhere in philosophy. Thus, the reality known by classical mathematics on things can be only a subset of the reality set, i.e., the relation (2.1)

$$\mathscr{MR} \subset \mathscr{R} \quad and \quad \mathscr{MR} \neq \mathscr{R}. \qquad \square$$

Although Proposition 2.1 is simple but it implies that for holding on reality of things, an envelope theory on classical mathematics, i.e., mathematics including contradictions is needed to establish for human beings.

2.1 Thinking Models

Let us discuss 3 thinking models on complex systems following.

T1. The Blind Men with an Elephant.   This is a famous story in Buddhism which implies the entire consisting of its parts but we always hold on parts. In this story, there are six blind men were asked to determine what an elephant looked like by feeling different parts of an elephant's body. The man touched the elephant's leg, tail, trunk, ear, belly or tusk respectively claims it's like a pillar, a rope, a tree branch, a hand fan, a wall or a solid pipe, such as those shown in Fig.6.
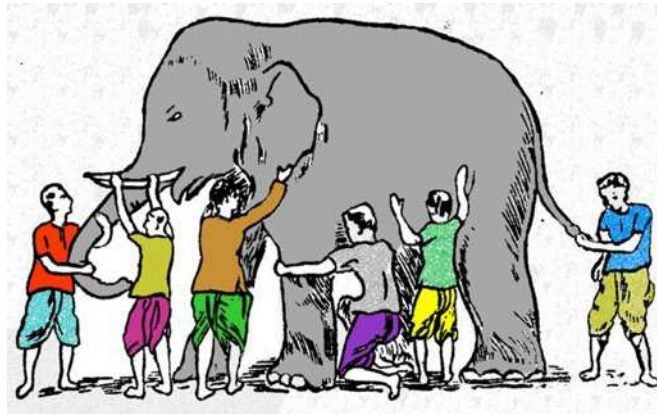
Fig.6

Each of these blind men insisted on his own's right, not accepted others, and then entered into an endless argument. *All of you are right*! A wise man explains to them: *why are you telling it differently is because each one of you touched the different part of the elephant. So, actually the elephant has all those features what you all said.* Hence, the wise man told these blind man that an elephant seemingly looked

$$
\text{An elephant} \quad = \quad \{4 \text{ pillars}\} \bigcup \{1 \text{ rope}\} \bigcup \{1 \text{ tree branch}\}
$$
$$
\bigcup \quad \{2 \text{ hand fans}\} \bigcup \{1 \text{ wall}\} \bigcup \{1 \text{ solid pipe}\} \tag{2.2}
$$

What is the implication of this story for human beings? It lies in the situation that human beings understand things in the world is analogous to these blind men. Usually, a thing T is understand by its known characters at one by one time and known gradually. For example, let $\mu_1, \mu_2, \cdots, \mu_n$ be known and $\nu_i, i \geq 1$ unknown characters on a thing T at time t. Then, T is understood by

$$
T = \left(\bigcup_{i=1}^{n} \{\mu_i\} \right) \bigcup \left(\bigcup_{k \geq 1} \{\nu_k\} \right) \tag{2.3}
$$

in logic and with an approximation $T^\circ = \bigcup_{i=1}^{n} \{\mu_i\}$ at time t. The equation (2.3) is called the *Smarandache multispace* ([8], [25]), a combination of discrete characters for understanding a thing T.

T2. Everett's Multiverse on Superposition. The multiverse interpretation by H.Everett [3] on wave function of equation (1.3) in 1957 answered the superposition of particles in machinery. By an assumption that the wave function of an observer would be interacted with a superposed object, he concluded different worlds in different quantum system obeying equation (1.3) and the superposition of a particle be liked those separate arms of a 2-branching universe ([16], [17]) such as those shown in Fig.7,

Fig.7

which revolutionary changed an ambiguous interpretation in quantum mechanics before him, i.e., an observer will cause the wave function to collapse randomly into one of the alternatives with all others disappearing. Everett's multiverse interpretation on the superposition of particle is in fact alluded in thinking model $T1$, i.e., the story of blind men with an elephant because

if one views each of these pillar, rope, tree branch, hand fan, wall and solid pipe by these blind men feeling on different parts of the elephant to be different spaces, then the looks of an elephant of the wise man told these blind men (2.2) is nothing else but an Everett's multiverse.

T3. Quarks Model. The divisibility of matter initiates human beings to search elementary constituting cells of matter, i.e., elementary particles such as those of quarks, leptons with interaction quanta including photons and other particles of mediated interactions, also with those of their antiparticles at present ([23], [24]), and unmatters between a matter and its antimatter which is partially consisted of matter but others antimatter ([26], [27]). For example, a baryon is predominantly formed by three quarks, and a meson is mainly composed of a quark and an antiquark in the models of Sakata, or Gell-Mann and Ne'eman, such as those shown in Fig.8, where there is also a particle composed of 5 quarks.
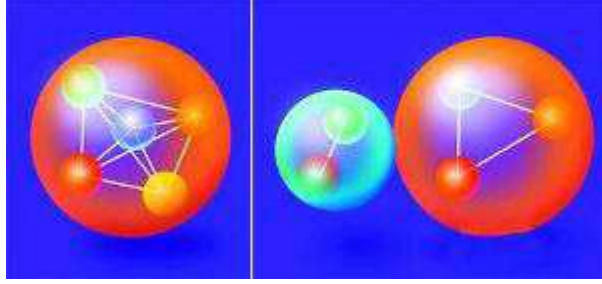
Fig.8

However, a free quark was never found in experiments. We can not even conclude the Schrödinger equations (1.3) is the right equation on quarks. But *why is it believed without a shadow of doubt that the dynamical equation of elementary particles such as those of quarks, leptons with interaction quanta is* (1.3) *in physics*? The reason is because that all observations come from a macro viewpoint, the human beings, not the quarks, and which can only lead to coherent behaviors, not the individuals. In mathematics, it is just an equation on those of particles viewed abstractly to be a geometrical point or an independent field from a macroscopic point, which results in physicists assuming the internal structures mechanically for understanding behaviors of particles, such as those shown in Fig.8. However, such an assumption is a little ambiguous in logic, i.e., we can not even distinguish who is the geometrical point or the field, the particle or its quark.

2.2 Contradiction Originated in Non-complete Recognizing

If we completely understand a thing T, i.e., $T = T^o$ in formula (2.3) at time t, there are no contradiction on T. However, this is nearly impossible for human beings, concluded in the first chapter of *TAO TEH KING* written by *Lao Zi*, a famous ideologist in China, i.e., "Name named is not the eternal; the without is the nature and naming the origin of things", which also implies the universality of contradiction and a generalization of equation (2.1).

Certainly, the looks (2.2) of the wise man on the elephant is a complete recognizing but these of the blind men is not. However, *which is the right way of recognizing*? The answer

depends on the standing view of observer. The observation of these blind men on the elephant are a microscopic or in-observing but the wise man is macroscopic or out-observing. If one needs only for the macroscopic of an elephant, the wise man is right, but for the microscopic, these blind men are right on the different parts of the elephant. For understanding the reality of a thing T, we need the complete by individual recognizing, i.e., the whole by its parts. Such an observing is called a *parallel observing* ([17]) for avoiding the defect that each observer can only observe one behavior of a thing, such as those shown in Fig.9 on the water molecule H_2O with 3 observers.
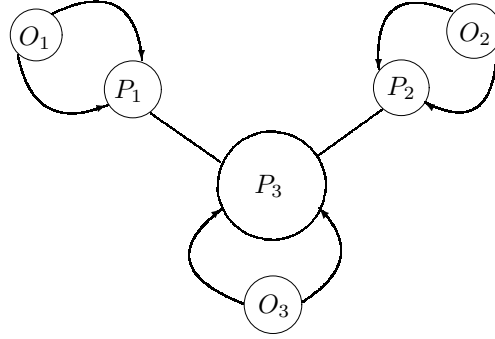
Fig.9

Thus, the looks of the wise man on an elephant is a collection of parallel observing by these 6 blind men and finally results in the recognizing (2.2), and also the Everett's multiverse interpretation on the superposition, the models of Sakata, or Gell-Mann and Ne'eman on particles. This also concludes that multiverse exists everywhere if we observing a thing T by in-observation, not only those levels of $I - IV$ classified by Max Tegmark in [28].

However, these equations (1.2) established on parallel observing datum of multiverse, for instance the equations (1.4) on 2 hydrogen atoms and 1 oxygen atom ([17]), and generally, differential equations (1.2) on population biology with more than 3 species are generally non-solvable. Then, *how to understand the reality of a thing T by mathematics holding with an equality $\mathcal{MR} = \mathcal{R}$?* The best answer on this question is the combination of continuous mathematics with that of the discrete, i.e., turn these non-mathematics in the classical to mathematics by a combinatorial manner ([13]), i.e., *mathematical combinatorics*, which is the appropriated way for understanding the reality because all things are in contradiction.

§3. Mathematical Combinatorics

3.1 Labeled Graphs

A *graph* G is an ordered 2-tuple (V, E) with $V \neq \emptyset$ and $E \subset V \times V$, where V and E are finite sets and respectively called the vertex set, the edge set of G, denoted by $V(G)$ or $E(G)$, and a graph G is said to be embeddable into a topological space $\mathcal{T}$ if there is a $1 - 1$ continuous mapping $\phi : G \to \mathcal{T}$ with $\phi(p) \neq \phi(q)$ if $p, q \notin V(G)$. Particularly, if $\mathcal{T} = \mathbb{R}^3$ such a topological

graph is called *spacial graph* such as those shown in Fig.10 for cube $C_4 \times C_4$,
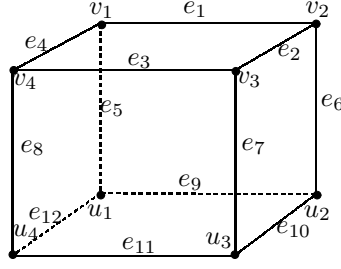
Fig.10

and a labeling on a graph G is a mapping $L: \; V(G) \bigcup E(G) \to \mathscr{L}$ with a labeling set $\mathscr{L}$. For example, $\mathscr{L} = \{v_i, u_i, e_j, 1 \le i \le 4, 1 \le j \le 12\}$ in Fig.10. Notice that the inherent structure of an elephant finding by these blind men is a labeled tree shown in Fig.11,
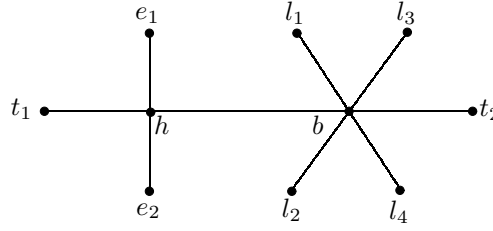
Fig.11

where, $\{t_1\}$ =tusk, $\{e_1, e_2\}$ =ears, $\{h\}$ =head, $\{b\}$ =belly, $\{l_1, l_2, l_3, l_4\}$ =legs and $\{t_2\}$ =tail. Then, *how can one rebuilt the geometric space of an elephant from the labeled tree in space* $\mathbb{R}^3$? First, one can blows up all edges, i.e., $e \to$ a cylinder for $\forall e \in E(G^L)$ and then, homeomorphically transforms these cylinders as parts of an elephant. After these transformations, a 3-dimensional elephant is built again in $\mathbb{R}^3$ such as those shown in Fig.12.

Fig.12

All of these discussions implies that labeled graph should be a mathematical element for understanding things ([20]), not only a labeling game because of

$$\text{Labeled Graphs in } \mathbb{R}^n \quad \Leftrightarrow \quad \text{Inherent Structure of Things.}$$

But *what are labels on labeled graphs, is it just different symbols?* And *are such labeled graphs a mechanism for the reality of things, or only a labeling game?* In fact, labeled graphs are researched mainly on symbols, not mathematical elements. If one puts off this assumption, i.e., labeling a graph by elements in mathematical systems, *what will happens? Are these resultants important for understanding things in the world?* The answer is certainly yes ([6], [7]) because this step will enable one to pullback more characters of things, particularly the metrics in physics, characterize things precisely and then holds on the reality of things.

3.2 G-Solutions on Equations

Let $\mathscr{F} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be a $\mathbb{C}^k$, $1 \le k \le \infty$ mapping with $\mathscr{F}(\overline{x}_0, \overline{y}_0) = \overline{0}$ for $\overline{x}_0 \in \mathbb{R}^n$, $\overline{y}_0 \in \mathbb{R}^m$ and a non-singular $m \times m$ matrix $\left(\partial \mathscr{F}^j / \partial y^i (\overline{x}_0, \overline{y}_0) \right)$. Then the implicit mapping theorem concludes that there exist opened neighborhoods $V \subset \mathbb{R}^n$ of $\overline{x}_0$, $W \subset \mathbb{R}^m$ of $\overline{y}_0$ and a $\mathbb{C}^k$ mapping $\phi : V \to W$ such that $T(\overline{x}, \phi(\overline{x})) = \overline{0}$, i.e.,equation (1.2) is always solvable.

Let $\mathscr{F}_1, \mathscr{F}_2, \cdots, \mathscr{F}_m$ be m mappings holding with conditions of the implicit mapping theorem and let $S_{\mathscr{F}_i} \subset \mathbb{R}^n$ be a manifold such that $\mathscr{F}_i : S_{\mathscr{F}_i} \to 0$ for integers $1 \le i \le m$. Consider the equations

$$
\begin{cases}
\mathscr{F}_1(x_1, x_2, \cdots, x_n) = 0 \\
\mathscr{F}_2(x_1, x_2, \cdots, x_n) = 0 \\
\qquad \cdots\cdots\cdots\cdots\cdots\cdots \\
\mathscr{F}_m(x_1, x_2, \cdots, x_n) = 0
\end{cases}
\tag{3.1}
$$

in Euclidean space $\mathbb{R}^n, n \ge 1$. Geometrically, the system (3.1) is non-solvable or not dependent on $\bigcap\limits_{i=1}^{m} S_{\mathscr{F}_i} = \emptyset$ or $\ne \emptyset$.

Now, *is the non-solvable case meaningless for understanding the reality of things?* Certainly not because the non-solvable case of (3.1) only concludes the intersection $\bigcap\limits_{i=1}^{m} S_{\mathscr{F}_i} = \emptyset$, the behavior of the solvable and non-solvable cases should be both characterized by the union $\bigcup\limits_{i=1}^{m} S_{\mathscr{F}_i}$ such as those shown in (2.2) for the elephant.

For example, if things T_1, T_2, T_3, T_4 and T_1', T_2', T_3', T_4' are respectively characterized by systems of equations following

$$
(LES_4^N) \quad
\begin{cases}
x + y &= \quad 1 \\
x + y &= \quad -1 \\
x - y &= \quad -1 \\
x - y &= \quad 1
\end{cases}
\qquad\qquad
(LES_4^S) \quad
\begin{cases}
x = y \\
x + y = 2 \\
x = 1 \\
y = 1
\end{cases}
$$

it is clear that (LES_4^N) is non-solvable because $x + y = -1$ is contradictious to $x + y = 1$, and so that for equations $x - y = -1$ and $x - y = 1$, i.e., there are no solutions x_0, y_0 hold with this system. But (LES_4^S) is solvable with $x = 1$ and $y = 1$. *Can we conclude that things T_1', T_2', T_3', T_4' are $x = 1, y = 1$ and T_1, T_2, T_3, T_4 are nothing?* Certainly not because

$(x,y) = (1,1)$ is the intersection of straight line behavior of things T_1', T_2', T_3', T_4' and there are no intersection of T_1, T_2, T_3, T_4 in plane $\mathbb{R}^2$. However, they are indeed exist in $\mathbb{R}^2$ such as those shown in Fig.13.

Fig.13

Let $L_{a,b,c} = \{(x,y)|ax + by = c, ab \neq 0\}$ be points in $\mathbb{R}^2$. We are easily know the straight line behaviors of T_1, T_2, T_3, T_4 and T_1', T_2', T_3', T_4' are nothings else but the unions $L_{1,-1,0} \bigcup L_{1,1,2} \bigcup L_{1,0,1} \bigcup L_{0,1,1}$ and $L_{1,1,1} \bigcup L_{1,1,-1} \bigcup L_{1,-1,-1} \bigcup L_{1,-1,1}$, respectively.
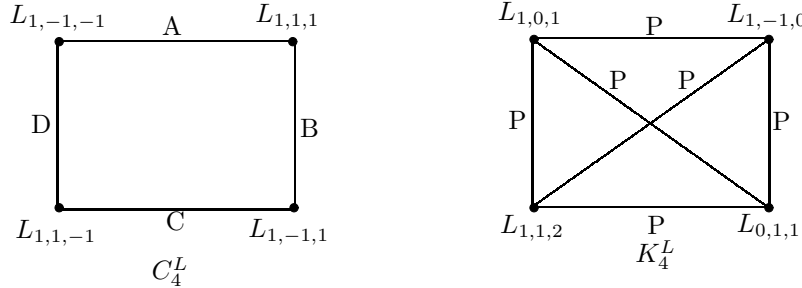
Fig.14

Definition 3.1 *A G-solution of system* (3.1) *is a labeling graph* G^L *defined by*

$$V(G) = \{S_{\mathscr{F}_i}, \ 1 \leq i \leq n\};$$

$$E(G) = \left\{(S_{\mathscr{F}_i}, S_{\mathscr{F}_j}) \ if \ S_{\mathscr{F}_i} \bigcap S_{\mathscr{F}_j} \neq \emptyset \ for \ integers \ 1 \leq i, j \leq n\right\} \ with \ a \ labeling$$

$$L: \ S_{\mathscr{F}_i} \to S_{\mathscr{F}_i}, \qquad (S_{\mathscr{F}_i}, S_{\mathscr{F}_j}) \to S_{\mathscr{F}_i} \bigcap S_{\mathscr{F}_j}.$$

For Example, the G-solutions of (LES_4^N) and (LES_4^S) are respectively labeling graphs C_4^L and K_4^L shown in Fig.14. Generally, we know the following result.

Theorem 3.2 *A system* (3.1) *of equations is G-solvable if* $\mathscr{F}_i \in \mathbb{C}^1$ *and* $\mathscr{F}_i|_{(x_1^0, x_2^0, \cdots, x_n^0)} = 0$ *but* $\dfrac{\partial \mathscr{F}_i}{\partial x_i}\bigg|_{(x_1^0, x_2^0, \cdots, x_n^0)} \neq 0$ *for any integer* i, $1 \leq i \leq n$.

More results on combinatorics of non-solvable algebraic, ordinary or partial differential equations can be found in references [9]-[14]. For example, let $(LDES_m^1)$ be a system of linear homogeneous differential equations

$$
\begin{cases}
\ddot{x} - 3\dot{x} + 2x = 0 & (1) \\
\ddot{x} - 5\dot{x} + 6x = 0 & (2) \\
\ddot{x} - 7\dot{x} + 12x = 0 & (3) \\
\ddot{x} - 9\dot{x} + 20x = 0 & (4) \\
\ddot{x} - 11\dot{x} + 30x = 0 & (5) \\
\ddot{x} - 7\dot{x} + 6x = 0 & (6)
\end{cases}
$$

where $\ddot{x} = \dfrac{d^2 x}{dt^2}$ and $\dot{x} = \dfrac{dx}{dt}$. Clearly, this system is non-solvable with solution bases $\{e^t, e^{2t}\}$, $\{e^{2t}, e^{3t}\}$, $\{e^{3t}, e^{4t}\}$, $\{e^{4t}, e^{5t}\}$, $\{e^{5t}, e^{6t}\}$, $\{e^{6t}, e^t\}$ respectively on equations $(1) - (6)$ and its G-solution is shown in Fig.15,
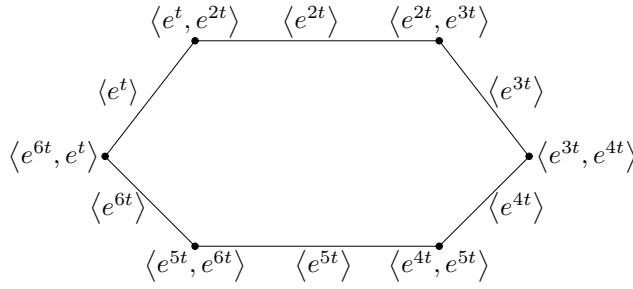
Fig.15

where $\langle \Delta \rangle$ denotes the linear space generalized by elements in Δ.

3.3 Mathematics Over Graph

Let $(\mathscr{A}; \circ_1, \circ_2, \cdots, \circ_k)$ be an algebraic system, i.e., $a \circ_i b \in \mathscr{A}$ for $\forall a, b \in \mathscr{A}$, $1 \leq i \leq k$ and let $\overrightarrow{G}$ be an oriented graph embedded in space $\mathcal{T}$. Denoted by $\overrightarrow{G}_{\mathscr{A}}^L$ all of those labeled graphs $\overrightarrow{G}^L$ with labeling $L: E\left(\overrightarrow{G}\right) \to \mathscr{A}$ constraint with ruler:

R1 : *For $\forall \overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2} \in \overrightarrow{G}_{\mathscr{A}}^L$, define $\overrightarrow{G}^{L_1} \circ_i \overrightarrow{G}^{L_2} = \overrightarrow{G}^{L_1 \circ_i L_2}$, where $L_1 \circ_i L_2: e \to L_1(e) \circ_i L_2(e)$ for $\forall e \in E\left(\overrightarrow{G}\right)$ and integers $1 \leq i \leq k$.*

For example, such a ruler on graph $\overrightarrow{C}_4$ is shown in Fig.16, where $\mathbf{a}_3 = \mathbf{a}_1 \circ_i \mathbf{a}_2$, $\mathbf{b}_3 = \mathbf{b}_1 \circ_i \mathbf{b}_2$, $\mathbf{c}_3 = \mathbf{c}_1 \circ_i \mathbf{c}_2$, $\mathbf{d}_3 = \mathbf{d}_1 \circ_i \mathbf{d}_2$.

Fig.16

Then, $\overrightarrow{G}^{L_1} \circ_i \overrightarrow{G}^{L_2} = \overrightarrow{G}^{L_1 \circ_i L_2} \in \overrightarrow{G}^L_{\mathscr{A}}$ by the ruler **R1**, and generally,

$$\overrightarrow{G}^{L_1} \circ_{i_1} \overrightarrow{G}^{L_2} \circ_{i_2} \cdots \circ_{i_s} \overrightarrow{G}^{L_{s+1}} \in \overrightarrow{G}^L_{\mathscr{A}}$$

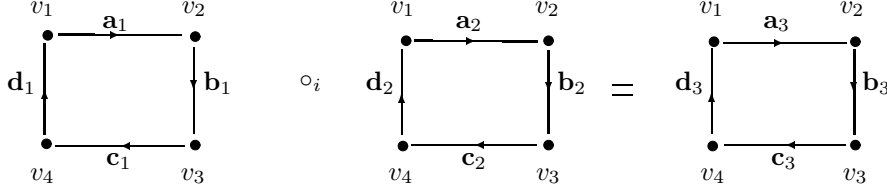for integers $1 \leq i_1, i_2, \cdots, i_s \leq k$, i.e., $\overrightarrow{G}^L_{\mathscr{A}}$ is also an algebraic system, and it is commutative on an operation $\circ_i$ if $(\mathscr{A}; \circ_1, \circ_2, \cdots, \circ_k)$ is commutative on an operation $\circ_i$ for an integer i, $1 \leq i \leq k$. Particularly, if $k = 1$, $\overrightarrow{G}^L_{\mathscr{A}}$ is a group if $(\mathscr{A}; \circ_1)$ is a group. Thus, we extend $(\mathscr{A}; \circ_1, \circ_2, \cdots, \circ_k)$ and obtain an algebraic system over graph $\overrightarrow{G}$ underlying a geometrical structure in space $\mathcal{T}$.

Notice that such an extension $\overrightarrow{G}^L_{\mathscr{A}}$ is only a pure extension of algebra over $\overrightarrow{G}$ without combining the nature of things, i.e., the conservation of matter which states that the amount of the conserved quantity at a point or within a volume can only change by the amount of the quantity which flows in or out of that volume. Thus, understanding the reality of things motives the extension of mathematical systems $(\mathscr{A}; \circ_1, \circ_2, \cdots, \circ_k)$ over graph $\overrightarrow{G}$ constrained also on the laws of conservation

R2 : $\displaystyle\sum_l \mathbf{F}(v)_l^- = \sum_s \mathbf{F}(v)_s^+$, *where* $\mathbf{F}(v)_l^-$, $l \geq 1$ *and* $\mathbf{F}(v)_s^+$, $s \geq 1$ *denote respectively the output and input amounts at vertex* $v \in E(\overrightarrow{G})$.

This notion brings about a new mathematical element finally, i.e., action flows, which combines well the continuous mathematics with that of the discrete.

Definition 3.3([19]) *An action flow* $\left(\overrightarrow{G}; L, A\right)$ *is an oriented embedded graph* $\overrightarrow{G}$ *in a topological space* $\mathscr{S}$ *associated with a mapping* $L : (v, u) \to L(v, u)$, 2 *end-operators* $A_{vu}^+ : L(v, u) \to L^{A_{vu}^+}(v, u)$ *and* $A_{uv}^+ : L(u, v) \to L^{A_{uv}^+}(u, v)$ *on a Banach space* $\mathscr{B}$
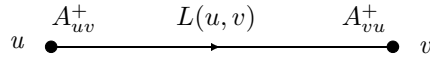
Fig.17

with $L(v, u) = -L(u, v)$ *and* $A_{vu}^+(-L(v, u)) = -L^{A_{vu}^+}(v, u)$ *for* $\forall(v, u) \in E\left(\overrightarrow{G}\right)$ *holding with conservation laws*

$$\sum_{u \in N_G(v)} L^{A_{vu}^+}(v, u) = \mathbf{c}_v \quad for \quad \forall v \in V\left(\overrightarrow{G}\right)$$

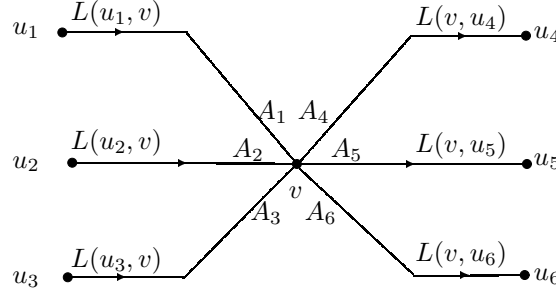*such as those shown for vertex* v *in Fig.*18 *following*

Fig.18

with a conservation law

$$-L^{A_1}(v, u_1) - L^{A_2}(v, u_2) - L^{A_4}(v, u_3) + L^{A_4}(v, u_4) + L^{A_5}(v, u_5) + L^{A_6}(v, u_6) = \mathbf{c}_v,$$

where $\mathbf{c}_v$ is the surplus flow on vertex v, and usually, let $\mathbf{c}_v = \mathbf{0}$.

Indeed, action flow is an element both with the character of continuous and discrete mathematics. For example, the conservation laws on an action flow over dipole shown in Fig.19
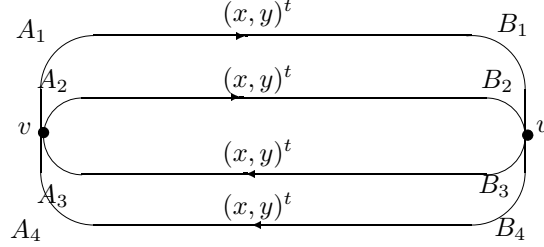
Fig.19

are partial differential equations

$$\begin{cases} a_1 \dfrac{\partial^2 x}{\partial t^2} + b_1 \dfrac{\partial^2 y}{\partial t^2} - a_3 \dfrac{\partial x}{\partial t} + (a_2 - a_4)x + (b_2 - b_3 - b_4)y = 0 \\ c_2 \dfrac{\partial^2 x}{\partial t^2} + d_2 \dfrac{\partial^2 y}{\partial t^2} - d_4 \dfrac{\partial y}{\partial t} + (c_1 - c_3 - c_4)x + (d_1 - d_3)y = 0 \end{cases},$$

where, $A_1 = (a_1 \partial^2/\partial t^2, b_1 \partial^2/\partial t^2)$, $A_2 = (a_2, b_2)$, $A_3 = (a_3 \partial/\partial t, b_3)$, $A_4 = (a_4, b_4)$, $B_1 = (c_1, d_1)$, $B_2 = (c_2 \partial^2/\partial t^2, d_2 \partial^2/\partial t^2)$, $B_3 = (c_3, d_3)$, $B_4 = (c_4, d_4 \partial/\partial t)$.

Certainly, not all mathematical systems can be extended over a graph $\overrightarrow{G}$ constraint with the laws of conservation at $v \in V(\overrightarrow{G})$ unless $\overrightarrow{G}$ with special structure but such an extension of linear space $\mathscr{A}$ can be always done.

Theorem 3.4([20]) *Let* $(\mathscr{A}; +, \cdot)$ *be a linear space,* $\overrightarrow{G}$ *an embedded graph in space* $\mathcal{T}$ *and* $A_{vu}^+ = A_{uv}^+ = \mathbf{1}_{\mathscr{A}}$ *for* $\forall(v, u) \in E(\overrightarrow{G})$. *Then,* $\left(\overrightarrow{G}_{\mathscr{A}}^L; +, \cdot\right)$ *is also a linear space under rulers* **R1** *and* **R2** *with dimension* $\dim \mathscr{A}^{\beta(\overrightarrow{G})}$ *if* $\dim \mathscr{V} < \infty$, *where* $\beta(\overrightarrow{G}) = |E(\overrightarrow{G})| - |V(\overrightarrow{G})| + 1$, *or infinite.*

An action flow $\left(\overrightarrow{G}; L, \mathbf{1}_{\mathscr{A}}\right)$, i.e., $A_{vu}^{+} = A_{uv}^{+} = \mathbf{1}_{\mathscr{A}}$ for $\forall (v, u) \in E(\overrightarrow{G})$ is usually called $\overrightarrow{G}$-*flows*, denoted by $\overrightarrow{G}^{L}$ and the linear space $\left(\overrightarrow{G}_{\mathscr{A}}^{L}; +, \cdot\right)$ extended over $\overrightarrow{G}$ by $\overrightarrow{G}^{\mathscr{A}}$ for simplicity.

§4. Banach $\overrightarrow{G}$-Flow Spaces with Multiverses

4.1 Banach $\overrightarrow{G}$-Flow Space

A Banach or Hilbert space is respectively a linear space $\mathscr{A}$ over a field $\mathbb{R}$ or $\mathbb{C}$ equipped with a complete norm $\|\cdot\|$ or inner product $\langle \cdot, \cdot \rangle$, i.e., for every Cauchy sequence $\{x_n\}$ in $\mathscr{A}$, there exists an element x in $\mathscr{A}$ such that

$$\lim_{n \to \infty} \|x_n - x\|_{\mathscr{A}} = 0 \quad \text{or} \quad \lim_{n \to \infty} \langle x_n - x, x_n - x \rangle_{\mathscr{A}} = 0,$$

which can be extended over graph $\overrightarrow{G}$ by introducing the norm of a $\overrightarrow{G}$-flow $\overrightarrow{G}^{L}$ following

$$\left\| \overrightarrow{G}^{L} \right\| = \sum_{(v,u) \in E(\overrightarrow{G})} \|L(v, u)\|,$$

where $\|L(v, u)\|$ is the norm of $L(v, u)$ in $\mathscr{A}$.

Theorem 4.1([15]) *For any graph $\overrightarrow{G}$, $\overrightarrow{G}^{\mathscr{A}}$ is a Banach space, and furthermore, if $\mathscr{A}$ is a Hilbert space, $\overrightarrow{G}^{\mathscr{A}}$ is a Hilbert space too.*

We can also consider operators action on the Banach or Hilbert $\overrightarrow{G}$-flow space $\overrightarrow{G}^{\mathscr{A}}$. Particularly, an operator $\mathbf{T}: \overrightarrow{G}^{\mathscr{A}} \to \overrightarrow{G}^{\mathscr{A}}$ is *linear* if

$$\mathbf{T}\left(\lambda \overrightarrow{G}^{L_1} + \mu \overrightarrow{G}^{L_2}\right) = \lambda \mathbf{T}\left(\overrightarrow{G}^{L_1}\right) + \mu \mathbf{T}\left(\overrightarrow{G}^{L_2}\right)$$

for $\forall \overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2} \in \overrightarrow{G}^{\mathscr{A}}$, $\lambda, \mu \in \mathscr{F}$, which enables one to generalize the representation theorem of Fréchet and Riesz on linear continuous functionals of Hilbert space to Hilbert $\overrightarrow{G}$-flow space $\overrightarrow{G}^{\mathscr{A}}$ following.

Theorem 4.2([15]) *Let $\mathbf{T}: \overrightarrow{G}^{\mathscr{A}} \to \mathbb{C}$ be a linear continuous functional. Then there is a unique $\overrightarrow{G}^{\hat{L}} \in \overrightarrow{G}^{\mathscr{A}}$ such that $\mathbf{T}\left(\overrightarrow{G}^{L}\right) = \left\langle \overrightarrow{G}^{L}, \overrightarrow{G}^{\hat{L}} \right\rangle$ for $\forall \overrightarrow{G}^{L} \in \overrightarrow{G}^{\mathscr{A}}$.*

Notice that linear continuous functionals exist everywhere in mathematics, particularly, the differential and integral operators. For example, let $\mathscr{A}$ be a Hilbert space consisting of measurable functions $f(x_1, x_2, \cdots, x_n)$ on a set

$$\Delta = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n | a_i \leq x_i \leq b_i, 1 \leq i \leq n\},$$

which is a functional space $L^2[\Delta]$ with inner product

$$\langle f(\mathbf{x}), g(\mathbf{x}) \rangle = \int_\Delta \overline{f(\mathbf{x})} g(\mathbf{x}) d\mathbf{x} \text{ for } f(\mathbf{x}), g(\mathbf{x}) \in L^2[\Delta],$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. By Theorem 4.1, $\mathscr{A}$ can be extended to Hilbert $\overrightarrow{G}$-flow space $\overrightarrow{G}^{\mathscr{A}}$, and the *differential* or *integral operators*

$$D = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} \quad \text{and} \quad \int_\Delta$$

on $\mathscr{A}$ are extended to $\overrightarrow{G}^{\mathscr{A}}$ respectively by $D\overrightarrow{G}^L = \overrightarrow{G}^{DL(v,u)}$ and

$$\int_\Delta \overrightarrow{G}^L = \int_\Delta K(\mathbf{x},\mathbf{y}) \overrightarrow{G}^{L[\mathbf{y}]} d\mathbf{y} = \overrightarrow{G}^{\int_\Delta K(\mathbf{x},\mathbf{y}) L(v,u)[\mathbf{y}] d\mathbf{y}}$$

for $\forall (v,u) \in E(\overrightarrow{G})$, where a_i, $\dfrac{\partial a_i}{\partial x_j} \in \mathbb{C}^0(\Delta)$ for integers $1 \leq i, j \leq n$ and $K(\mathbf{x}, \mathbf{y}) : \Delta \times \Delta \to \mathbb{C} \in L^2(\Delta \times \Delta, \mathbb{C})$ with

$$\int_{\Delta \times \Delta} K(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} < \infty.$$

Theorem 4.3([15]) *The differential or integral operator* $D : \overrightarrow{G}^{\mathscr{A}} \to \overrightarrow{G}^{\mathscr{A}}$, $\displaystyle\int_\Delta : \overrightarrow{G}^{\mathscr{A}} \to \overrightarrow{G}^{\mathscr{A}}$ *both are linear operators on* $\overrightarrow{G}^{\mathscr{A}}$.

For example, let $f(t) = t$, $g(t) = e^t$, $K(t, \tau) = t^2 + \tau^2$ for $\Delta = [0,1]$ and let $\overrightarrow{G}^L$ be the $\overrightarrow{G}$-flow shown on the left in Fig.20. Then we get the $\overrightarrow{G}$-flows on the right in Fig.20 by the differential or integral operator action on.
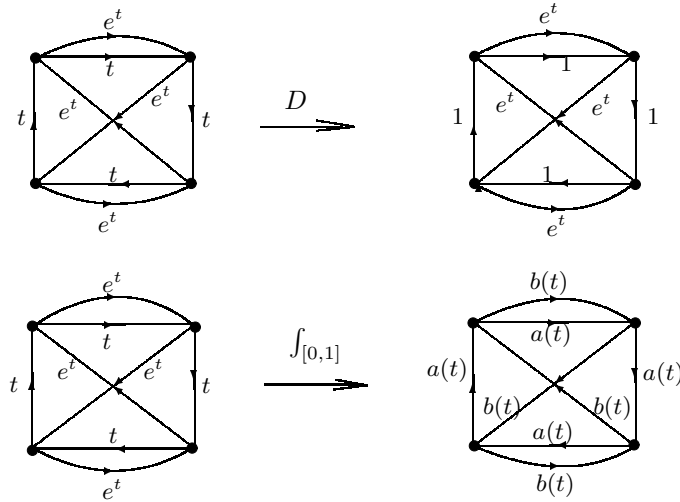

Fig.20

where $a(t) = \dfrac{t^2}{2} + \dfrac{1}{4}$ and $b(t) = (e-1)t^2 + e - 2$.

4.2 Multiverses on Equations

Notice that solving Schrödinger equation (1.3) with initial data only get one state of a particle P but the particle is in superposition, which brought the H.Everett multiverse on superposition and the quark model of Sakata, or Gell-Mann and Ne'eman on particles machinery. However, Theorems $4.1 - 4.3$ enables one to get multiverses constraint with linear equations (3.1) in $\overrightarrow{G}^{\mathscr{A}}$.

For example, we can consider the Cauchy problem

$$\frac{\partial X}{\partial t} = c^2 \sum_{i=1}^{n} \frac{\partial^2 X}{\partial x_i^2}$$

with initial values $X|_{t=t_0}$ in $\overrightarrow{G}^{\mathbb{R}^n \times \mathbb{R}}$, i.e., Hilbert space $\mathbb{R}^n \times \mathbb{R}$ over graph $\overrightarrow{G}$, and get multiverse solutions of heat equation following.

Theorem 4.4([15]) *For $\forall \overrightarrow{G}^{L'} \in \overrightarrow{G}^{\mathbb{R}^n \times \mathbb{R}}$ and a non-zero constant c in $\mathbb{R}$, the Cauchy problems on differential equation*

$$\frac{\partial X}{\partial t} = c^2 \sum_{i=1}^{n} \frac{\partial^2 X}{\partial x_i^2}$$

with initial value $X|_{t=t_0} = \overrightarrow{G}^{L'} \in \overrightarrow{G}^{\mathbb{R}^n \times \mathbb{R}}$ is solvable in $\overrightarrow{G}^{\mathbb{R}^n \times \mathbb{R}}$ if $L'(v,u)$ is continuous and bounded in $\mathbb{R}^n$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$.

And then, the H.Everett's multiverse on the Schrödinger equation (1.3) is nothing else but a 2-branch tree
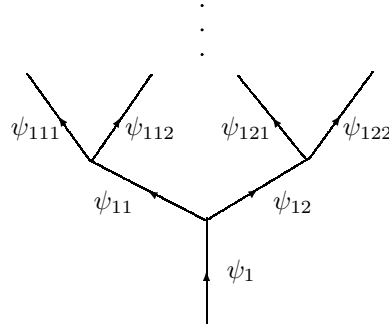
Fig.21

with equalities $\psi_1 = \psi_{11} + \psi_{12}$, $\psi_{11} = \psi_{111} + \psi_{112}$, $\psi_{12} = \psi_{121} + \psi_{122}$, $\cdots$ ([16], [17]).

If the equations (3.1) is not linear, we can not immediately apply Theorems $4.1 - 4.3$ to get multiverse over any graphs $\overrightarrow{G}$. However, if the graph $\overrightarrow{G}$ is prescribed with special structures, for instance the circuit decomposable, we can also solve the Cauchy problem on an equation in Hilbert $\overrightarrow{G}$-flow space $\overrightarrow{G}^{\mathscr{A}}$ if it is solvable in $\mathscr{A}$ and obtain a general conclusion following, which enable us to interpret also the superposition of particles ([17]), biological diversity and establish multiverse model of spacetime in Einstein's gravitation.

Theorem 4.5([15]) *If the graph $\overrightarrow{G}$ is strong-connected with circuit decomposition $\overrightarrow{G} = \bigcup\limits_{i=1}^{l} \overrightarrow{C}_i$ such that $L(v, u) = L_i(\boldsymbol{x})$ for $\forall (v, u) \in E\left(\overrightarrow{C}_i\right)$, $1 \leq i \leq l$ and the Cauchy problem*

$$\begin{cases} \mathscr{F}_i\left(\boldsymbol{x}, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1 x_2}, \cdots\right) = 0 \\ u|_{\boldsymbol{x}_0} = L_i(\boldsymbol{x}) \end{cases}$$

is solvable in a Hilbert space $\mathscr{A}$ on domain $\Delta \subset \mathbb{R}^n$ for integers $1 \leq i \leq l$, then the Cauchy problem

$$\begin{cases} \mathscr{F}_i\left(\boldsymbol{x}, X, X_{x_1}, \cdots, X_{x_n}, X_{x_1 x_2}, \cdots\right) = 0 \\ X|_{\boldsymbol{x}_0} = \overrightarrow{G}^L \end{cases}$$

such that $L(v, u) = L_i(\boldsymbol{x})$ for $\forall (v, u) \in E\left(\overrightarrow{C}_i\right)$ is solvable for $\overrightarrow{G}^L \in \overrightarrow{G}^{\mathscr{A}}$.

Theorem 4.5 enables one to explore the multiverse, particularly, the solutions of Einstein's gravitational equations

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} + \lambda g^{\mu\nu} = -8\pi G T^{\mu\nu},$$

where $R^{\mu\nu} = R^{\mu\alpha\nu}_{\alpha} = g_{\alpha\beta} R^{\alpha\mu\beta\nu}$, $R = g_{\mu\nu} R^{\mu\nu}$ are the respective Ricci tensor, Ricci scalar curvature, $G = 6.673 \times 10^{-8} cm^3/gs^2$, $\kappa = 8\pi G/c^4 = 2.08 \times 10^{-48} cm^{-1} \cdot g^{-1} \cdot s^2$. In fact, Einstein's general relativity is established on $\mathbb{R}^4$. However, *if the dimension n of the universe> 4, how can we characterize the structure of spacetime for the universe?* In fact, if the dimension of the universe> 4, all observations are nothing else but a projection of the true faces on our six organs because the dimension of human beings is 3 hold with

$$\mathbb{R}^n = \bigcup_{i=1}^{m} \mathbb{R}^4_i \quad \text{and} \quad \left| \bigcap_{i=1}^{m} \mathbb{R}^4_i \right| = 1$$

such as those shown in Fig.22 for a projection of 3-dimensional objects on Euclidean plane $\mathbb{R}^2$.
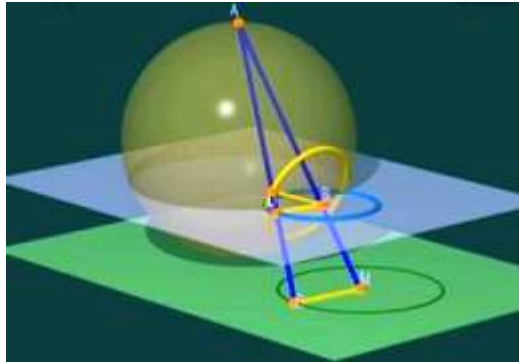
Fig.22

In this case, we can characterize the spacetime with a complete graph K_m^L labeled by $\mathbb{R}^4$ ([7]-[8]). For example, if $m = 4$ there are 4 Einstein's gravitational equations respectively on

$v \in V\left(K_4^L\right)$. We can solve them one by one by applying the spherically symmetric solution in $\mathbb{R}^4$ and construct K_4^L shown in Fig.23,
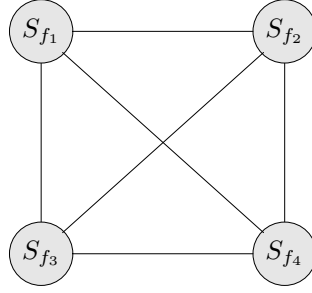

Fig.23

where, each $S_{\mathscr{F}_i}$ is the geometrical space of the Schwarzschild spacetime

$$ds^2 = f(t)\left(1 - \frac{r_s}{r}\right)dt^2 - \frac{1}{1 - \frac{r_s}{r}}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

for integers $1 \le i \le 4$.

Notice that $m = 4$ is only an assumption. We do not know the exact value of m at present. Similarly, by Theorem 4.5 we can also get a conclusion on multiverse of the Einstein's gravitational equations, even in $\mathbb{R}^4$. Certainly, we do not know also which is the real spacetime of the universe.

Theorem 4.6([15], [19]) *There are infinite many $\overrightarrow{G}$-flow solutions on Einstein's gravitational equations*

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = -8\pi GT^{\mu\nu}$$

in $\overrightarrow{G}^{\mathbb{C}}$, particularly on those graphs with circuit-decomposition $\overrightarrow{G} = \bigcup_{i=1}^{m} \overrightarrow{C}_i$ labeled with Schwarzschild spacetime on their edges.

For example, let $\overrightarrow{G} = \overrightarrow{C}_4$. We are easily find $\overrightarrow{C}_4$-flow solution of Einstein's gravitational equations such as those shown in Fig.24.
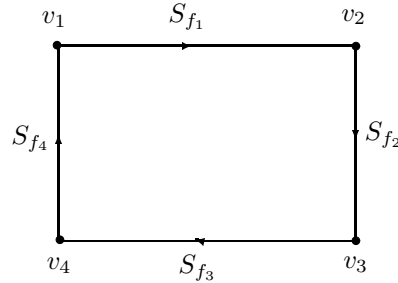

Fig.24

Then, the spacetime of the universe is nothing else but a curved ring in space such as those shown in Fig.25.
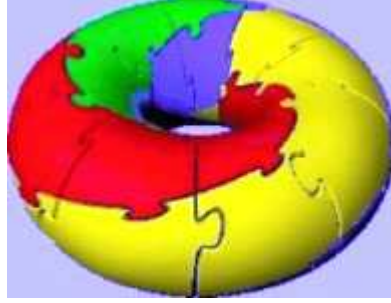
Fig.25

Generally, if $\overrightarrow{G}$ is the union of m orientated circuits $\overrightarrow{C}_i$, $1 \le i \le m$, Theorem 4.6 implies the spacetime of Einstein's gravitational equations is a multiverse consisting of m curved rings over graph $\overrightarrow{G}$ in space.

Notice that a graph $\overrightarrow{G}$ is circuit decomposable if and only if it is an Eulerian graph. Thus, Theorems $4.1 - 4.5$ can be also applied to biology with global stability of food webs on n species following.

Theorem 4.7([21]) *A food web $\overrightarrow{G}^L$ with initial value $\overrightarrow{G}^{L_0}$ is globally stable or asymptotically stable if and only if there is an Eulerian multi-decomposition*

$$\left(\overrightarrow{G} \bigcup \overleftarrow{G}\right)^{\widehat{L}} = \bigoplus_{i=1}^{s} \overrightarrow{H}_i^L$$

with solvable stable or asymptotically stable conservative equations on Eulerian subgraphs $\overrightarrow{H}_i^L$ for integers $1 \le i \le s$, where $\left(\overrightarrow{G} \bigcup \overleftarrow{G}\right)^{\widehat{L}}$ is the bi-digraph of $\overrightarrow{G}$ defined by $\overrightarrow{G} \bigcup \overleftarrow{G}$ with a labeling $\widehat{L} : V(\overrightarrow{G} \bigcup \overleftarrow{G}) \to L\left(V(\overrightarrow{G})\right)$, $\widehat{L} : E\left(\overrightarrow{G} \bigcup \overleftarrow{G}\right) \to L\left(E\left(\overrightarrow{G} \bigcup \overleftarrow{G}\right)\right)$ by $\widehat{L} : (u,v) \to \{0, (x,y), yf'\}$, $(v,u) \to \{xf, (x,y), 0\}$ if $L : (u,v) \to \{xf, (x,y), yf'\}$ for $\forall (u,v) \in E(\overrightarrow{G})$, such as those shown in Fig.26,
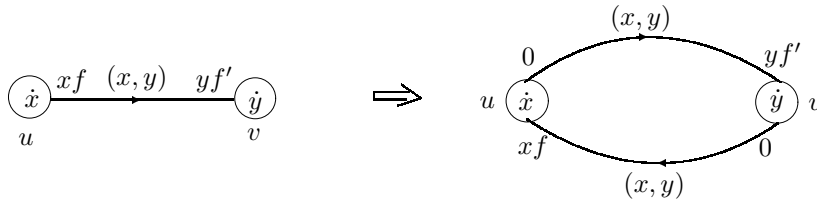
Fig.26

and a multi-decomposition $\bigoplus_{i=1}^{s} \overrightarrow{H}_i^L$ of $\left(\overrightarrow{G} \bigcup \overleftarrow{G}\right)^{\widehat{L}}$ is defined by

$$\left(\overrightarrow{G} \bigcup \overleftarrow{G}\right)^{\widehat{L}} = \bigcup_{i=1}^{s} \overrightarrow{H}_i$$

with $\overrightarrow{H}_i \neq \overrightarrow{H}_j$, $\overrightarrow{H}_i \bigcap \overrightarrow{H}_j = \emptyset$ or $\neq \emptyset$ for integers $1 \leq \neq j \leq s$.

Theorem 4.8([21]) *A food web $\overrightarrow{G}^L$ with initial value $\overrightarrow{G}^{L_0}$ is globally asymptotically stable if there is an Eulerian multi-decomposition*

$$\left(\overrightarrow{G} \bigcup \overleftarrow{G} \right)^{\widehat{L}} = \bigoplus_{k=1}^{s} \overrightarrow{H}_k^L$$

with solvable conservative equations such that $Re\lambda_i < 0$ for characteristic roots λ_i of A_v in the linearization

$$A_v X_v = 0_{h_v \times h_v}$$

of conservative equations at an equilibrium point $\overrightarrow{H}_k^{L_0}$ in $\overrightarrow{H}_k^L$ for integers $1 \leq i \leq h_v$ and $v \in V(\overrightarrow{H}_k^L)$, where $V(\overrightarrow{H}_k^L) = \{v_1, v_2, \cdots, v_{h_v}\}$,

$$A_v = \begin{pmatrix} a_{11}^v & a_{12}^v & \cdots & a_{1h_v}^v \\ a_{21}^v & a_{22}^v & \cdots & a_{2h_v}^v \\ \cdots & \cdots & \cdots & \cdots \\ a_{h1}^v & a_{h2}^v & \cdots & a_{hh_v}^v \end{pmatrix}$$

a constant matrix and $X_k = (x_{v_1}, x_{v_2}, \cdots, x_{v_{h_v}})^T$ for integers $1 \leq k \leq l$.

§5. Conclusion

Answer the question which is better to the reality of things on the continuous or discrete mathematics is not easy because our world appears both with the continuous and discrete characters. However, contradictions exist everywhere which are all artificial, not the nature of things. Thus, holding on the reality of things which are in fact being a complex system motivates one to turn mathematical contradictory systems to compatible systems, i.e., giving up the notion that contradiction is meaningless and establish an envelope theory on mathematics, which needs the combination of the continuous mathematics with that of discrete, i.e., mathematical combinatorics because a non-mathematics in classical is in fact a mathematics over a graph $\overrightarrow{G}$ (See [13] for details).

References

[1] R.Abraham and J.E.Marsden, *Foundation of Mechanics* (2nd edition), Addison-Wesley, Reading, Mass, 1978.

[2] A.Eistein, *Relativity: the Special and General Theory*, Methuen & Co Ltd., 1916.

[3] H.Everett, Relative state formulation of quantum mechanics, *Rev.Mod.Phys.*, 29(1957), 454-462.

[4] Fred Brauer and Carlos Castillo-Chaver, *Mathematical Models in Population Biology and Epidemiology*(2nd Edition), Springer, 2012.

[5] Y.Lou, Some reaction diffusion models in spatial ecology (in Chinese), *Sci.Sin. Math.*, Vol.45(2015), 1619-1634.

[6] Linfan Mao, Combinatorial speculation and combinatorial conjecture for mathematics, *International J.Math. Combin.* Vol.1(2007), No.1, 1-19.

[7] Linfan Mao, Geometrical theory on combinatorial manifolds, *JP J.Geometry and Topology*, Vol.7, No.1(2007),65-114.

[8] Linfan Mao, *Combinatorial Geometry with Applications to Field Theory*, The Education Publisher Inc., USA, 2011.

[9] Linfan Mao, Non-solvable spaces of linear equation systems, *International J.Math. Combin.*, Vol.2(2012), 9-23.

[10] Linfan Mao, Global stability of non-solvable ordinary differential equations with applications, *International J.Math. Combin.*, Vol.1 (2013), 1-37.

[11] Linfan Mao, Non-solvable equation systems with graphs embedded in $\mathbf{R}^n$, *Proceedings of the First International Conference on Smarandache Multispace and Multistructure*, The Education Publisher Inc. July, 2013

[12] Linfan Mao, Geometry on G^L-systems of homogenous polynomials, *International J.Contemp. Math. Sciences*, Vol.9 (2014), No.6, 287-308.

[13] Linfan Mao, Mathematics on non-mathematics - A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.

[14] Linfan Mao, Cauchy problem on non-solvable system of first order partial differential equations with applications, *Methods and Applications of Analysis*, Vol.22, 2(2015), 171-200.

[15] Linfan Mao, Extended Banach $\overrightarrow{G}$-flow spaces on differential equations with applications, *Electronic J.Mathematical Analysis and Applications*, Vol.3, No.2 (2015), 59-91.

[16] Linfan Mao, A new understanding of particles by $\overrightarrow{G}$-flow interpretation of differential equation, *Progress in Physics*, Vol.11(2015), 193-201.

[17] Linfan Mao, A review on natural reality with physical equation, *Progress in Physics*, Vol.11(2015), 276-282.

[18] Linfan Mao, Mathematics after CC conjecture – combinatorial notions and achievements, *International J.Math. Combin.*, Vol.2(2015), 1-31.

[19] Linfan Mao, Mathematics with natural reality–action flows, *Bull.Cal.Math.Soc.*, Vol.107, 6(2015), 443-474.

[20] Linfan Mao, Labeled graph – A mathematical element, *International J.Math. Combin.*, Vol.3(2016), 27-56.

[21] Linfan Mao, Biological n-system with global stability, *Bull.Cal.Math.Soc.*, Vol.108, 6(2016), 403-430.

[22] J.D.Murray, *Mathematical Biology I: An Introduction* (3rd Edition), Springer-Verlag Berlin Heidelberg, 2002.

[23] Y.Nambu, *Quarks: Frontiers in Elementary Particle Physics*, World Scientific Publishing Co.Pte.Ltd, 1985.

[24] Quang Ho-Kim and Pham Xuan Yem, *Elementary Particles and Their Interactions*, Springer-Verlag Berlin Heidelberg, 1998.

[25] F.Smarandache, *Paradoxist Geometry*, State Archives from Valcea, Rm. Valcea, Romania,

1969, and in *Paradoxist Mathematics*, Collected Papers (Vol. II), Kishinev University Press, Kishinev, 5-28, 1997.

[26] F.Smarandache, A new form of matter – unmatter, composed of particles and anti-particles, *Progess in Physics*, Vol.1(2005), 9-11.

[27] F.Smarandache and D.Rabounski, Unmatter entities inside nuclei, predicted by the Brightsen nucleon cluster model, *Progress in Physics*, Vol.1(2006), 14-18.

[28] M.Tegmark, Parallel universes, in *Science and Ultimate Reality: From Quantum to Cosmos*, ed. by J.D.Barrow, P.C.W.Davies and C.L.Harper, Cambridge University Press, 2003.

The First Zagreb Index, Vertex-Connectivity, Minimum Degree

And Independent Number in Graphs

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Abstract: Let G be a simple, undirected and connected graph. Defined by $M_1(G)$ and $RMTI(G)$ the first Zagreb index and the reciprocal Schultz molecular topological index of G, respectively. In this paper, we determined the graphs with maximal M_1 among all graphs having prescribed vertex-connectivity and minimum degree, vertex-connectivity and bipartition, vertex-connectivity and vertex-independent number, respectively. As applications, all maximal elements with respect to $RMTI$ are also determined among the above mentioned graph families, respectively.

Key Words: Zagreb index, reciprocal molecular topological index, vertex connectivity, bipartite graph, independent number.

AMS(2010): 05C12.

§1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex v is the number of edges incident to v and denoted by $d(v)$. One of the most important topological indices is the well-known Zagreb indices introduced in [8, 10], the first and second Zagreb indices M_1 and M_2 of G, respectively, are defined as follows:

$$M_1(G) = \sum_{v \in V(G)} d(v)^2, M_2(G) = \sum_{uv \in E(G)} d(u)d(v).$$

They reflect the extent of branching of the underlying molecular structure [8, 10, 20]. Their main properties were recently summarized in [1, 4, 6, 7, 9, 11, 12, 13, 15, 16, 23, 24, 25, 26].

Let G be a connected graph with n vertices. The distance matrix $\mathbf{D} = (D_{ij})_{n \times n}$ of G is an $n \times n$ matrix such that D_{ij} is the distance between vertices i and j in G [18]. The reciprocal distance matrix $\mathbf{R}$, also called the Harary matrix (see [14, 18]), is defined as an $n \times n$ matrix $R = (R_{ij})$ such that $R_{ij} = \frac{1}{D_{ij}}$ if $i \neq j$ and 0 otherwise. Let $R_i = \sum_{j=1}^n R_{ij}$. Then the

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reciprocal molecular topological index $RMTI$ [20] of G is defined as

$$RMTI(G) = \sum_{i=1}^{n} R_i^2 + \sum_{i=1}^{n} d_i R_i.$$

Some formulations of reciprocal and constant-interval reciprocal Schultz-type topological indices, included RMTI, have been discussed in [20], and they were illustrated by the QSPR, which studied on physical constants of alkanes and cycloalkanes.

Recently, Zhou and Trinajstić [27] reported some properties of the reciprocal molecular topological index RMTI. They also derived the upper bounds for RMTI in terms of the number of vertices and the number of edges for various classes of graphs under some restricted conditions.

In this paper, we determined, respectively, the graphs with maximal value of M_1 among all graphs having prescribed graph invariants, such as, vertex-connectivity and minimum degree, vertex-connectivity and vertex-independent number. As applications, all maximum elements with respect to $RMTI(G)$ are also determined among the above mentioned graph families, respectively.

§2. Preliminaries

Denoted by $\delta(G)$ the minimum degree of G, and by $Diam(G)$ the diameter of a graph G, i.e., the maximum cardinality among all distance of any one pair of vertices in G. Let K_n be the complete graph with n vertices. Suppose that G_1 and G_2 are graphs with $V(G_1) \cap V(G_2) = \emptyset$. Denoted by $G_1 \cup G_2$ the new graph with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The join of G_1 and G_2, denoted by $G_1 \vee G_2$, is the new graph with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2) \cup \{xy | x \in V(G_1) \ and \ y \in V(G_2)\}$.

For $S, S' \subseteq V(G)$, the induced subgraph of S, denoted by $G[S]$, is the graph whose vertex set is S and edge set is composed of those edges with both ends in S. The induced subgraph of S and S', denoted by $G[S, S']$, is the graph whose vertex set is $S_1 \cup S_2$ and edge set is composed of those edges with one end in S and another end in S'.

The bipartite graph is the graph whose vertices can be divided into two disjoint sets U and V, such that every edge connects a vertex in U to one in V. Vertex sets U and V usually called the parts of the graph. A vertex cut of a connected graph G is a set of vertices whose removal renders G disconnected. The vertex-connectivity $\kappa(G)$ is the size of a minimal vertex cut. An independent set of G is a set of vertices in a graph G, no two of which are adjacent. A maximum independent set is an independent set of largest possible size for a given graph G. This size is called the independence number of G, and denoted by $\alpha(G)$.

For other notations and terminology not defined here, see [5].

By the definition of the first Zagreb index, the lemma follows immediately.

Lemma 2.1 *Let G be a simple graph with $u, v \in V(G)$ and $uv \notin E(G)$. Then*

$$M_1(G + uv) > M_1(G).$$

Lemma 2.2([27]) *Let G be a connected simple graph with n vertices and m edges. Then*

$$RMTI(G) \leq \frac{3}{2}M_1(G) + (n-1)m,$$

with equality holds if and if $Diam(G) \leq 2$.

§3. Graphs with Given Connectivity and Minimum Degree

Let n, k and δ be integers such that $n \geq \delta \geq k \geq 1$. Denoted by $\mathcal{G}(n, k, \delta)$ the set of n-vertex connected graphs with vertex-connectivity k and minimum degree δ, where $1 \leq k \leq \delta$ and $2 \leq \delta \leq n$.

Theorem 3.1 *If $G \in \mathcal{G}(n, k, \delta)$ with $k \leq \delta \leq n - 1$. Then*

$$M_1(G) \leq n(n-1)^2 + (n-k)(k+\delta-2n+3)(k+\delta+1)$$

with equality holds if and only if $G = K_k \vee (K_{\delta-k+1} \bigcup K_{n-\delta-1})$.

Proof If $n = k+1$, then $k = \delta = n-1$, i.e., $\mathcal{G}(n, k, \delta) = \{K_{k+1}\}$. Suppose that $n \geq k+2$. Let G_{max} be graph in $\mathcal{G}(n, k, \delta)$ with maximal M_1- value in $\mathcal{G}(n, k, \delta)$, that is, $M_1(G) \leq M_1(G_{max})$ for all $G \in \mathcal{G}(n, k, \delta)$. Denoted by $S \subset V(G_{max})$ the vertex cut and $|S| = k$. We will prove the three claims as follows.

Claim 1. $G_{max} - S$ contain exactly two components.

Proof of Claim 1: Suppose by contrary that $G_{max} - S$ contain at least three components. Denoted two components of $G_{max} - S$ by C_1 and C_2. There exist vertices $u \in V(C_1), v \in V(C_2)$ such that $G_{max} + uv \in \mathcal{G}(n, k, \delta)$. By Lemma 2.1, $M_1(G_{max} + uv) > M_1(G_{max})$, which contradicts the choice of G_{max}. This completes the proof of Claim 1.

Therefore, we assume that $G_{max} - S$ contain exactly two connected components, denoted by C_1 and C_2. Denoted by $|V(C_1)| = n_1, |V(C_2)| = n_2$. Since $\delta \leq d(u) \leq n_1 - 1 + k$ and $\delta \leq d(v) \leq n_2 - 1 + k$ for $u \in V(C_1), v \in V(C_2)$, we have $n_1, n_2 \geq \delta - k + 1$.

Claim 2. $G_{max}[S \cup V(C_1)]$ and $G_{max}[S \cup V(C_2)]$ are cliques.

Proof of Claim 2: Without loss of generality, suppose by contrary that $G_{max}[S \bigcup V(C_1)]$ is not a clique. There are two cases as follows:

Case 1. There exists nonadjacent vertices $u, v \in S \bigcup V(C_1)$ such that $G_{max} + uv \in \mathcal{G}(n, k, \delta)$, then by Lemma 2.1, $M_1(G_{max} + uv) > M_1(G_{max})$, which contradicts the choice of G_{max}.

Case 2. Otherwise, adding a new edge to G_{max} will increase the minimum degree of G. From Eqn.(1) in the proof of Claim 3, we have

$$M_1(G_{max}) < M_1(K_s \vee (K_{n_1} \bigcup K_{n_2})) \leq M_1(K_k \vee (K_{\delta-k+1} \bigcup K_{n-\delta-1})),$$

which contradicts the choice of G_{max} since $K_k \vee (K_{\delta-k+1} \bigcup K_{n-\delta-1}) \in \mathcal{G}(n, k, \delta)$.

This complete the proof of Claim 2.

From Claim 2, we suppose that $G_{max} = K_k \vee (K_{n_1} \bigcup K_{n_2})$, where $n_1, n_2 \geq 1$ and $n_1 + n_2 = n - k$.

Claim 3. $n_1 = \delta - k + 1$ or $n_2 = \delta - k + 1$.

Proof of Claim 3: Consider the graph $K_k \vee (K_{n_1} \bigcup K_{n_2})$. Suppose by contrary that $n_1 \geq n_2 > \delta - k + 1$, by direct calculation, we have

$$M_1(K_k \vee (K_{n_1+1} \bigcup K_{n_2-1})) > M_1(K_k \vee (K_{n_1} \bigcup K_{n_2})) \tag{1}$$
$$= k(n-1)^2 + n_1(k + n_1 - 1)^2 + (n - k - n_1)(n - n_1 - 1)^2,$$

which implies that $M_1(K_k \vee (K_{\delta-k+1} \bigcup K_{n-\delta-1})) > M_1(K_k \vee (K_{n_1} \bigcup K_{n_2}))$ if $n_1, n_2 > min\{\delta - k + 1, n - \delta - 1\}$. This complete the proof of Claim 3.

By combine above claims, we have $G_{max} = K_k \vee (K_{\delta-k+1} \bigcup K_{n-\delta-1})$. Then the result holds. $\square$

Corollary 3.1 *Let* $G \in \mathcal{G}(n, k, \delta)$ *with* m *edges and* $k \leq \delta \leq n - 1$. *Then*

$$RMTI(G) \leq \frac{3}{2}n(n-1)^2 + \frac{3}{2}n(n-k)(k + \delta - 2n + 3)(k + \delta + 1) + (n-1)m$$

with equality if and only if $k = n - 1$ *and* $G = K_{k+1}$.

§4. Bipartite Graphs with Given Connectivity

Let $\mathcal{B}(n, k)$ be the set of bipartite graphs with n vertices and $\kappa(G) = k$, and $B_{n,x}$ the graph obtained from $K_{x,n-x-1}$ by adding a new vertex v to k vertices of degree x of $K_{x,n-x-1}$.

Theorem 4.1 *Let* $G \in \mathcal{B}(n, k)$ *with* $1 \leq k \leq n - 1$. *Then*

$$M_1(G) \leq max\{f(a), f(b)\},$$

with equality if and only if $G \in \{B_{n,a}, B_{n,b}\}$, *where*

$$f(x) = nx(n - x) - x(2n + 2k + 1) + k(k - 1),$$
$$a = \left\lfloor \frac{(n-1)^2 - 2(k+1)}{2n} \right\rfloor \quad and$$
$$b = \left\lceil \frac{(n-1)^2 - 2(k+1)}{2n} \right\rceil .$$

Proof If $k = 1$, then $\mathcal{B}(n, k) = \{K_{1,n-1}\}$. Suppose that $1 < k \leq \frac{n}{2}$, let G_{max} be the graph with the maximal M_1- value in $\mathcal{B}(n, k)$, and S a $k-$vertex cut of G_{max}. Let A, B be vertex parts of $V(G_{max})$ such that $A \cup B = V(G_{max})$. Denoted by $S_A = S \cap A, S_B = S \cap B$. We will

prove the four claims as follows.

Claim 1. $G_{max}[S]$ and $G_{max}[C \cup S]$ are complete bipartite graphs, where C is one of components in $G_{max} - S$.

Proof of Claim 1: Suppose by contrary that $G_{max}[S]$ or $G_{max}[C \cup S]$ is not a complete bipartite graph. There exist vertices $u, v \in V(G_{max})$ and $uv \notin E(G_{max})$ such that $G_{max} + uv \in \mathcal{B}(n, k)$. By Lemma 2.1, we have $M_1(G_{max} + uv) > M_1(G_{max})$, which contradicts the choice of G_{max}. This complete the proof of Claim 1.

Claim 2. If $S_A \neq \emptyset$ and $S_B \neq \emptyset$, then $G_{max} - S$ have exactly two components.

Proof of Claim 2: Suppose by contrary that $G_{max} - S$ contain at least three components. Let C_1 and C_2 be two components of $G_{max} - S$. Then there exist vertices $u \in V(C_1) \cap A$, $v \in V(C_2) \cap B$ such that $G_{max} + uv \in \mathcal{B}(n, k)$ and S is also a $k-$vertex cut of $G_{max} + uv$. By Lemma 2.1, $M_1(G_{max} + uv) > M_1(G_{max})$, which contradicts the choice of G_{max}. Thus Claim 2 holds.

Claim 3. $S_A = \emptyset$ or $S_B = \emptyset$.

Proof of Claim 3: Suppose by contrary that $S_A \neq \emptyset$ and $S_B \neq \emptyset$. From Claim 2, $G_{max} - S$ contain exactly two components, denoted by C_1, C_2. Let $u \in V(C_1) \cap A$ and $v \in V(C_2) \cap A$. Without loss of generality, we assume that $a = d(u) \geq d(v) = b > 0$ and $|N_{C_2}(v)| = c > 0$. Taking transformations on G_{max} as follows.

(1) Let $G_1 = G_{max} - \{wv : w \in N_{C_2}(v)\} + \{wu : w \in N_{C_2}(v)\}$. Then

$$M_1(G_1) - M_1(G_{max}) \quad = \quad (b + c)^2 + (a - c)^2 - (b^2 + a^2) = 2c(b - a + c) > 0,$$

(2) Consider the graph G_1. Using the definitions from G_{max}. Let $|S_B| = s$, and choose arbitrary vertices $v_1, v_2, \cdots, v_{k-s} \in B - S$. Let G_2 be the graph obtained from G_1 by adding more edges between $A - \{v\}$ and B as possible, and then adding edges $v_1 v, v_2 v, \cdots, v_{k-s} v$. It is obviously that $N_{G_2}(v)$ is the vertex cut of G_2 and $|N_{G_2}(v)| = k$, i.e., $G_2 \in \mathcal{B}(n, k)$. From 1) of Claim 3 and Lemma 2.1, we have

$$M_1(G_2) > M_1(G_{max}),$$

which contradicts the choice of G_{max}. Thus Claim 3 holds.

From Claim 3, without loss of generality, let $S \subset A$ be the $k-$vertex cut of G_{max}.

Claim 4. $G_{max} - S$ contains a isolated vertex.

Proof of Claim 4: From Claim 1, suppose by contrary that the components of $G_{max} - S$, denoted by C_1, C_2, are complete bipartite graphs. Let $V(C_1) = A_1 \cup B_1$ and $V(C_2) = A_2 \cup B_2$, where A_i, B_i are vertex parts of C_i, i.e., $A_i \subset A, B_i \subset B$, $i = 1, 2$.

Without loss of generality, suppose that $S \subset A$. Let $u \in B_1$. Let G^* be the graph obtained from G_{max} by deleting the edges connecting u and vertices in A_1, and adding more edges between $A - S$ and $B - \{u\}$ as possible, i.e., $G^* = G_{max} - \{xu : x \in A_1\} + \{xy : x \in A - S, y \in$

$B - \{u\}, xy \notin E(G_{max})\}$. Then S is also a $k-$vertex cut of G_2, and $G_2 \in \mathcal{B}(n, k)$. Similar to Claim 3(1), we have

$$M_1(G^*) - M_1(G_{max}) > 0,$$

which contradicts the choice of G_{max}. Thus $G_{max} = B_{n,x}$.

By calculation, we have

$$f(x) = M_1(B_{n,x}) = nx(n - x) - x(2n + 2k + 1) + k(k - 1)$$

and $x = \frac{(n-1)^2 - 2(k+1)}{2n}$, and can obtains its maximal value by differentiating $f(x)$ on x. Since $k \le \frac{(n-1)^2 - 2(k+1)}{2n} \le n - 2$, let

$$a = \left\lfloor \frac{(n-1)^2 - 2(k+1)}{2n} \right\rfloor \text{ and } b = \left\lceil \frac{(n-1)^2 - 2(k+1)}{2n} \right\rceil.$$

Then by Claim 4, we have $G_{max} \in \{B_{n,a}, B_{n,b}\}$. This completes the proof. $\qquad \square$

Corollary 4.1 *Let $G \in \mathcal{B}(n, k)$ with m edges and $1 \le k \le n - 1$. Then*

$$RMTI(G) \le \frac{3}{2}T + (n - 1)m,$$

where

$$
\begin{aligned}
T &= max\{f(a), f(b)\}, \quad f(x) = nx(n - x) - x(2n + 2k + 1) + k(k - 1), \\
a &= \left\lfloor \frac{(n-1)^2 - 2(k+1)}{2n} \right\rfloor \text{ and} \\
b &= \left\lceil \frac{(n-1)^2 - 2(k+1)}{2n} \right\rceil.
\end{aligned}
$$

§5. Graphs with Given Connectivity and Independent Number

Let $\mathcal{D}(n, k, r)$ be the set of n-vertex graphs with $\kappa(G) = k$ and $\alpha(G) = r$.

Theorem 5.1 *Let $G \in \mathcal{D}(n, k, r)$ with $r \ge 1$ and $1 \le k \le n - 1$. Then*

$$M_1(G) \le (r - 1)(n - r)^2 + (n - r)(n - 2)^2 + k^2 + k(2n - 3),$$

with equalities hold if and only if $G = K_k \vee (K_1 \cup (K_{n-k-r} \vee (r - 1)K_1))$.

Proof If $r = 2$, then $M_1(G) \le M_1(K_k \vee (K_1 \cup K_{n-k-1}))$ from [11]. We assume that $2 \le r \le n - 1$, let G_{max} be the graph with the maximal M_1- value in $\mathcal{D}(n, k, r)$. Denoted by S the $k-$vertex cut of G_{max}, and by D the maximum independent set of G_{max}. We will prove three claims as follows.

Claim 1. $G_{max}[C] = K_a \vee (c-a)K_1$, $G_{max}[S] = K_b \vee (k-b)K_1$ and $G_{max}[S \cup V(C)] = K_{a+b} \cup (k+c-a-b)K_1$, where C is one of components of $G_{max} - S$, $|V(C)| = c$, $|V(C) - D| = a$ and $|S - D| = b$.

Proof of Claim 1: By Lemma 2.1 and the definition of G_{max}, it clear that $G_{max} = G_{max}[S] \vee G_{max}[V(G) - S]$. Now suppose by contrary that $G_{max}[C] \neq K_a \vee (c-a)K_1$. There exist $u, v \in V(C) - D$ and $w \in V(C) \cap D$, such that $uv \notin E(G_{max})$ or $uw \notin E(G_{max})$. It is clear that $G_{max} + uv, G_{max} + uw \in \mathcal{D}(n, k, r)$. By Lemma 2.1, we have $M_1(G_{max} + uv) > M_1(G_{max})$ or $M_1(G_{max} + uw) > M_1(G_{max})$, which contradicts the choice of G_{max}. Similarly to S, we have $G_{max}[S] = K_b \vee (k-b)K_1$. Thus Claim 1 holds.

Claim 2. $G_{max} - S$ contain exactly two components.

Proof of Claim 2: Suppose by contrary that $G_{max} - S$ contain at least three components. Let C_1 and C_2 be two of components, and $u \in V(C_1) - D, v \in V(C_2) - D$. Then $G_{max} + uv \in \mathcal{D}(n, k, r)$. By Lemma 2.1, $M_1(G_{max} + uv) > M_1(G_{max})$, which contradicts the choice of G_{max}. Thus Claim 2 holds.

By Claim 2, $G_{max} - S = C_1 \cup C_2$, where C_1 and C_2 are components of $G_{max} - S$.

Claim 3. If $V(C_1) \geq V(C_2)$, then $|V(C_2)| = 1$.

Proof of Claim 3: Suppose by contrary that $|V(C_2)| \geq 2$. If $V(C_2) - D = \emptyset$, then $|V(C_2)| = 1$ since C_2 is a connected components. Suppose that $V(C_2) - D \neq \emptyset$, then $V(C_2) \cap D \neq \emptyset$. Otherwise, $V(C_2) \cap D = \emptyset$, choose $u \in V(C_2)$, and $D \cup \{u\}$ is a independent set, which contradicts the definition of D.

Using the definitions from G_{max} and constructing a new graph G^* as follows. Let $v \in V(C_2) \cap D$. Then

$$G^* = G_{max} - \{xv : x \in V(C_2) - D\} + \{xy : x \in V(C_2) - \{v\}, y \in V(C_1)\},$$

it is clear that S and D are also minimal vertex cut and maximal independent set of G^*, respectively. Thus $G^* \in \mathcal{D}(n, k, r)$.

Let $u \in V(G_{max}) - S - \{v\}$ and $w \in V(C_1) - D$. Then $d_{G^*}(u) > d_{G_{max}}(u)$ and

$$M_1(G^*) - M_1(G_{max}) > d_{G^*}(w)^2 + d_{G^*}(v)^2 - d_{G_{max}}(w)^2 - d_{G_{max}}(v)^2 > 0,$$

which contradicts the choice of G_{max}. Thus Claim 3 holds.

By combine above claims, we have

$$G_{max} \in \{G' : G' = (K_{n-r} \vee (r-1)K_1) \cup \{v\} \cup \{u_i v : u_i \in S, i = 1, 2, \cdots, k\},$$

where v is a isolated vertex of $G_{max} - S$. Let $|S \cap D| = a$. Then

$$
\begin{aligned}
M_1(G') &= a(n-r+1)^2 + (r-a-1)(n-r)^2 + (k-a)(n-1)^3 \\
&\quad + (n-r-k+a)(n-2)^2 + k^2 \\
&= (r-1)(n-r)^2 + (n-r)(n-2)^2 + k^2 + k(2n-3) - a(2r-4),
\end{aligned}
$$

and the point $a = 0$ attains the maximal value of $M_1(G)$. Therefore, $M_1(G_{max}) = (r-1)(n-r)^2 + (n-r)(n-2)^2 + k^2 + k(2n-3)$ and $G_{max} = K_k \vee (K_1 \cup (K_{n-k-1} \vee (r-1)K_1))$. This complete the proof. $\qquad\square$

Corollary 5.1 *Let $G \in \mathcal{D}(n, k, r)$ with m edges, $r \geq 1$ and $1 \leq k \leq n-1$. Then*

$$RMTI(G) \leq \frac{3}{2} \left[(r-1)(n-r)^2 + (n-r)(n-2)^2 + k^2 + k(2n-3) \right] + (n-1)m.$$

Acknowledgement

References

[1] B. B. Anin, On the extremal Zagreb indices of trees with given number of segments or given number of branching vertices, *MATCH Commun. Math. Comput. Chem.*, 74 (2015) 57–79.

[2] B. B. Anin, T. A.Lampert, On the maximum and minimum Zagreb indices of trees with a given number of vertices of maximum degree, *MATCH Commun. Math. Comput. Chem.*, 74 (2015) 81–96.

[3] A. Behtoei, M. Jannesari, B.Taeri, Maximum Zagreb index, minimum hyper-Wiener index and graph connectivity, *Applied Mathematics Letters*, 22 (2009) 1571–1576.

[4] M.Bianchi, A.Cornaro, J.L.Palacios, A.Torriero,New bounds of degree-based topological indices for some classes of c -cyclic graphs, *Discr. Appl. Math.* , 194 (2015) 62–75.

[5] J. A. Bondy, U. S. Murty, Graph theory with applications, Macmillan, London and Elsevier, New York, 1976.

[6] S. Chen, W. Liu, Tricyclic graphs with minimum modified Schultz index and maximum Zagreb indices, *Ars Comb.*, 122 (2015) 379–397.

[7] H. Deng, A unified approach to the extremal Zagreb indices for trees, unicyclic graphs and bicyclic graphs,*MATCH Commun. Math. Comput. Chem.* , 57 (2007) 597–616.

[8] I. Gutman, N. Trinajstic,Graph theory and molecular orbitals. Total-electron energy of alternant hydrocarbons, *Chem Phys. Lett.* , 17 (1972) 535–538.

[9] I.Gutman, M.K.Jamil,N.Akhter, Graphs with fixed number of pendent vertices and minimal first Zagreb index, *Trans. Comb.*, 4 (2015) 43–48.

[10] I. Gutman, B.Ruscic, N. Trinajstiic, C.F.Wilcox, Graph theory and molecular orbitals. XII. Acyclic polyenes, *J. Chem. Phys.*, 62 (1975), 3399–3405.

[11] H. Hua, Zagreb M_1 index, independence number and connectivity in graphs, *MATCH Commun. Math. Comput. Chem.*, 60 (2008) 45–56.

[12] S. M. Hosamani, B. Basavanagoud, New upper bounds for the first Zagreb index, *MATCH

Commun. Math. Comput. Chem., 74 (2015) 97–101.

[13] P. Hansen, D. Vukičevic, Comparing Zagreb indices, *Croat. Chem. Acta*, 80 (2007), 165–168.

[14] O. Ivanciuc, T. S. Balaban, A. T. Balaban, Reciprocal distance matrix, related local vertex invariants and topological indices, *J. Math. Chem.*, 12 (1993) 309–318.

[15] S. Li, M. Zhang, Sharp upper bounds for Zagreb indices of bipartite graphs with a given diameter, *Applied Math. Lett.*, 24 (2011)131–137.

[16] S. Li, H. Zhou, On the maximum and minimum Zagreb indices of graphs with connectivity at most k, *Applied Math. Lett.*, 23 (2010) 128–132.

[17] H. Lu, Y. Lin, Maximum spectral radius of graphs with given connectivity, minimum degree and independence number, *J. Discr. Algorithms*, 31 (2015) 113-119.

[18] Z. Mihalić, D. Veljan, D. Amić, S. Nikolić, D. Plavšić, N. Trinajstić, The distance matrix in chemistry, *J. Math. Chem.*, 11 (1992) 223–258.

[9] S. Nikolic, G. Kovačevic, A. Milicevic, N. Trinajstic, The Zagreb indices 30 years after, *Croat. Chem. Acta*, 76 (2003) 113–124.

[20] H. P. Schultz, T. P. Schultz, Topological organic chemistry. 11. Graph theory and reciprocal Schultz-type molecular topological indices of alkanes and cycloalkanes, *J. Chem. Inf. Comput. Sci.*, 38 (1998) 853–857.

[21] D. Vukičevic, Comparing variable Zagreb indices, *MATCH Commun. Math. Comput. Chem.*, 57 (2007) 633–641.

[22] D. Vukičevic, A. Graovac, Comparing Zagreb M_1 and M_2 indices for acyclic molecules, *MATCH Commun. Math. Comput. Chem.*, 57 (2007) 587–590.

[23] J. F. Wang, F. Belardo, A lower bound for the first Zagreb index and its application, *MATCH Commun. Math. Comput. Chem.*, 74 (2015) 35–56.

[24] B. Zhou, Zagreb indices, *MATCH Commun. Math. Comput. Chem.*, 52 (2004) 113–118.

[25] B. Zhou, Upper bounds for the Zagreb indices and the spectral radius of series-parallel graphs, *Int. J. Quantum Chem.*, 107 (2007) 875–878 .

[26] B. Zhou, D. Stevanović, A note on Zagreb indices, *MATCH Commun. Math. Comput. Chem.* ,56 (2006) 571–578.

[27] B. Zhou, N. Trinajstić, On reciprocal molecular topological index, *J. Math. Chem.*, 44 (2008) 235–243.

[28] B. Zhou, N. Trinajstić, Some properties of the reformulated Zagreb index, *J. Math. Chem.*, 48 (2010) 714–719.

D-Conformal Curvature Tensor in Generalized (κ, μ)-Space Forms

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Abstract: The object of the first two sections is to give brief history of generalized (κ, μ) space forms and some basic results related to such manifold. In the last section we have derived few results regarding D-conformal curvature tensor in generalized (κ, μ) space-forms.

Key Words: Generalized (κ, μ)-space form, D-conformal curvature tensor, η-Einstein manifold.

AMS(2010): 53B25, 53C15, 53C25, 53D15.

§1. Introduction

In [1], Carriazo jointly with P. Alegre and D.E. Blair defined a generalized Sasakian space form as an almost contact metric manifold (M, ϕ, ξ, η, g) whose curvature tensor R is given by

$$
\begin{aligned}
R(X,Y)Z \quad = \quad & f_1\{g(Y,Z)X - g(X,Z)Y\} \\
& + f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\
& + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}
\end{aligned}
\tag{1.1}
$$

for any vector fields X, Y, Z on M.

In particular a Sasakian manifold $M(\phi, \xi, \eta, g)$ is said to be a Sasakian space form if all the $\phi-$sectional curvatures $K(X \wedge \phi X)$ are equal to a constant c, where $K(X \wedge \phi X)$ denotes the sectional curvature of the section spanned by the unit vector field X, orthogonal to ξ and ϕX. Later on many scientists R. Al-Ghefari, F. R. Alsomy [2],[5], M. H. Shahid have studied the CR-submanifolds of generalized Sasakian space forms. After them Ricci curvature of contact CR-submanifolds of such space were studied in [6].

In [2] authors studied contact metric and generalized Sasakian-space forms. In [7] and [8] authors studied locally ϕ-symmetric and η-recurrent Ricci tensor and also studied the projective curvature tensor respectively. Generalized Sasakian space form with few properties like conformally flat, locally symmetric were studied by Kim [9].

In recent paper [10], the authors (jointly with M. M. Tripathi) defined a generalized (κ, μ)-space form as an almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ whose curvature tensor is

given as
$$R = f_1 R_1 + f_2 R_2 + f_3 R_3 + f_4 R_4 + f_5 R_5 + f_6 R_6, \tag{1.2}$$

where $f_1, f_2, f_3, f_4, f_5, f_6$ are differentiable functions on M, and $R_1, R_2, R_3, R_4, R_5, R_6$ are tensors defined as follows:
$$R_1(X,Y)Z = g(Y,Z)X - g(X,Z)Y, \tag{1.3}$$

$$R_2(X,Y)Z = g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z, \tag{1.4}$$

$$R_3(X,Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi, \tag{1.5}$$

$$R_4(X,Y)Z = g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y, \tag{1.6}$$

$$R_5(X,Y)Z = g(hY,Z)hX - g(hX,Z)hY + g(\phi hX,Z)\phi hY - g(\phi hY,Z)\phi hX, \tag{1.7}$$

$$R_6(X,Y)Z = \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi, \tag{1.8}$$

where $2h = \mathcal{L}_\xi \phi$ and L is the usual Lie derivative. Usually, this manifold is denoted by $M(f_1, f_2, f_3, f_4, f_5, f_6)$. If $f_4 = f_5 = f_6 = 0$ then the manifold is the usual Sasakian space form.

Again, (κ, μ)-space forms are natural examples of generalized (κ, μ) space forms for constant functions ([10])

$$f_1 = \frac{c+3}{4}, \ f_2 = \frac{c-1}{4}, \ f_3 = \frac{c+3}{4}, \ f_4 = 1, \ f_5 = \frac{1}{2}, \ f_6 = 1 - \mu. \tag{1.9}$$

In this paper we have established few conditions related to D-conformal curvature tensor.

§2. Preliminaries

An almost contact metric manifold is a $(2n+1)$-dimensional manifold endowed with an almost contact structure (ϕ, ξ, η) consisting of a tensor field ϕ of type $(1,1)$, a structure vector field ξ and 1-form η satisfying:

$$\phi^2 = -I + \eta \otimes \xi, \eta(\xi) = 1, \phi(\xi) = 0, \eta \circ \phi = 0, \tag{2.1}$$

for any vector field $X, Y \in \tilde{M}$ and a Riemannian metric g defined as

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.2}$$

From above equation we can easily derive

$$g(X, \xi) = \eta(X). \tag{2.3}$$

The metric tensor satisfies the following properties:

$$g(\phi X, Y) = -g(X, \phi Y), \tag{2.4}$$

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y). \tag{2.5}$$

In a $(2n + 1)$ dimensional generalized (κ, μ)-space form we obtain from (1.2)

$$
\begin{aligned}
R(X,Y)\xi \;=\; & f_1\{g(Y,\xi)X - g(X,\xi)Y\} \\
& + f_2\{g(X,\phi\xi)\phi Y - g(Y,\phi\xi)\phi X + 2g(X,\phi Y)\phi\xi\} \\
& + f_3\{\eta(X)\eta(\xi)Y - \eta(Y)\eta(\xi)X + g(X,\xi)\eta(Y)\xi - g(Y,\xi)\eta(X)\xi\} \\
& + f_4\{g(Y,\xi)hX - g(X,\xi)hY + g(hY,\xi)X - g(hX,\xi)Y\} \\
& + f_5 g(hY,\xi)hX - g(hX,\xi)hY + g(\phi hX,\xi)\phi hY - g(\phi hY,\xi)\phi hX \\
& + f_6\eta(X)\eta(\xi)hY - \eta(Y)\eta(\xi)hX + g(hX,\xi)\eta(Y)\xi - g(hY,\xi)\eta(X)\xi
\end{aligned}
\tag{2.6}
$$

After some brief calculations we obtain from [4]

$$
R(X,Y)\xi = (f_1 - f_3)\{\eta(X)Y - \eta(Y)X\} + (f_4 - f_6)\{\eta(Y)hX - \eta(X)hY\}.
\tag{2.7}
$$

Now putting $X = \xi$, $Y = X$, $Z = Y$ we get

$$
R(\xi,X)Y = (f_1 - f_3)\{g(X,Y)\xi - \eta(Y)X\} + (f_4 - f_6)\{g(hX,Y) - \eta(Y)hX\}.
\tag{2.8}
$$

Again putting $Y = \xi$ in (2.7) we get

$$
R(\xi,X)\xi = (f_1 - f_3)\{\eta(X)\xi - X\} - (f_4 - f_6)\{hX\}.
\tag{2.9}
$$

Applying η on both side of the equation (1.2) we calculate

$$
\begin{aligned}
\eta(R(X,Y)Z) \;=\; & (f_1 - f_3)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} \\
& + (f_4 - f_6)\{g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)\}.
\end{aligned}
\tag{2.10}
$$

Putting $Z = \xi$ we can easily write

$$
\eta(R(X,Y)\xi) = 0.
\tag{2.11}
$$

Applying η on both side of equation (2.7) we can get the following equations

$$
\begin{aligned}
\eta(R(\xi,X)Y) \;=\; & (f_1 - f_3)\{g(X,Y) - \eta(Y)\eta(X)\} \\
& + (f_4 - f_6)g(hX,Y),
\end{aligned}
\tag{2.12}
$$

$$
\begin{aligned}
S(X,Y) \;=\; & \{2nf_1 + 3f_2 - f_3\}g(X,Y)\{(2n-1)f_4 - f_6\}g(hX,Y) \\
& - \{3f_2 + (2n-1)f_3\}\eta(X)\eta(Y).
\end{aligned}
\tag{2.13}
$$

From (2.13) we obtain

$$
S(X,\xi) = 2n(f_1 - f_3)\eta(X),
\tag{2.14}
$$

$$
r = 2n\{(2n+1)f_1 + 3f_2 - 2f_3\},
\tag{2.15}
$$

$$
\begin{aligned}
S(\phi X, \phi Y) \;=\; & S(X,Y) - 2n(f_1 - f_3)\eta(X)\eta(Y) \\
& -\{(2n-1)f_4 - f_6\}g(hX,Y),
\end{aligned}
\tag{2.16}
$$

$$
\begin{aligned}
QX \;=\; & \{2nf_1 + 3f_2 - f_3\}X + \{(2n-1)f_4 - f_6\}hX - \{3f_2 \\
& +(2n-1)f_3\}\eta(X)\xi,
\end{aligned}
\tag{2.17}
$$

$$
Q\xi = 2n(f_1 - f_3)\xi.
\tag{2.18}
$$

From [12], D-conformal curvature tensor on a Riemannian manifold (M^{2n+1}, g) is defined as

$$
\begin{aligned}
B(X,Y)Z \;=\; & R(X,Y)Z + \frac{1}{2(n-1)}\{S(X,Z)Y - S(Y,Z)X + g(X,Z)QY \\
& -g(Y,Z)QX\} - S(X,Z)\eta(Y)\xi + S(Y,Z)\eta(X)\xi - \eta(X)\eta(Z)QY \\
& -\eta(Y)\eta(Z)QX\} - \frac{k-2}{2(n-1)}\{g(X,Z)Y - g(Y,Z)X\} \\
& +\frac{k}{2(n-1)}\{g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y \\
& -\eta(Y)\eta(Z)X\},
\end{aligned}
\tag{2.19}
$$

where $k = \frac{r+4n}{2n-1}$, R is the curvature tensor, S is the Ricci tensor and r is the scalar curvature.

Now we give the definition of D-conformally flat generalized (κ, μ) space form following.

Definition 2.1 *A* $(2n+1)$*-dimensional generalized* (κ, μ) *space form* $M(f_1, f_2, f_3, f_4, f_5, f_6)$ *is said to be* D*-conformally flat if*

$$
B(X,Y)Z = 0.
\tag{2.20}
$$

We give the definition of $\xi - D$-conformally flat generalized (κ, μ) space form following.

Definition 2.2 *A* $(2n+1)$*-dimensional generalized* (κ, μ) *space form* $M(f_1, f_2, f_3, f_4, f_5, f_6)$ *is said to be* $\xi - D$*-conformally flat if*

$$
B(X,Y)\xi = 0.
\tag{2.21}
$$

Also we mention the following definition.

Definition 2.3 *A* $(2n+1)$*-dimensional generalized* (κ, μ) *space form* $M(f_1, f_2, f_3, f_4, f_5, f_6)$ *is said to be* $\phi - D$*-conformally flat if*

$$
g(B(\phi X, \phi Y)\phi Z, \phi W) = 0.
\tag{2.22}
$$

§3. Main Results

From Definition 2.1 we can draw the following theorem.

Theorem 3.1 *If a $(2n+1)$-dimensional generalized (κ, μ) space form $M(f_1, f_2, f_3, f_4, f_5, f_6)$ is D-conformally flat then $f_3 = f_1 + 1$ and $f_4 = f_6$.*

Proof Let us consider a $(2n+1)$-dimensional generalized (κ, μ) space form which satisfy the condition $B(X, Y)Z = 0$ then from (2.19) we obtain on using Definition 2.1 and taking inner product with W we obtain

$$
\begin{aligned}
0 = {} & R(X, Y, Z, W) + \frac{1}{2(n-1)} \{ S(X, Z)g(Y, W) - S(Y, Z)g(X, W) + g(X, Z)g(QY, W) \\
& - g(Y, Z)g(QX, W) \} - S(X, Z)\eta(Y)\eta(W) + S(Y, Z)\eta(X)\eta(W) - \eta(X)\eta(Z)g(QY, W) \\
& - \eta(Y)\eta(Z)g(QX, W) \} - \frac{k-2}{2(n-1)} \{ g(X, Z)g(Y, W) - g(Y, Z)g(X, W) \} \\
& + \frac{k}{2(n-1)} \{ g(X, Z)\eta(Y)\eta(W) - g(Y, Z)\eta(X)\eta(W) + \eta(X)\eta(Z)g(Y, W) \\
& - \eta(Y)\eta(Z)g(X, W) \}
\end{aligned}
\tag{3.1}
$$

because of $R(X, Y, Z, W) = g(R(X, Y)Z, W)$.

Now setting $W = \xi$ in (3.1) and using (2.1) and (2.2), we have

$$
\begin{aligned}
0 = {} & \eta(R(X, Y)Z) + \frac{1}{2(n-1)} \{ S(Y, \xi)g(X, Z) - S(X, \xi)g(Y, Z) - S(Y, \xi)\eta(X)\eta(Z) \\
& + S(X, \xi)\eta(Y)\eta(Z) + 2\{ g(X, Z)\eta(Y) - g(Y, Z)\eta(X) \}.
\end{aligned}
\tag{3.2}
$$

On using (2.10) and (2.14) we get on brief calculation

$$
\begin{aligned}
0 = {} & \frac{f_3 - f_1 - 1}{n - 1} \{ g(Y, Z)\eta(X) - g(X, Z)\eta(Y) \} + (f_4 - f_6)\{ g(hY, Z)\eta(X) \\
& - g(hX, Z)\eta(Y) \}.
\end{aligned}
\tag{3.3}
$$

Since L.H.S. is equal to zero and

$$
\{ g(Y, Z)\eta(X) - g(X, Z)\eta(Y) \}, \ \{ g(hY, Z)\eta(X) - g(hX, Z)\eta(Y) \} \neq 0
$$

we must have $f_3 - f_1 - 1 = 0$ and $f_4 - f_6 = 0$.

Hence

$$
f_3 = f_1 + 1 = 0, \quad f_4 = f_6.
\tag{3.4}
$$

Therefore the above equation proves our theorem. □

Now on basis of the definition (2.2) we give our next theorem.

Corollary 3.1 *If a $(2n+1)$-dimensional generalized (κ, μ) space form $M(f_1, f_2, f_3, f_4, f_5, f_6)$ is said to be $\xi - D$-conformally flat then $f_3 = f_1 + 1, f_4 = f_6$.*

Proof Suppose the condition $B(X, Y)\xi = 0$ holds in a $(2n+1)-$dimensional generalized

(κ, μ) space form. We have from (2.1), (2.2) in (2.19)

$$
\begin{aligned}
0 \;=\; & R(X,Y)\xi + \frac{1}{2(n-1)}\{S(X,\xi)Y - S(Y,\xi)X - S(X,\xi)\eta(Y)\xi + S(Y,\xi)\eta(X)\xi\} \\
& +2\{\eta(X)Y - \eta(Y)X\}.
\end{aligned}
\tag{3.5}
$$

Using (2.7) and (2.15) we calculate

$$
0 = \frac{f_3 - f_1 - 1}{n-1}\{\eta(Y)X-)\eta(X)Y\} + (f_4 - f_6)\{\eta(Y)hX - \eta(X)hY\}
\tag{3.6}
$$

because of $\{\eta(Y)X-)\eta(X)Y\} \neq 0$ we must have $f_3 - f_1 - 1 = 0$ and $f_4 - f_6 = 0$. Hence we obtain our proof. $\square$

From Definition 2.3 we can state our next theorem.

Theorem 3.2 *If a $(2n+1)$-dimensional generalized (κ, μ) space form $M(f_1, f_2, f_3, f_4, f_5, f_6)$ is $\phi - D$-conformally flat then Ricci tensor reduces to the form*

$$
S(Y,Z) = \alpha g(Y,Z) + \beta\eta(Y)\eta(Z) + \gamma g(hY, Z)
\tag{3.7}
$$

under the condition $Tr.\phi = 0$, and $\mu = 0$, where α, β, γ are constants.

Proof Let $M(f_1, f_2, f_3, f_4, f_5, f_6)$ be a $(2n+1)$-dimensional generalized (κ, μ) space form. Suppose M satisfies $g(B(\phi X, \phi Y)\phi Z, \phi W) = 0$ then from $(2.1), (2.19)$ we obtain

$$
\begin{aligned}
0 \;=\; & g(R(\phi X, \phi Y)\phi Z, \phi W) \\
& +\frac{1}{2(n-1)}\{S(\phi X, \phi Z)g(\phi Y, \phi W) - S(\phi Y, \phi Z)g(\phi X, \phi W) \\
& +S(\phi Y, \phi W)g(\phi X, \phi Z) - S(\phi X, \phi W)g(\phi Y, \phi Z)\} \\
& -\frac{k-2}{2(n-1)}\{g(\phi X, \phi Z)g(\phi Y, \phi W) - g(\phi Y, \phi Z)g(\phi X, \phi W)\}
\end{aligned}
\tag{3.8}
$$

In view of (2.22) and (3.8) and having few steps of calculations we get

$$
\begin{aligned}
0 \;=\; & f_1\{g(Y,Z)g(X,W) - g(Y,Z)\eta(X)\eta(W) - g(X,W)\eta(Y)\eta(Z) \\
& -g(X,Z)g(Y,W) - g(Y,W)\eta(X)\eta(Z) + g(X,Z)\eta(Y)\eta(W)\} \\
& +f_2\{g(X, \phi Z)g(\phi Y, W) - g(\phi Y, Z)g(X, \phi W) + 2g(X, \phi Y)g(\phi Z, W)\} \\
& f_4\{-g(Y,Z)g(hX, W) + \eta(Y)\eta(Z)g(hX, W) + g(X,Z)g(hY, W) \\
& -\eta(X)\eta(Z)g(hY, W) - g(hY, Z)g(X, W) + \eta(X)\eta(W)g(hY, Z) + g(Y,W)g(hX, Z) \\
& -g(hX, Z)\eta(Y)\eta(W)\} \\
& +f_5\{g(hY, Z)g(hX, W) - g(hX, Z)g(hY, W) + g(hX, \phi Z)g(hY, \phi W) \\
& -g(hY, Z)g(hX, W)\} + \frac{1}{2(n-1)}\{S(X,Z)g(Y,W) - S(X,Z)\eta(Y)\eta(W)
\end{aligned}
\tag{3.9}
$$

$$-2n(f_1 - f_3)g(Y,W)\eta(X)\eta(Z) - S(Y,Z)g(X,W) + S(Y,Z)\eta(X)\eta(W)$$

$$+2n(f_1 - f_3)g(X,W)\eta(Y)\eta(Z) + S(Y,W)g(X,Z) - S(Y,W)\eta(X)\eta(Z)$$

$$-2n(f_1 - f_3)g(X,Z)\eta(Y)\eta(W) - S(X,W)g(Y,Z) + S(X,W)\eta(Y)\eta(Z)$$

$$+2n(f_1 - f_3)g(Y,Z)\eta(X)\eta(W)\} + \frac{(2n-1)f_4 - f_6}{2(n-1)}\{-g(hX,Z)g(Y,W)$$

$$+g(hX,Z)\eta(Y)\eta(W) + g(hY,Z)g(X,W)$$

$$-g(hY,Z)\eta(X)\eta(W) - g(hY,Z)g(X,Z) + g(hY,Z)\eta(X)\eta(Z)$$

$$+g(hX,W)g(Y,Z) - g(hX,W)\eta(Y)\eta(Z)\}$$

$$-\frac{k-2}{2(n-1)}\{g(X,Z)g(Y,W) - g(X,Z)\eta(Y)\eta(W) - g(Y,W)\eta(X)\eta(Z)$$

$$-g(Y,Z)g(X,W) + g(Y,Z)\eta(X)\eta(W) + g(X,W)\eta(Y)\eta(Z)\}$$

Let $\{e_i : i = 1, 2, \cdots, 2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Putting $X = W = e_i$ in (3.14) and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$
\begin{aligned}
0 ={}& (2n-1)f_1\{g(Y,Z) - \eta(Y)\eta(Z)\} + f_2\{3g(\phi Y, \phi Z) - g(\phi Y, Z)Tr\phi\} \\
& -(2n-1)f_4 g(hY,Z) + \frac{1}{2(n-1)}[-2(n-1)S(Y,Z) - S(Z,\xi)\eta(Y) \\
& +\{2n(2n-1)(f_1 - f_3) + r\}\eta(Y)\eta(Z) - S(Y,\xi)\eta(Z) \\
& +\{2n(2n-1)(f_1 - f_3) - r\}g(Y,Z)] + \frac{r+2}{2(n-1)}[g(Y,Z) - \eta(Y)\eta(Z)] \\
& +0\frac{[(2n-1)f_4 - f_6](2n-1)}{2(n-1)}g(hY,Z) \qquad\qquad (3.10)
\end{aligned}
$$

By using (2.3) and (2.14) we obtain

$$
\begin{aligned}
S(Y,Z) ={}& \frac{[(2n^2 - 2n + 1) + 3(n-1)f_2 - nf_3 + 1]}{n-1}g(Y,Z) \\
& +\frac{[n(3-2n)f_3 - f_1 - 3(n-1)f_2 - 1]}{n-1}\eta(Y)\eta(Z) \\
& +\frac{(f_4 - f_6)(2n-1)}{2(n-1)}g(hY,Z). \qquad\qquad (3.11)
\end{aligned}
$$

Assuming

$$\alpha = \frac{[(2n^2 - 2n + 1) + 3(n-1)f_2 - nf_3 + 1]}{n-1},$$

$$\beta = \frac{[n(3-2n)f_3 - f_1 - 3(n-1)f_2 - 1]}{n-1},$$

$$\gamma = \frac{(f_4 - f_6)(2n-1)}{2(n-1)}.$$

Hence we arrive at our proposed result. $\square$

Theorem 3.3 *If a $(2n+1)$-dimensional generalized (κ, μ) space form $M(f_1, f_2, f_3, f_4, f_5, f_6)$*

satisfies the condition $B(\xi, X).S = 0$, then the scalar curvature is given by

$$r = 2n(f_1 - f_3)(2n + 1) - 2n(f_4 - f_6)(f_1 - f_3 - 1)[(2n - 1)f_4 - f_6]. \qquad (3.12)$$

Proof Let $M(f_1, f_2, f_3, f_4, f_5, f_6)$ be a $(2n + 1)$-dimensional generalized (κ, μ) space form. we suppose that M satisfies the condition $(B(\xi, X).S)(U, V) = 0$, where S is the Ricci tensor. Then we get

$$S(B(\xi, X)U, V) + S(U, B(\xi, X)V) = 0. \qquad (3.13)$$

From equation (2.19) we can write after having few steps of calculations

$$
\begin{aligned}
B(\xi, Y)Z &= (f_1 - f_3 - \frac{1}{n-1})\{g(Y, Z)\xi - \eta(Z)Y\} + (f_4 - f_6)\{g(hY, Z)\xi - \eta(Z)hY\} \\
&\quad + \frac{1}{2(n-1)}\{2n(f_1 - f_3)[Y - \eta(Y)\xi]\eta(Z) \\
&\quad - [g(Y, Z) - \eta(Y)\eta(Z)]Q\xi\}.
\end{aligned}
\qquad (3.14)
$$

Similarly replacing Y with X and Z with U we obtain

$$
\begin{aligned}
B(\xi, X)U &= (f_1 - f_3 - \frac{1}{n-1})\{g(X, U)\xi - \eta(U)X\} + (f_4 - f_6)\{g(hX, U)\xi - \eta(U)hX\} \\
&\quad + \frac{1}{2(n-1)}\{2n(f_1 - f_3)[X - \eta(X)\xi]\eta(U) \\
&\quad - [g(X, U) - \eta(X)\eta(U)]Q\xi\}.
\end{aligned}
\qquad (3.15)
$$

Putting equations (3.14), (3.15) in (3.13) and replacing V with ξ we infer

$$
\begin{aligned}
0 &= (f_1 - f_3 - \frac{1}{n-1})\{2n(f_1 - f_3)g(X, U) - S(X, \xi)\eta(U)\} \\
&\quad \frac{n(f_1 - f_3)}{n-1}[\{S(X, \xi) - 2n(f_1 - f_3)\eta(X)\}\eta(U) - 2n(f_1 - f_3)\{g(X, U) - \eta(x)\eta(U)\}] \\
&\quad + (f_1 - f_3 - \frac{1}{n-1})\{2n(f_1 - f_3)\eta(X)\eta(U) - S(X, U)\} \\
&\quad + \frac{n(f_1 - f_3)}{n-1}[S(X, U) - 2n(f_1 - f_3)\{g(X, U) - \eta(x)\eta(U)] \\
&\quad + (f_4 - f_6)\{2n(f_1 - f_3)g(hX, U) - S(hX, U)\}.
\end{aligned}
\qquad (3.16)
$$

Using equations (2.1), (2.2), (2.12) we obtain

$$S(X, U) = 2n(f_1 - f_3)g(X, U) + \frac{1}{a}[(f_4 - f_6)\{2n(f_1 - f_3)g(hX, U) - S(hX, U)\}], \qquad (3.17)$$

where $a = f_1 - f_3 - \frac{2}{n-1}$. Again taking the orthonormal frame field at any point of the manifold and contracting over X and U we get from above equation

$$r = 2n(f_1 - f_3)(2n + 1) - 2n(f_4 - f_6)(f_1 - f_3 - 1)[(2n - 1)f_4 - f_6]. \qquad (3.18)$$

Hence we get the result.   □

References

[1] P. Alegre, D.E.Blair and A. Carriazo, Generalized Sasakian-space form, *Israel J. Math.*, 141(2004), 157-183.

[2] P. Alegre and A. Carriazo, Structures on Generalized Sasakian space forms, *Differential Geom. Appl.*, 26(2008), 656-666.

[3] P. Alegre and A. Carriazo, Submanifolds of generalized Sasakian space forms, *Taiwanese J. Math.*, 13(2009), 923-941.

[4] A. Carriazo, V. Martin-Molina and M. M. Tripathi, Generalized (κ, μ) space forms, Available on *arXiv: 0812.2605v1*,(2012), 1-20.

[5] R. Al-Ghefari, F. R. Al-Solamy and M. H. Shahid, Cr-submanifolds of generalized Sasakian space forms, *JP J. Geom. Topol.*, 6 (2006), 151-166.

[6] I. Mihai, M. H. Shahid and F. R. Al-Solamy ,Ricci curvature of a contact CR-submanifolds of generalized Sasakian space forms, *Rev. Bull. Calcutta Math. Soc.*, 13(2005), 89-94.

[7] U. C. De and A. Sarkar Some results on generalized Sasakian space forms, *Thai J. Math.*, 8(2010), No. 1, 1-10.

[8] U. C. De and A. Sarkar, On the projective tensor of generalized Sasakian space forms, *Quaestiones Mathematicae*, 33(2010), No. 2, 245-252.

[9] UK. Kim, Conformally flat generalized Sasakian space forms and locally symmetric generalized Sasakian space forms, *Note Di Matematica*, 26(2006), No. 1, 55-67.

[10] T. Koufogiorgos, Contact Riemannian manifolds with constant ϕ-sectional curvature, *Tokyo. J. Math.*, 20(1997), 55-67.

[11] A. Carriazo, V. L. Molina, Generalized (κ, μ) space forms and D_a-homothetic deformations, *Balkan Journal of Geometry and Applications*, Vol.16, No.1,2011, pp.37-47.

[12] G. Chuman, On the D-conformal curvature tensor, *Tensor N. S.*, Vol. 2, 46(1983), 125-134.

Spectrum of (k, r) - Regular Hypergraphs

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Abstract: We present a spectral theory of uniform, regular and linear hypergraph. The main result are the nature of the eigenvalues of (k, r) - regular linear hypergraph and the relation between its dual and line graph. We also discuss some properties of Laplacian spectrum of a (k, r) - regular hypergraphs.

Key Words: Hypergraph, spectrum, dual of hypergraph, line graph, Laplacian spectrum, Smarandachely linear.

AMS(2010): 05C12.

§1. Introduction

The spectral graph theory [2, 3] is the study of the relation between eigenvalues and eigenvectors of certain associated matrices of a graph and its combinatorial properties. There is some relation between the size of the eigenvalues and maximum degree of the graph [8, 1]. Connection between spectral characteristics of a graph and other graph theoretic parameters is a well explored area.

A graph structure is obtained when a non empty set (set of vertices) and a subset of all unordered pairs of elements of the set of vertices (this subset is called the set of edges) are taken together. When the set of unordered pairs is replaced by order pairs we get directed graphs. For a regular graph G, eigenvalues of the adjacency matrix are well studied. The second largest eigenvalue of the adjacency matrix is related to quantities such as diameter [6], chromatic number [4] etc. The regular graph with small non trivial eigenvalue can be used as good expanders and superconductors in communication network [12].

From the point of view of applications in social network and allied disciplines a more general structure is very useful. This structure, called *hypergraph* is obtained by taking a subset of the set of all proper subsets of the given set. The elements of the second set are called hyperedges. A hypergraph is denoted by H. For basic ideas and definitions on hypergraph readers may refer the text by [2], Chung [6] and Wen - Ching et al. [11] proposed the operator attached to a regular and uniform hypergraph and obtained some estimate of the eigenvalues of the operation.

K. Feng and W. C. Li [8] studeid the growth of the second largest eigenvalue and distribution of the eigenvalues of the adjacency matrix attached to a regular hypergraph.

In the next section we go through the main definitions and important results needed to understand the concepts included in the subsequent sections.

§2. Preliminaries

We now discuss some basic terminology of hypergraphs, which we require in the sequel. A hypergraph H is a pair (X, E) where $X = \{v_1, v_2, \cdots, v_n\}$ is a finite set and $E = \{E_1, E_2, \cdots, E_m\}$ of subsets of X, such that $E_j \neq \phi$, $j = 1, 2, 3 \cdots m$ and $\bigcup_{i=1}^{m} E_i = X$. The elements of X are called vertices or hypervertices and the elements of the sets $\{E_1, E_2, \cdots, E_m\}$ are called hyperedges of H. The cardinality of X is called the *order* of H and cardinality of E is called the *size* of H. In a hypergraph two vertices are adjacent if there is a hyperedge that containing both of these vertices. Two hyperedge are said to be adjacent if $E_i \cap E_j \neq \phi$. A *simple hypergraph* is a hypergraph such that $E_i \subset E_j \Rightarrow i = j$. A hypergraph H is said to be k - *uniform* for an integer $k \geq 2$, if for all E_i in E, $|E_i| = k$ for all i.

A hypergraph H is said to be *Smarandachely linear* respect to a subset $E' \subset E(H)$ if $|E_i \bigcap E_j| \leq 1$ for all $E_i, E_j \in E'$ but $|E_i \bigcap E_j| > 1$ if $E_i, E_j \in E \setminus E'$. Particularly, a hypergraph H is said to be *linear* if $|E_i \cap E_j| \leq 1$ for all $i \neq j$, i.e., a Smarandachely linear in case of $E' = E$. A hypergraph in which all vertices have the same degree is said to be regular. An r - *regular hypergraph* is a hypergraph with $d(v_i) = r$ for all $i \leq n$ and an (r, k) - *regular hypergraph* is a hypergraph which is r - regular and k - uniform. The following results are important.

Theorem 2.1([4]) *Let T be a real $n \times n$ matrix with non negative entries and irreducible then there exists a unique positive real number θ_0 with the following properties:*

(1) there is a real number $x_0 > 0$ with $Tx_0 = \theta_0 x_0$;

(2) any non negative right or left eigenvector of T has eigenvalue θ_0. Suppose $t \in R$ and $x \geq 0$, $x \neq 0$. If $Tx \leq tx$, then $x \geq 0$ and $t \geq \theta_0$ and moreover $t = \theta_0$ iff $Tx = tx$. If $Tx \geq tx$, then $t \leq \theta_0$ and moreover $t = \theta_0$ iff $Tx = tx$.

Theorem 2.2([4]) *Consider two sequence of real numbers $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$ and $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_m$ with $m < n$. The second is said to interlace the first one whenever $\theta_i \geq \eta_{n-m+i}$ for $i = 1, 2, \cdots, m$.*

Theorem 2.3([4]) *Let C be the quotient matrix of a symmetric matrix A whose rows and columns are partitioned according to the partitioning $\{x_1, x_2, \cdots, x_m\}$ then the eigenvalues of C interlace the eigenvalues of A.*

Let A be a real symmetric matrix and u be a non zero vector. The *Rayleigh Quotient* of u [2] with respect to A is defined as $\frac{u^T A u}{u^T u}$. The *dual* of the hypergraph $H(X, E)$ [?] is a hypergraph $H^* = H(X^*, E^*)$ where $X^* = \{e_1, e_2, \cdots, e_m\}$ corresponding to the edges of H and $E^* = \{X_1, X_2, \cdots, X_n\}$ where $X_i = \{e_j : x_i \in E_j \text{ in } H\}$. Also $(H^*)^* = H$. Given a hypergraph $H = (X, E)$, where $X = \{E_1, E_2, ..., E_m\}$. Its *representative* graph or *line graph*

$\mathcal{L}(H)$, is a graph whose vertices are points $\{e_1, e_2, \cdots, e_m\}$ representing the edges of H and the vertices e_i and e_j being adjacent if and only if $E_i \cap E_j \neq \phi$.

Theorem 2.4([2]) *The dual of a linear hypergraph is also linear.*

Proof Given H is linear. Suppose H^* is not linear. Let X_1 and X_2 in H^* intersect at two distinct points e_1 and e_2. Hence $\{e_1, e_2\} \subset E_1$ and E_2. Therefore $|E_1 \cap E_2| \geq 2$, which contradicts $|E_i \cap E_j| \leq 1$. So H* is linear. $\square$
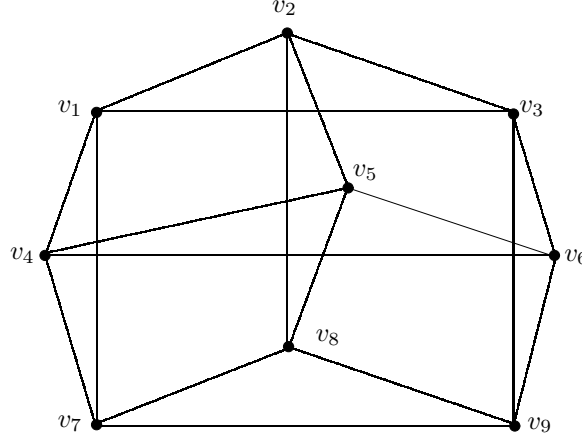
Figure 1 An example of (2,3)- regular and linear hypergraph on 9 vertices

Now we define the *adjacency matrix* [1] of a hypergraph H. Adjacency matrix of H is denoted by $A_H = (a_{ij})$, where

$$a_{ij} = \begin{cases} |\{E_k \in E : \{v_i, v_j\} \subseteq E_k\}| & \text{if } i \neq j \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of the adjacency matrix A_H is called the eigenvalues of H. Since A_H is a real symmetric matrix, all the eigenvalues are real. The spectrum of H is the set of all eigenvalues of A_H together with their multiplicities. Spectrum of H is denoted by $Spec(H)$ or $Spec(A_H)$. Let $\lambda_1, \lambda_2, \cdots \lambda_s$ be distinct eigenvalues of H with multiplicities $m_1, m_2, \cdots m_s$ then we write

$$Spec(H) = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_s \\ m_1 & m_2 & \dots & m_s \end{pmatrix}$$

In the next section we derive some properties of the spectrum and Laplacian spectrum of (r, k) - regular hypergraphs. Also discuss the relation between the Line graph and dual of 2 - regular hypergraphs.

§3. **Spectrum of (r, k) - Regular Hypergraphs**

We know that the sum of the entries in each row and column of the adjacency matrix of an

(r, k) - regular hypergraph H is $\theta = r(k-1)$. Thus θ is an eigenvalue of $A(H)$ [9, 11].

Theorem 3.1 *Let H be an (r, k) - regular linear hypergraph then*

(1) $\theta = r(k-1)$ *is an eigen value of H;*
(2) *if H is connected, then multiplicity of θ is one;*
(3) *for any eigenvalue λ of H we have $|\lambda| \leq \theta$.*

Proof (1) Let $A_H = A$ be the adjacency matrix of H. Also let $u = (1, 1, 1, \cdots, 1)^t$. Since H is k-uniform, each edge has exactly k vertices. ie $|E_j| \leq k$ for $i \leq m$. Since H is r- regular each vertex x_i belongs to exactly r hyperedge. Let $v_i \in E_1, E_2, \cdots, E_k$ Then k hyper edge consist of rk vertices. So different pair of vertices with x_i as one factor is $rk - r = r(k-1)$. Let $\theta = r(k-1)$. Then $Au = \theta u$ ie θ is an eigenvalue of H.

(2) let $z = (z_1, z_2, \cdots, z_n)^t$, $z \neq 0$ be such that $Az = \theta z$ Let z_i be the entry of z having the largest absolute value $(Az)_i = \theta z_i$

$$\sum_{j=1}^{m} a_{ij} z_j = \theta z_i.$$

We have each vertex x_i is associated with $r(k-1)$ other vertices through a hyperedge. By the maximality property of z_i, $z_j = z_i$ for all these vertices. Since H is connected all the vertices of z are equal. ie z is a multiple of $u = (1, 1, \cdots, 1)^t$. Therefore the space of eigenvector associated with the eigen value θ has dimension one. So the multiplicity of θ is one.

(3) Suppose $Ay = \lambda y, y \neq 0$. Let y_i denote the entry of y which is the largest in absolute value. Then,

$$(Ay)_i = \lambda y_i$$

implies that

$$\sum_{j=1}^{n} a_{ij} y_j = \lambda y_i.$$

Thus,

$$
\begin{aligned}
|\lambda y_i| &= \left| \sum_{j=1}^{n} a_{ij} y_j \right| \leq \sum_{j=1}^{n} a_{ij} |y_j| \\
&\leq |y_i| \sum_{j=1}^{n} a_{ij} = \theta |y_i|,
\end{aligned}
$$

i.e.,

$$|\lambda| \leq \theta. \qquad \square$$

Proposition 3.2 *Let H be a k - uniform hypergraph with largest eigenvalue λ_1. If H is regular of degree r, then $\lambda_1 = \theta$ where $\theta = r(k-1)$. Otherwise $(k-1)\delta_{min} \leq \overline{\delta} \leq \lambda_1 \leq (k-1)\delta_{max}$, where $\delta_{min}, \delta_{max}$ and $\overline{\delta}$ are the minimum, maximum and average degree respectively.*

Proof Let **1** be the vector with all entries equal to 1. Then $A\mathbf{1} \leq (k-1)\delta_{max}\mathbf{1}$. By Theorem

2.1, $\lambda_1 \leq \theta_{max}$ where $\theta_{max} = (k-1)\delta_{max}$ and equality happens if and only if $A\mathbf{1} = \lambda_1\mathbf{1}$, ie if and only if $\lambda_1 = r(k-1)$ where r is the degree of the vertices. Consider the partition of the vertex set consisting of a single part. By Theorem 2.3 we have $\overline{\delta} \leq \lambda_1$. Equality happens if and only if H is regular. $\qquad\square$

Proposition 3.3 *Let H be a linear hypergraph with eigenvalues $\theta = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, then the following are equivalent.*

    (1) *H is an (r, k) - regular hypergraph;*

    (2) *$AJ = \theta J$ where $\theta = r(k-1)$ and J is an $n \times n$ matrix with all entries equal to 1;*

    (3) *$\sum_{i=1}^{n} \lambda_i^2 = \theta\, n$.*

Proof The statements (1) and (2) are equivalent. In order to complete the theorem, we prove that that (1) and (3) are equivalent. First assume H is an (r, k) - regular hypergraph. Then

$$\sum_{i=1}^{n} \lambda_i^2 = tr(A^2) = \theta\, n.$$

Conversely, assume (3) is true. Then

$$\frac{1}{n}\sum_{i=1}^{n} \lambda_i^2 = \theta = r(k-1) = \lambda_1.$$

By Proposition 3.2, H is regular. $\qquad\square$

Theorem 3.4 *The dual H^* of a (2,k) - regular hypergraph H is k - regular. Hence k is an eigenvalue of H^*.*

Proof Let H be a $(2, k)$ - regular hypergraph. We know that $d(v_i) = 2$ for $i \leq n$ and $|E_j| = k$, for $j \leq m$. $X_i = \{e_j / x_i \in E_j \text{ in } H\}$. Each x_i belongs to exactly two edges in H. So $|x_j| \leq 2$ for $j \leq n$. Since $|E_j| = k$, each e_i is adjacent to exactly k, $e_j's$ hence $d(e_j) = k$. Therefore H^* is a k - regular simple graph. Hence k is an eigenvalue of H^*. $\qquad\square$

Theorem 3.5 *Let H be a $(2, k)$ - regular linear hypergraph. Its line graph $\mathcal{L} \cong H^*$.*

Proof By Theorem 3.4 , H^* is a k - regular simple graph. Let $\{x_1, x_2, \cdots, x_m\}$ be the vertices of $\mathcal{L}$. In H, each edge E_j has k - vertices. Since the vertex x_j is adjacent to k other vertices, $d(x_j) = k$ for $j \leq m$. $\mathcal{L}$ is a k - regular simple graph. Hence $\mathcal{L}$ and H^* have same number of vertices and edges. Also incidence relation is preserved. Therefore $L(H) \cong H^*$. $\quad\square$

§4. Laplacian Spectrum of (r, k) - Regular Hypergraphs

We define the itLaplacian degree of a vertex $v_i \in X(H)$ as $\delta_l(v_i) = \sum_{j=1}^{n} a_{ij}$. We say that the hypergraph H is Laplacian regular of degree δ_l if any vertex $v \in X(H)$ has Laplacian degree $\delta_l(v) = \delta_l$. If H is a simple graph, then $\delta_l(v_i) = \delta(v_i)$. The Laplacian matrix of a hypergraph H

is denoted by $L = L(H)$ and is defined as $L = D - A$ where $D = diag\{\delta_l(v_1), \delta_l(v_2), \cdots, \delta_l(v_n)\}$. The matrix L is symmetric and positive semi definite. All the eigen values are real and non negative. The smallest eigen value is 0 and the corresponding eigen vector is $j = (1, 1, \cdots, 1)$. Moreover the multiplicity is the number of connected components of H. Second smallest and largest Laplacian eigen values and parameters related are studied by Rodrigues [10]. The eigen values of L are denoted by $0 = \mu_1 < \mu_2 < \cdots < \mu_b$ and their multiplicities $m_1, m_2, m_3, \cdots, m_b$. Thus the Laplacian spectrum of H and Laplacian degree of its vertices are related by

$$\sum_{l=1}^{b} m_l \mu_l = tr(L(H)) = \sum_{i=1}^{n} \delta_l(v_i). \tag{4.1}$$

If H is a regular hypergraph of degree $\delta(l)$, then $L = D - A = \delta_l I - A$. Thus the eigenvalues of A and L are related by $\mu_i = \delta_i - \lambda_i, \quad i = 1, 2, \cdots, b$.

The second smallest eigenvalue of the graph gives the most important information contained in the spectrum. This eigenvalue is called the *algebraic connectivity* and is related to several graph invariants and imposes reasonably good bounds on the value of several parameters of graphs which are very hard to compute. The concept of algeraic connectivity was introduced by Fiedler [7]. Also $\mu_2 \geq 0$ with equality if and only if H is disconnected. Algebraic connectivity is monotone. It does not decrease when edges are added to the graph.

Theorem 4.1 *Let H be a k - uniform hypergraph. Also let μ_2 be the algebraic connectivity of H. For any non adjacent vertices s and t in H we have*

$$\mu_2(H) \leq \frac{k-1}{2}(deg(s) + deg(t)).$$

Proof We have,
$$\mu_2(H) = \overset{Min}{u} \left\{ \frac{< Lu, u >}{< u, u >} :< u, \mathbf{1} >= 0 \right\}.$$
The vector u is defined by
$$u_x = \begin{cases} 1 & \text{if } x = s \\ -1 & \text{if } x = t \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $< u, \mathbf{1} >= 0$ and by definition,

$$\begin{aligned} \mu_2(H) &\leq \frac{< Lu, u >}{< u, u >} = \frac{\sum_{xy \in E_j} a_{xy}(u_x - u_y)^2}{\sum_{x \in V} u_x^2} \\ &= \frac{(k-1)(\sum_x a_{xs} + \sum_x a_{tx})}{2} = \frac{(k-1)}{2}(deg(s) + deg(t)). \quad \square \end{aligned}$$

Let G be the weighted graph on the same vertex set X. Two vertices x and y are adjacent in X if they are adjacent in H also. The edge weight of xy is equal to (a_{ij}), the number of edges in H containing both x and y. Clearly $L(G) = L(H)$. Rodriguez [10] obtain the result

from Fiedler [7] on weighted graph as

$$\mu_2 = 2n \ min \left\{ \frac{\sum_{xy \in E_j} a_{xy}(u_x - u_y)^2}{\sum_{x \in X} \sum_{x \in X}(u_x - u_y)^2} : u \neq \alpha \mathbf{1} \ for \ \alpha \in R \right\}, \qquad (4.2)$$

$$\mu_b = 2n \ max \left\{ \frac{\sum_{xy \in E_j} a_{xy}(u_x - u_y)^2}{\sum_{x \in X} \sum_{x \in X}(u_x - u_y)^2} : u \neq \alpha \mathbf{1} \ for \ \alpha \in R \right\}. \qquad (4.3)$$

This eigenvalues μ_2 and μ_b are bounded in terms of maximum and minimum degree of H. For any vertex x, e_x denote the corresponding unit vector of the canonical basis of R^n by putting $u = e_x$ in equations (4.2) and (4.3) we get,

$$\mu_2 \leq \frac{n}{n-1}\delta_x \leq \mu_b.$$

This leads to

$$\mu_2 \leq \frac{n}{n-1}\delta_{l \ min} \leq \frac{n}{n-1}\delta_{l \ max} \leq \mu_b.$$

Theorem 4.2 *Let H_1 and H_2 be two edge disjoint hypergraphs on the same vertex set and $H_1 \cup H_2$ be their union. Then*

$$\mu_2(H_1 \cup H_2) \geq \mu_2(H_1) + \mu_2(H_2) \geq \mu_2(H_1).$$

Proof Let L_1, L_2 and L be the Laplace adjacency matrix of H_1, H_2 and $H = H_1 \cup H_2$ respectively. We have,

$$
\begin{aligned}
\mu_2(H) \ &= \ \underset{u}{Min} \left\{ \frac{< Lu, u >}{< u, u >} :< u, \mathbf{1} >= 0 \right\} \\
&= \ \underset{u}{Min} \left\{ \frac{< (L_1 + L_2)u, u >}{< u, u >} :< u, \mathbf{1} >= 0 \right\} \\
&= \ \underset{u}{Min} \left\{ \frac{< (L_1 u + L_2 u), u >}{< u, u >} :< u, \mathbf{1} >= 0 \right\} \\
&\geq \ \underset{u}{Min} \left\{ \frac{< L_1 u, u >}{< u, u >} :< u, \mathbf{1} >= 0 \right\} \\
&\quad + \underset{u}{Min} \left\{ \frac{< L_2 u, u >}{< u, u >} :< u, \mathbf{1} >= 0 \right\} \\
&\geq \ \mu_1(H_1) + \mu_2(H_2) \geq \mu_1(H_1). \qquad \square
\end{aligned}
$$

Acknowledgments

References

[1] M. Astuti, A.N.M. Salman, Hami Garminia and Irawati, The properties of some coefficients of the characteristic and Laplace polynomials of a hypergraph, *International Journel of Contemperary Mathematical Science*, Vol. 7, 21(2012), 1029 - 1035.

[2] C. Berge, *Graphs and Hypergraphs*, North - Holland Publishing Company, 1989.

[3] N. Biggs, *Algebraic Graph Theory*, Cambridge Mathematical Library, Cambridge University Press, 1983.

[4] A. E. Brouwer and W. H.Haemers, *Spectra of Graphs*, Springer 2012.

[5] M. Fiedler, Algebraic connectivity of graphs, *Czech. Math. J.*, 23 (98), 1973, 298 - 305.

[6] F. R. K. Chung, Diameter and eigenvalues, *J. Am. Math. Soc.*, 2(1989), 187 - 196.

[7] F. Friedman, On the second eigenvalue and random walks in random d-regular graphs, *Combinatorica*, 11(1991), 331 - 362.

[8] Keqin Feng and Wen Ching Li, Spectra of hypergraph and applications, *Journal of Number Theory*, 60(1996), 1-22.

[9] Maria G. Martinez, Harold M. Stark and Audrcy A. Terras, Some Ramanujan Hypergraphs Associated to $GL(n, F_q)$' *Proceedings of the American Mathematical Society*, Vol. 129, No.6, Jan. 2001, 1623-1629.

[10] J. A. Rodriguez, Laplacian eigenvalues and partition problem in hypergraphs, *Applied Mathematics Letters*, 22(2009), 916 - 921.

[11] Wen-Ching, Winnie Li and Patrick Sole, Spectra of regular graphs and hypergraphs and orthogonal polynomials, *European J. Combinatorics*, 17(1996), 461 - 477.

[12] Xiaobo Wang, Xianwei Zhou and Junde Song, Hypergraph based modal and architecture for planet surface network and orbit access, *Journal of Network*, Vol.7, No.4(2012), 723-729.

On the Spacelike Parallel Ruled Surfaces with Darboux Frame

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Abstract: In this study, the spacelike parallel ruled surfaces with Darboux frame are introduced in Minkowski 3-space. Then some characteristic properties of the spacelike parallel ruled surfaces with Darboux frame such as developability, the striction point and the distribution parameter are obtained in Minkowski 3-space.

Key Words: Ruled surface, parallel surface, Darboux frame.

AMS(2010): 53A05, 53A15.

§1. Introduction

In differential geometry, ruled surface is a special type of surface which can be defined by choosing a curve and a line along that curve. The ruled surfaces are one of the easiest of all surfaces to parametrize. That surface was found and investigated by Gaspard Monge who established the partial differential equation that satisfies all ruled surface. V. Hlavaty [9] also investigated ruled surfaces which are formed by one parameter set of lines.

A surface and another surface which have constant distance with the reference surface along its surface normal have a relationship between their parametric representations. Such surfaces are called parallel surface [8]. By this definition, it is convenient to carry the points of a surface to the points of another surface. Since the curves are set of points, then the curves lying fully on a reference surface can be carry to another surface.

Another one of the most important subjects of the differential geometry is the Darboux frame which is a natural moving frame constructed on a surface. It is the version of the Frenet frame as applied to surface geometry. A Darboux frame exists on a surface in a Euclidean or non-Euclidean spaces. It is named after the French mathematician Jean Gaston Darboux, in the four volume collection of studies published between 1887 and 1896. Since that time, there have been many important repercussions of Darboux frame, having been examined for example in [2], [3].

The Minkowski space E_1^3 is the Euclidean space E^3 provided with the Lorentzian inner product

$$\langle u, v \rangle = u_1 v_1 + u_2 v_2 - u_3 v_3,$$

where

$$u = (u_1, u_2, u_3), \ v = (v_1, v_2, v_3) \in E^3.$$

We say that a vector u in E_1^3 is spacelike, lightlike or timelike if

$$\langle u, u \rangle > 0, \ \langle u, u \rangle = 0 \text{ or } \langle u, u \rangle < 0,$$

respectively. The norm of the vector $u \in E_1^3$ is defined by

$$\|u\| = \sqrt{|\langle u, u \rangle|}.$$

G.Y. Senturk and S. Yucel have taken into a consideration about the ruled surface with Darboux frame in E^3 ([4]).

By making use of the paper of Unluturk et al. [10], we describe our general approach to compute the spacelike parallel ruled surfaces with Darboux frame, and give some theorems for these kinds of surfaces.

§2. **Preliminaries**

A ruled surface M in $\mathbb{R}^3$ is generated by a one-parameter family of straight lines which are called the rulings. The equation of the ruled surface can be written as

$$\varphi(s, v) = \alpha(s) + vX(s),$$

where (α) is curve which is called the base curve of the ruled surface and (X) is called the unit direction vector of the straight line.

An unit direction vector of straight line X is stretched by the system $\{T, g\}$. So it can be written as

$$X = T \sin \phi + g \cos \phi,$$

where ϕ is the angle between T and X vectors ([10]).

The striction point on the ruled surface is the foot of the common perpendicular line successive rulings on the main ruling. The set of the striction points of the ruled surface generates its striction curve. It is given by ([10])

$$c(s) = \alpha(s) - \frac{\langle \alpha_s, X_s \rangle}{\langle X_s, X_s \rangle} X(s).$$

Theorem 2.1([7]) *If successive rulings intersect, the ruled surface is called developable. The unit tangent vector of the striction curve of a developable ruled surface is the unit vector with direction X.*

The distribution of the ruled surface is identified by

$$P_X = \frac{\det(\alpha_s, X, X_s)}{\langle X_s, X_s \rangle}.$$

Theorem 2.2([7]) *The ruled surface is developable if and only if $P_X = 0$.*

The ruled surface is said to be a noncylindrical ruled surface provided that ([1])

$$\langle X_s, X_s \rangle \neq 0$$

Theorem 2.3([1]) *Let M be a noncylindrical ruled surface and defined by its striction curve. The Gaussian curvature of M is given by its distribution parameter by*

$$K = -\frac{P_X^2}{(P_X^2 + v^2)^2}$$

.

Definition 2.1([5]) *Let M and $\overline{M}$ be two surfaces in Euclidean space. The function*

$$
\begin{aligned}
f: \quad M \quad &\rightarrow \quad \overline{M} \\
P \quad &\rightarrow \quad f(P) = P + rN_P
\end{aligned}
\tag{2.1}
$$

is called the parallelization function between M and $\overline{M}$ and furthermore $\overline{M}$ is called parallel surface to M, where N is the unit normal vector field on M and r is a given real number.

Theorem 2.4([5]) *Let M and $\overline{M}$ be two parallel surfaces in Euclidean space and*

$$f: \quad M \quad \rightarrow \quad \overline{M}$$

be the parallelization function. Then for $X \in \chi(M)$

    1. $f_*(X) = X - rS(X)$
    2. $S^r\left(f_*(X)\right) = S(X)$
    3. f preserves principal directions of curvature, that is

$$S^r(f_*(X)) = \frac{\kappa}{1 - r\kappa} f_*(X),$$

where S^r is the shape operator on $\overline{M}$, and κ is a principal curvature of M at P in direction X.

Theorem 2.5([11]) *Let Darboux frame of curve β at $f(\alpha(t_0)) = f(P)$ on $\overline{M}$ be $\{\overline{T}, \overline{g}, \overline{N}\}$, then*

$$
\begin{aligned}
\overline{T} &= \frac{1}{v}\left[(1 - r\kappa_n)T - r\tau_g g\right], \\
\overline{g} &= \frac{1}{v}\left[(1 - r\kappa_n)g + r\tau_g T\right], \quad \cdot \\
\overline{N} &= N
\end{aligned}
\tag{2.2}
$$

Definition 2.2([2]) *Let x be a spacelike vector and y be a timelike vector in E_1^3. Then there is an unique real number $\theta \geq 0$ such that $\langle x, y \rangle = |x|\,|y|\sinh\theta$. This number is called the Lorentzian timelike angle between the vectors x and y.*

Definition 2.3([2]) *Let x and y be future pointing (or past pointing) timelike vectors in E_1^3. Then there is an unique real number $\theta \geq 0$ such that $\langle x, y \rangle =$- $|x|\,|y|\cosh\theta$. This number is called the hyperbolic angle between the vectors x and y.*

We denote by $\{T, n, b\}$ the moving Frenet frame along the unit speed curve $\alpha(s)$ in the Minkowski space E_1^3, the following Frenet formulae are given

$$
\frac{d}{ds}\begin{bmatrix} T \\ n \\ b \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\varepsilon\kappa_g & 0 & \tau_g \\ \kappa_n & \tau_g & 0 \end{bmatrix}\begin{bmatrix} T \\ n \\ b \end{bmatrix},
\tag{2.3}
$$

in [6], where $\langle T, T \rangle = 1$, $\langle n, n \rangle = \varepsilon = \pm 1$, $\langle b, b \rangle = -1$.

If the surface M is a spacelike surface, then the curve $\alpha(s)$ lying on surface M is a spacelike curve. So, the relations between the frames can be given as follows ([6]):

$$
\begin{bmatrix} T \\ g \\ N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix}\begin{bmatrix} T \\ n \\ b \end{bmatrix},
\tag{2.4}
$$

where $\{T, g, N\}$ is Darboux frame.

Besides, the derivative formulae of the Darboux frame of $\alpha(s)$ is given by ([6])

$$
\frac{d}{ds}\begin{bmatrix} T \\ g \\ N \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ \kappa_n & \tau_g & 0 \end{bmatrix}\begin{bmatrix} T \\ g \\ N \end{bmatrix},
\tag{2.5}
$$

where $\langle T, T \rangle = \langle g, g \rangle = 1$ and $\langle N, N \rangle = -1$.

Theorem 2.6([2]) *Let $\alpha(s)$ be non-unit speed curve on surface M in E^3. The Darboux frame of curve $\alpha(s)$ where $\|\alpha'(s)\| = v$, is $\{T, g, N\}$. Geodesic, normal curvatures and geodesic torsion of this curve-surface pair which are denoted by κ_g, κ_n, τ_g respectively are defined as follows:*

$$
\begin{aligned}
\kappa_g &= \frac{1}{v^2}\langle \alpha'', g \rangle, \\
\kappa_n &= \frac{1}{v^2}\langle \alpha'', N \rangle, \; . \\
\tau_g &= -\frac{1}{v}\langle N', g \rangle
\end{aligned}
\tag{2.6}
$$

§3. On the Spacelike Parallel Ruled Surfaces with Darboux Frame

A spacelike parallel ruled surface can be given as in the following parametrization:

$$\overline{\varphi}(s,\overline{v}) = \overline{\alpha}(s) + \overline{v}\,\overline{X}(s), \tag{3.1}$$

where the curve $\overline{\alpha}(s)$ is lying on $\overline{\varphi}(s,\overline{v})$ is a spacelike curve.

Darboux frame is obtained by rotating Frenet frame around $\overline{T}$ as far as $\overline{\theta} = \overline{\theta}(s)$ while the hyperbolic angle $\overline{\theta}$ is between the timelike unit vector $\overline{b}$ and the timelike normal vector field $\overline{N}$ of $\overline{\varphi}(s,\overline{v})$.

By Definitions 2.2 and 2.3, the spacelike unit vector $\overline{g}$ and the timelike vector $\overline{N}$ are written, in terms of $\overline{\theta}$, as:

$$\begin{array}{rcl} \overline{g} & = & \overline{n}\cosh\overline{\theta} + \overline{b}\sinh\overline{\theta}, \\ \overline{N} & = & \overline{n}\sinh\overline{\theta} + \overline{b}\cosh\overline{\theta}. \end{array} \tag{3.2}$$

From the expression (3.2), the spacelike unit vector $\overline{n}$ and timelike unit binormal vector $\overline{b}$ are obtained

$$\begin{array}{rcl} \overline{n} & = & \overline{g}\cosh\overline{\theta} - \overline{N}\sinh\overline{\theta}, \\ \overline{b} & = & -\overline{g}\sinh\overline{\theta} + \overline{N}\cosh\overline{\theta}. \end{array} \tag{3.3}$$

Taking into the expressions (3.2) and (3.3), Darboux derivative formulae of the spacelike parallel ruled surface can be found as

$$\frac{d}{ds}\begin{bmatrix} \overline{T} \\ \overline{g} \\ \overline{N} \end{bmatrix} = \begin{bmatrix} 0 & \overline{\kappa}_g & \overline{\kappa}_n \\ -\overline{\kappa}_g & 0 & \overline{\tau}_g \\ \overline{\kappa}_n & \overline{\tau}_g & 0 \end{bmatrix} \begin{bmatrix} \overline{T} \\ \overline{g} \\ \overline{N} \end{bmatrix}, \tag{3.4}$$

where

$$\overline{\kappa}_g = \overline{\kappa}\cosh\overline{\theta}, \; \overline{\kappa}_n = -\overline{\kappa}\sinh\overline{\theta} \text{ and } \overline{\tau}_g = \overline{\tau} + \overline{\theta}'. \tag{3.5}$$

Differentiating the expression (3.3) and using the matrix representation (3.4), we have the Frenet derivative formulae of the spacelike parallel ruled surface as:

$$\begin{array}{rcl} \overline{T}' & = & \overline{\kappa}\overline{n}, \\ \overline{n}' & = & -\overline{\kappa}\overline{T} + \overline{\tau}\overline{b}, \\ \overline{b}' & = & \overline{\tau}\overline{n}. \end{array} \tag{3.6}$$

Theorem 3.1 *Let $\overline{M}$ be a spacelike parallel ruled surface and Darboux frame of curve β be*

$\left\{ \overline{T}, \overline{g}, \overline{N} \right\}$ *at* $f(\alpha(t_0)) = f(P)$ *on* $\overline{M}$, *then we have*

$$
\begin{aligned}
\overline{T} &= \frac{1}{v}\left[(1 + r\kappa_n)T + r\tau_g g\right], \\
\overline{g} &= \frac{1}{v}\left[-(1 + r\kappa_n)g + r\tau_g T\right], \\
\overline{N} &= N.
\end{aligned}
\tag{3.7}
$$

Proof From Theorem 2.4 and the matrix representation (2.5), tangent vector of $(f \circ \alpha) = \beta$ of the curve at $f(\alpha(t_0))$ on the spacelike parallel ruled surface $\overline{M}$ is

$$
\begin{aligned}
\beta' = \overline{T} &= \frac{1}{v}\left[T + r(\kappa_n T + \tau_g g)\right] \\
&= \frac{1}{v}\left[(1 + r\kappa_n)T + r\tau_g g\right],
\end{aligned}
\tag{3.8}
$$

where the norm of β' is

$$
\|\beta'\| = \sqrt{(1 + r\kappa_n)^2 + r^2\tau_g^2} = v.
\tag{3.9}
$$

Same as, we find $\overline{g}$. And also there is the equation $\overline{N} = N$ between normal vectors of surfaces M and $\overline{M}$. $\qquad\square$

Theorem 3.2 *Let α be a regular curve on the surface M. Then the geodesic curvature, the normal curvature and the geodesic torsion of the curve $(f \circ \alpha) = \beta$ are respectively;*

$$
\begin{aligned}
\overline{\kappa}_g &= -\frac{\kappa_g}{v^3} + \frac{r}{v^3}\left[((1 + r\kappa_n)\tau_g' + r\tau_g\kappa_n')\right], \\
\overline{\kappa}_n &= \frac{1}{v^2}\left[\kappa_n + r(\kappa_n^2 + \tau_g^2)\right], \\
\overline{\tau}_g &= \frac{\tau_g}{v^2},
\end{aligned}
\tag{3.10}
$$

at the point $f(\alpha(t_0))$ on the spacelike parallel ruled surface $\overline{M}$.

Proof Because β spacelike curve is a non-unit speed curve, we use the Theorems 2.6, 2.4 and the equation (3.8), hence the following equation is obtained

$$
\beta'' = (r\kappa_n' - r\tau_g\kappa_g)T + (\kappa_g(1 + r\kappa_n) + r\tau_g')g + (\kappa_n(1 + r\kappa_n) + r\tau_g^2)N
\tag{3.11}
$$

Using the expressions (2.6), (3.7) and (3.11), we find $\overline{\kappa}_g$, $\overline{\kappa}_n$, $\overline{\tau}_g$ for spacelike parallel ruled surface $\overline{M}$. $\qquad\square$

Theorem 3.3 *Let Frenet frame of the curve $(f \circ \alpha) = \beta$ be $\left\{ \overline{T}, \overline{n}, \overline{b} \right\}$ at $f(\alpha(t_0)) = f(P)$ on*

the surface $\overline{M}$, then Frenet frame of the spacelike parallel ruled surface is as follows:

$$\overline{T} = \frac{1}{v}\left[(1 + r\overline{\kappa}_n)T + r\overline{\tau}_g g\right],$$
(3.12)

$$\overline{n} = \frac{1}{v\sqrt{\left(\overline{\kappa}_n\right)^2 - \left(\overline{\kappa}_g\right)^2}}\left[-(r\overline{\tau}_g\overline{\kappa}_g)T + (1 + r\overline{\kappa}_n)\overline{\kappa}_g g + (v\overline{\kappa}_n)N\right],$$

$$\overline{b} = \frac{1}{v^3\sqrt{\left(\overline{\kappa}_n\right)^2 - \left(\overline{\kappa}_g\right)^2}}\left[(v^2 r\overline{\tau}_g\overline{\kappa}_n)T - (v^2(1 + r\overline{\kappa}_n)\overline{\kappa}_n)g - (v^3\overline{\kappa}_g)N\right].$$

Proof If we use the equations (3.8) and (3.11), the following equation is obtained

$$\beta' \wedge \beta'' = \left(r\overline{\tau}_g\overline{\kappa}_n v^2\right)T - \left((1 + r\overline{\kappa}_n)\overline{\kappa}_n v^2\right)g - (\overline{\kappa}_g v^3)N.$$
(3.13)

The norm of the equation (3.13) is

$$\|\beta' \wedge \beta''\| = v^3\sqrt{(\overline{\kappa}_n)^2 - (\overline{\kappa}_g)^2}.$$
(3.14)

By the equations (3.13) and (3.14), we obtain the expression (3.12). □

Let $\overline{\phi}$ be the angle between direction vector $\overline{X}$ and tangent vector $\overline{T}$ at $\overline{\alpha} \in \overline{\varphi}(s, \overline{v})$. If we choose the direction vector $\overline{X}$, then we get

$$\overline{X} = \overline{T}\cos\overline{\phi} + \overline{g}\sin\overline{\phi},$$
(3.15)

where $\|\overline{X}\| = 1$.

Differentiating (3.15) and using (3.4), we find

$$\overline{X}' = -(\overline{\phi}' + \overline{\kappa}_g)\sin\overline{\phi}\,\overline{T} + (\overline{\phi}' + \overline{\kappa}_g)\cos\overline{\phi}\,\overline{g} + (\overline{\kappa}_n\cos\overline{\phi} + \overline{\tau}_g\sin\overline{\phi})\overline{N}.$$
(3.16)

Holding $\overline{v} =$ constant, we obtain the curve on the spacelike parallel ruled surface whose vector field as follows:

$$\begin{aligned}\overline{T}^* &= \overline{T} + \overline{v}\overline{X}' \\ &= (1 - \overline{v}(\overline{\phi}' + \overline{\kappa}_g)\sin\overline{\phi}\,\overline{T} + (\overline{\phi}' + \overline{\kappa}_g)\cos\overline{\phi}\,\overline{g} + (\overline{\kappa}_n\cos\overline{\phi} + \overline{\tau}_g\sin\overline{\phi})\overline{N}.\end{aligned}$$
(3.17)

Substituting the expression (2.6) into (3.5), we have

$$\begin{aligned}\overline{T}^* &= \left(1 - \frac{\overline{v}}{v^3}(-\kappa_g + r((r\kappa_n + 1)\tau_g)' + v^3\overline{\phi}')\sin\overline{\phi}\right)\overline{T} \\ &\quad + \frac{\overline{v}}{v^3}\left((-\kappa_g + r((r\kappa_n + 1)\tau_g)' + v^3\overline{\phi}')(\kappa_n + r(\kappa_n^2 + \tau_g^2) + v^2\overline{\phi}')\cos\overline{\phi}\right)\overline{g} \\ &\quad + \frac{\overline{v}}{v^2}\left((\kappa_n + r(\kappa_n^2 + \tau_g^2))\cos\overline{\phi} + \tau_g\sin\overline{\phi}\right)\overline{N}.\end{aligned}$$

The distribution parameter of the spacelike parallel ruled surface is defined by

$$\overline{P}_{\overline{X}} = \frac{-\sin\overline{\phi}(\overline{\kappa}_n\cos\overline{\phi} + \overline{\tau}_g\sin\overline{\phi})}{(\overline{\phi}' + \overline{\kappa}_g)^2 - (\overline{\kappa}_n\cos\overline{\phi} + \overline{\tau}_g\sin\overline{\phi})^2}. \tag{3.18}$$

By the expression (3.10), we obtain $\overline{P}_{\overline{X}}$ for spacelike parallel ruled surface as follows:

$$\overline{P}_{\overline{X}} = \frac{-v^4\sin\overline{\phi}\left((\kappa_n + r(\kappa_n^2 + \tau_g^2))\cos\overline{\phi} + \tau_g\sin\overline{\phi}\right)}{(-\kappa_g + r((r\kappa_n + 1)\tau_g)' + v^3\overline{\phi}')^2 + v^2\left((\kappa_n + r(\kappa_n^2 + \tau_g^2))\cos\overline{\phi} + \tau_g\sin\overline{\phi}\right)^2}.$$

Theorem 3.4 *The spacelike parallel ruled surface with Darboux frame is developable surface if and only if*

$$\sin\overline{\phi}(\overline{\kappa}_n\cos\overline{\phi} + \overline{\tau}_g\sin\overline{\phi}) = 0. \tag{3.19}$$

Proof Supposing that the spacelike parallel ruled surface with Darboux frame is developable surface, then $\overline{P}_{\overline{X}} = 0$. In this case, let us study the following subcases related to the equation (3.19) vanishing:

(1) If $\sin\overline{\phi} = 0$, then from (3.15), we obtain $\overline{X} = \overline{T}\cos\overline{\phi}$. So $\overline{T}^* = \overline{T}$. It means that tangent plane is constant along the main ruling on the spacelike parallel ruled surface.

(2) If $(\overline{\kappa}_n\cos\overline{\phi} + \overline{\tau}_g\sin\overline{\phi}) = 0$, then from the equation (3.17), it is seen that the tangent plane and the normal vector of spacelike parallel ruled surface with Darboux frame are orthogonal vectors. Therefore the spacelike parallel ruled surface with Darboux frame is developable surface.

Conversely, if

$$\sin\overline{\phi}(\overline{\kappa}_n\cos\overline{\phi} + \overline{\tau}_g\sin\overline{\phi}) = 0,$$

then from (3.18), $\overline{P}_{\overline{X}} = 0$.

The striction curve of the spacelike parallel ruled surface with Darboux frame is calculated as follows:

$$\overline{c}(s) = \overline{\alpha}(s) + \frac{\sin\overline{\phi}(\overline{\phi}' + \overline{\kappa}_g)}{(\overline{\phi}' + \overline{\kappa}_g)^2 + (\overline{\kappa}_n\cos\overline{\phi} + \overline{\tau}_g\sin\overline{\phi})^2}\overline{X}$$

Using the expression (3.10), we find $\overline{c}(s)$ for the spacelike parallel ruled surface as:

$$\overline{c}(s) = \overline{\alpha}(s) + \frac{-v^4\sin\overline{\phi}(-\kappa_g + r((r\kappa_n + 1)\tau_g)' + v^3\overline{\phi}')}{(-\kappa_g + r((r\kappa_n + 1)\tau_g)' + v^3\overline{\phi}')^2 - v^2\left((\kappa_n + r(\kappa_n^2 + \tau_g^2))\cos\overline{\phi} + \tau_g\sin\overline{\phi}\right)^2}\overline{X} \tag{3.20}$$

$$\square$$

Theorem 3.5 *Let $\overline{M}$ be a spacelike parallel ruled surface with Darboux frame as in (3.1). Then the shortest distance between the rulings of $\overline{M}$ along the orthogonal trajectories is the distance measured equaled to the value:*

$$\overline{v} = \frac{-v^4\sin\overline{\phi}(-\kappa_g + r((r\kappa_n + 1)\tau_g)' + v^3\overline{\phi}')}{(-\kappa_g + r((r\kappa_n + 1)\tau_g)' + v^3\overline{\phi}')^2 - v^2\left((\kappa_n + r(\kappa_n^2 + \tau_g^2))\cos\overline{\phi} + \tau_g\sin\overline{\phi}\right)^2}. \tag{3.21}$$

Proof From (3.17), we have

$$\overline{J}(\overline{v}) = \int_{s_1}^{s_2} \left(1\text{-}2\overline{v}(\overline{\phi}' + \overline{\kappa}_g)\sin\overline{\phi} + \overline{v}^2(\overline{\phi}' + \overline{\kappa}_g)^2 + \overline{v}^2(\overline{\kappa}_n\cos\overline{\phi} + \overline{\tau}_g\sin\overline{\phi})^2 \right)^{\frac{1}{2}} ds, \qquad (3.22)$$

where $s_1 < s_2$.

Differentiating the expression (3.22) according to the parameter $\overline{v}$ which gives the minimal value of $\overline{J}(\overline{v})$, we get

$$\overline{v} = \frac{\sin\overline{\phi}(\overline{\phi}' + \overline{\kappa}_g)}{(\overline{\phi}' + \overline{\kappa}_g)^2 + (\overline{\kappa}_n\cos\overline{\phi} + \overline{\tau}_g\sin\overline{\phi})^2}. \qquad (3.23)$$

Using the expression (3.10) in (3.23), the parameter $\overline{v}$ turns into the equation (3.21). □

Theorem 3.6 *Let $\overline{M}$ be a spacelike parallel ruled surface with Darboux frame as in (3.1), the absolute value of Gauss curvature K of the spacelike parallel ruled surface $\overline{M}$ along a ruling takes the maximum value at the striction point on that ruling.*

Proof Calculating Gauss curvature of the spacelike parallel ruled surface with Darboux frame, we get

$$K(s, \overline{v}) = \frac{(\overline{\kappa}_n\cos\overline{\phi} + \overline{\tau}_g\sin\overline{\phi})^2\sin^2\overline{\phi}}{\left(1\text{-}2\overline{v}(\overline{\phi}' + \overline{\kappa}_g)\sin\overline{\phi} + \overline{v}^2(\overline{\phi}' + \overline{\kappa}_g)^2 + \overline{v}^2(\overline{\kappa}_n\cos\overline{\phi} + \overline{\tau}_g\sin\overline{\phi})^2 - \cos^2\overline{\phi} \right)^2}. \qquad (3.24)$$

Differentiating the equation (3.24) with respect to $\overline{v}$, we have

$$\overline{v} = \frac{\sin\overline{\phi}(\overline{\phi}' + \overline{\kappa}_g)}{(\overline{\phi}' + \overline{\kappa}_g)^2 + (\overline{\kappa}_n\cos\overline{\phi} + \overline{\tau}_g\sin\overline{\phi})^2}.$$

Therefore, the absolute value of Gauss curvature K of the spacelike parallel ruled surface $\overline{M}$ along a ruling takes the maximum value at the striction point on that ruling. □

References

[1] A. Gray, S. Salamon and E. Abbena, *Modern Differential Geometry of Curves and Surfaces with Mathematica,* Chapman and Hall/CRC, 2006.

[2] B. O'Neill, *Elemantary Differential Geometry.* Academic Press Inc, New York, 1996.

[3] G. Darboux, *Lecons sur la Theorie generale des Sufaces I-II-III-IV.*, Gauthier-Villars, Paris, 1896.

[4] G. Y. Senturk, S. Yuce, Characteristic Properties of the Ruled Surface with Darboux Frame in E^3, *Kuwait Journal of Science*, 42(2), (2015), 14-33.

[5] H. H. Hacısalihoğlu, *Diferensiyel Geometri*, İnönü Üniv. Fen Edebiyat Fak. Yayınlar, 2, 1983.

[6] S. Kızıltuğ and Y. Yaylı, Timelike Curves on Timelike Parallel Surfaces in Minkowski 3-Space E_1^3, *Mathematica Aeterna*, 2(8), (2012), 689-700.

[7] T. Ravani and S. Ku, Bertrand Offsets of Ruled Surface and Developable Surface, *Computer-Aided Design*, 23(2), (1991), 145-152.

[8] T. Craig, Note on Parallel Surfaces, *Journal für die Reine and Angewandte Mathematik (Crelle's J.)*, 94, (1883), 162-170.

[9] V. Hlavaty, *Differentielle Linien Geometrie.* Uitg P. Noorfhoff, Groningen, 1945.

[10] Y. Ünlütürk, M. Çimdiker and C. Ekici, Characteristic properties of the parallel ruled surfaces with Darboux frame in Euclidean 3-space, *Communication in Mathematical Modeling and Applications*, 1(1), (2016), 26-43.

[11] Z. Savcı, A. Görgülü and C. Ekici, On Meusnier theorem for parallel surfaces, *Thai Journal of Mathematics* (in press), 2016.

Rainbow Connection Number in the Brick Product

Graphs $C(2n, m, r)$

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Abstract: Let G be a nontrivial connected graph on which is defined a coloring $c : E(G) \to \{1, 2, \cdots, k\}$, $k \in N$ of the edges of G, where adjacent edges may be colored the same. A path in G is called a rainbow path if no two edges of it are colored the same. G is rainbow connected if G contains a rainbow $u - v$ path for every two vertices u and v in it. The minimum k for which there exists such a k-edge coloring is called the rainbow connection number of G, denoted by $rc(G)$. In this paper we determine $rc(G)$ for the brick product $C(2n, m, r)$ associated with the even cycle C_{2n} for $m = 1$ and odd $r \geq 5$ such that $n = r + 1, r + 2$ and $n \geq r + 3$. We also discuss the critical property of the graphs with $n = r + 1$ and $n = r + 2$ with respect to rainbow coloring.

Key Words: Diameter, edge-coloring, rainbow path, Smarandacely H-rainbow connected, rainbow connected, rainbow connection number, rainbow critical graph, brick product.

AMS(2010): 05C15.

§1. Introduction

All graphs considered in this paper are simple, finite and undirected. Let G be a nontrivial connected graph with an edge coloring $c : E(G) \to \{1, 2, \cdots, k\}$, $k \in N$, where adjacent edges may be colored the same. A path in G is called a *rainbow path* if no two edges of it are colored the same. An edge colored graph G is said to be *Smarandacely H-rainbow connected to a graph H* if for any two vertices in a subgraph $G' \preceq G$ isomorphic to H, there is always a rainbow path in G connecting them. Particularly, an edge colored graph G is said to be *rainbow connected* if for any two vertices in G, there is a rainbow path in G connecting them, i.e., Smarandacely G-rainbow connected. Clearly, if a graph is rainbow connected, it must be connected. Conversely, any connected graph has a trivial edge coloring that makes it rainbow connected, i.e., a coloring such that each edge has a distinct color. The minimum k for which there exist a rainbow k-coloring of G is called the *rainbow connection number* of G, denoted by $rc(G)$.

Let c be a rainbow coloring of G. For any two vertices u and v of G a *rainbow $u - v$ geodesic* in G is a rainbow $u - v$ path of length $d(u, v)$ where $d(u, v)$ is the distance between u and v. G is termed *strongly rainbow connected* if G contains a rainbow $u - v$ geodesic for every two vertices u and v in G and in this case the coloring c is called a *strong rainbow coloring* of G. The minimum k for which there exists a coloring $c : E(G) \to \{1, 2, \cdots, k\},\ k \in N$ of the edges of G such that G is strongly rainbow connected is called the *strong rainbow connection number* of G, denoted by $src(G)$. Thus $rc(G) \le src(G)$ for every connected graph G.

The rainbow connection number and the strong rainbow connection number are defined for every connected graph G, since every coloring that assigns distinct colors to the edges of G is both a rainbow coloring and a strong rainbow coloring and G is rainbow connected and strongly rainbow connected with respect to some coloring of the edges of G.

The concept of rainbow connectivity and strong rainbow connectivity were first introduced by Chartrand et al. [2] in 2008 as a means of strengthening the connectivity. In [2], the authors computed the rainbow connection numbers of several graph classes (complete graphs, trees, cycles, wheels and complete bipartite graphs). In [5] and [6] K.Srinivasa Rao and R.Murali, determined $rc(G)$ and $src(G)$ of the stacked book graph, the grid graph, the prism graph etc. They also discussed the critical property of these graphs with respect to rainbow coloring. In [7], the authors determined $rc(G)$ of brick product graphs associated with even cycles and also discussed the critical property of these graphs with respect to rainbow coloring. An overview about rainbow connection number can be found in a book of Li and Sun in [4] and a survey by Li et.al. in [3].

§2. Definitions

Definition 2.1 *A graph G is said to be rainbow critical if the removal of any edge from G increases the rainbow connection number of G, i.e. if $rc(G) = k$ for some positive integer k, then $rc(G - e) > k$ for any edge e in G.*

The brick product of even cycles was introduced in a paper by B.Alspach et.al. [1] in which the Hamiltonian laceability properties of brick products was explored.

Definition 2.2 (Brick product of even cycle) *Let m, n and r be positive integers. Let $C_{2n} = v_0, v_1, v_2, \cdots, v_{(2n-1)}, v_0$ denote a cycle of order $2n$. The (m, r)-brick product of C_{2n} denoted by $C(2n, m, r)$ is defined as follows:*

For $m = 1$, we require that r be odd and greater than 1. Then, $C(2n, m, r)$ is obtained from C_{2n} by adding chords $v_{2k}(v_{2k+r})$, $k = 1, 2, \cdots, n$, where the computation is performed under modulo $2n$.

For $m > 1$, we require that $m + r$ be even. Then, $C(2n, m, r)$ is obtained by first taking disjoint union of m copies of C_{2n} namely, $C_{2n}(1), C_{2n}(2), C_{2n}(3), \cdots, C_{2n}(m)$ where for each $i = 1, 2, \cdots, m$, $C_{2n}(i) = v_{i1}, v_{i2}, v_{i3}, , \cdots, v_{i(2n)}$. Next, for each odd $i = 1, 2, \cdots, m - 1$ and each even $k = 0, 1, \cdots, 2n - 2$, an edge(called a brick edge) drawn to join v_{ik} to $v_{(i+1)k}$, whereas, for each even $i = 1, 2, \cdots, m - 1$ and each odd $k = 1, 2, \cdots, 2n - 1$, an edge (also called a brick

edge) is drawn to join v_{ik} *to* $v_{(i+1)k}$.

Finally, for each odd $k = 1, 2, \cdots, 2n - 1$, an edge (called a hooking edge)is drawn to join v_{1k} to $v_{m(k+r)}$. An edge in $C(2n, m, r)$ which is neither a brick edge nor a hooking edge is called a flat edge.

The brick products $C(10, 1, 5)$ and $C(14, 1, 5)$ are shown in the Figure 1.
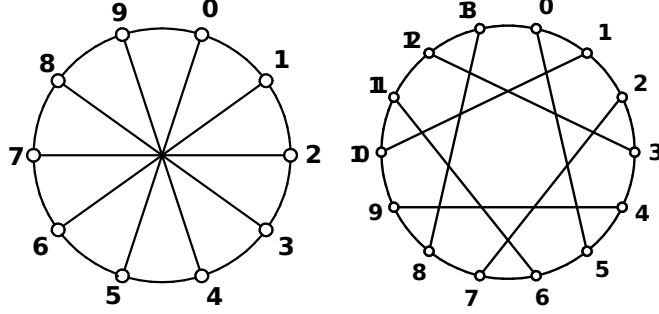
Figure 1 The brick product $C(10, 1, 5)$ and $C(14, 1, 5)$.

In the next section, we determine the values of $rc(G)$ for the brick product graph $C(2n, m, r)$ for $m = 1$ and $n = r + 1$, $n = r + 2$ and $n \geq r + 3$ for odd $r \geq 5$. In our results, we denote the vertices of the cycle C_{2n} as $v_0, v_1, \cdots, v_{2n-1}, v_{2n} = v_0$.

§3. Main Results

Theorem 3.1 Let $G = C(2n, m, r)$. Then for $m = 1$ and odd $r \geq 5$,

$$rc(G) = \begin{cases} \frac{n}{2} + 1 & \text{for} \quad n = r + 1 \\ \lceil \frac{n}{2} \rceil & \text{for} \quad n = r + 2 \end{cases}$$

Proof We consider the vertex set of G as $V(G) = \{v_0, v_1, \cdots, v_{2n-1}, v_{2n} = v_0\}$ and the edge set of G as $E(G) = \{e_i : 1 \leq i \leq 2n\} \cup \{e'_i : 1 \leq i \leq n\}$, where e_i is the *cycle edge* (v_{i-1}, v_i) and e'_i is the *brick edge* (v_{2k}, v_{2k+r}), $k = 0, 1, \cdots, n$. Here $2k + r$ is computed modulo $2n$. We prove this result in different cases as follows.

Case 1. $n = r + 1$.

Since $diam(G) = \frac{n}{2}$, it follows that $rc(G) \geq \frac{n}{2}$. We define a $\frac{n}{2}$ coloring $c : E(G) \to \{1, 2, \cdots, \frac{n}{2}\}$ on the cycle edges of G as

$$c(e_i) = \begin{cases} i & \text{if} \quad 1 \leq i \leq \frac{n}{2} \\ i - \frac{n}{2} & \text{if} \quad \frac{n}{2} + 1 \leq i \leq n \\ i - n & \text{if} \quad n + 1 \leq i \leq \frac{3n}{2} \\ i - \frac{3n}{2} & \text{if} \quad \frac{3n}{2} + 1 \leq i \leq 2n \end{cases}$$

Also, we consider the assignment of colors to the brick edges in two different ways.

1.
$$c(e_i') = \begin{cases} i & \text{if } 1 \leq i \leq \frac{n}{2} \\ i - \frac{n}{2} & \text{if } \frac{n}{2} + 1 \leq i \leq n \end{cases}$$

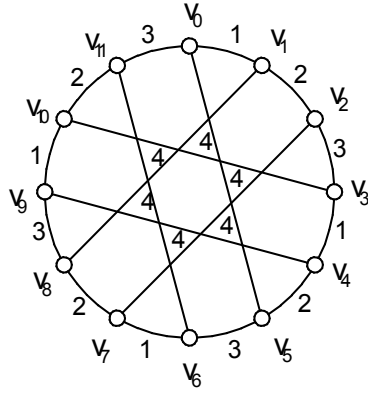This coloring will not give a rainbow $v_0 - v_{n+1}$, $v_1 - v_n$ path $\forall n$ and this is true for other pair of vertices.

2. $c(e_i') = \frac{n}{2}$ if $1 \leq i \leq n$.

This coloring also will not give a rainbow $v_0 - v_n$ path $\forall n$ and this is true for other pair of vertices. (This is illustrated in Figure 2).
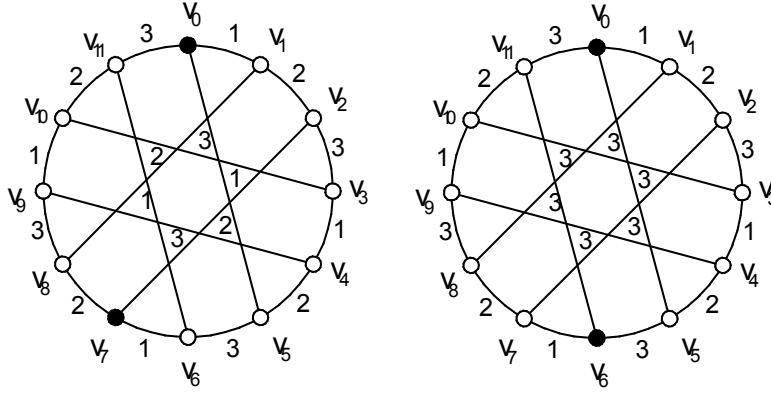
Figure 2 Assignment of colors in $C(12, 1, 5)$ in two different ways.

Figure 3 Assignment of colors in $C(12, 1, 5)$.

Accordingly, we define a coloring $c : E(G) \rightarrow \{1, 2, \cdots, \frac{n}{2} + 1\}$ and assign the colors to the edges of G as

$$c(e_i) = \begin{cases} i & \text{if } 1 \le i \le \frac{n}{2} \\ i - \frac{n}{2} & \text{if } \frac{n}{2} + 1 \le i \le n \\ i - n & \text{if } n + 1 \le i \le \frac{3n}{2} \\ i - \frac{3n}{2} & \text{if } \frac{3n}{2} + 1 \le i \le 2n \end{cases}$$

and

$$c(e_i^{'}) = \frac{n}{2} + 1 \ \text{ if } \ 1 \le i \le n$$

From the above assignment, it is clear that for every two distinct vertices $x, y \in V(G)$, there exists an $x - y$ rainbow path. Hence $rc(G) \le \frac{n}{2} + 1$. This proves $rc(G) = \frac{n}{2} + 1$. An illustration for the assignment of colors in $C(12, 1, 5)$ is provided in Figure 3.

Case 2. $n = r + 2$.

Since $diam(G) = \lceil \frac{n}{2} \rceil$, it follows that $rc(G) \ge \lceil \frac{n}{2} \rceil$. In order to show that $rc(G) \le \lceil \frac{n}{2} \rceil$, we construct an edge coloring $c : E(G) \to \{1, 2, \cdots, \lceil \frac{n}{2} \rceil\}$ and assign the colors to the edges of G as

$$c(e_i) = \begin{cases} i & \text{if } 1 \le i \le \lfloor \frac{n}{2} \rfloor \\ i - \lfloor \frac{n}{2} \rfloor & \text{if } \lceil \frac{n}{2} \rceil \le i \le n - 1 \\ i - n & \text{if } n + 1 \le i \le \lfloor \frac{3n}{2} \rfloor \\ i - \lfloor \frac{3n}{2} \rfloor & \text{if } \lceil \frac{3n}{2} \rceil \le i \le 2n - 1 \\ \lceil \frac{n}{2} \rceil & \text{elsewhere} \end{cases}$$

and

$$c(e_i^{'}) = \lceil \frac{n}{2} \rceil \ \text{ if } \ 1 \le i \le n$$

From the above assignment, for any two vertices $x, y \in V(G)$, we obtain a rainbow $x - y$ path in G. Hence $rc(G) \le \lceil \frac{n}{2} \rceil$.

This proves $rc(G) = \lceil \frac{n}{2} \rceil$. An illustration for the assignment of colors in $C(14, 1, 5)$ is provided in Figure 4.   □
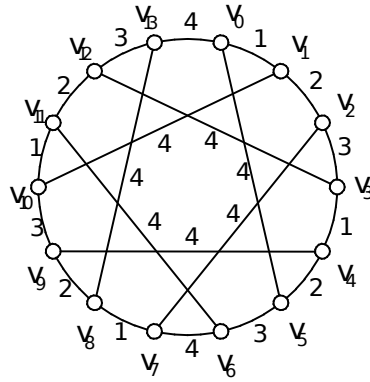
Figure 4 Assignment of colors in $C(14, 1, 5)$.

The critical nature of the brick product graph in Theorem 3.1 has been observed for particular values of n. This is illustrated in our next result.
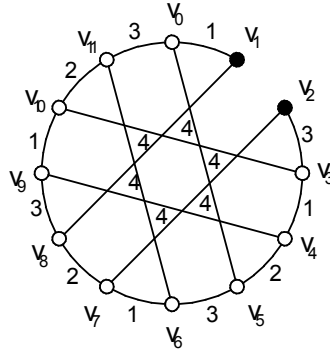
Lemma 3.2 *Let $G = C(2n, m, r)$, where $m = 1$ and odd $r \geq 5$. Then G is rainbow critical for $n = r + 1$ and $r + 2$.*

$$i.e., \quad rc(G - e) = n \quad for \quad n = r + 1 \ and \ n = r + 2.$$

Proof We prove this result in two cases.

Case 1. $n = r + 1$.

From the Theorem 3.1, $rc(G) = \frac{n}{2} + 1$ for $n = r + 1$. Consider the graph G and let $e \in E(G)$ be the any edge in G. Deletion of the edge e in G does not yield a rainbow path between the end vertices of e in $G - e$. For illustration see Figure 5.

Figure 5 Assignment of colors in $C(12, 1, 5)$.

This shows $rc(G - e) > \frac{n}{2} + 1$. We define a rainbow coloring $c : E(G) \rightarrow \{1, 2, \cdots, \frac{n}{2} + 2\}$ to the edges of G as defined in Theorem 3.1. i.e.,

$$c(e_i) = \begin{cases} i & \text{if} \ \ 1 \leq i \leq \frac{n}{2} + 1 \\ i - (\frac{n}{2} + 1) & \text{if} \ \ \frac{n}{2} + 2 \leq i \leq n + 2 \\ i - (n + 2) & \text{if} \ \ n + 3 \leq i \leq \frac{3n}{2} + 3 \\ i - (\frac{3n}{2} + 3) & \text{if} \ \ \frac{3n}{2} + 4 \leq i \leq 2n \end{cases}$$

and

$$c(e_i') = \tfrac{n}{2} + 2 \ \text{ if } \ \ 1 \leq i \leq n$$

From the above assignment of colors, we fail to get a rainbow path between any two arbitrary vertices in $G - e$. Without loss of generality let $e = v_1 v_2$. An illustration is provided in Figure 6.
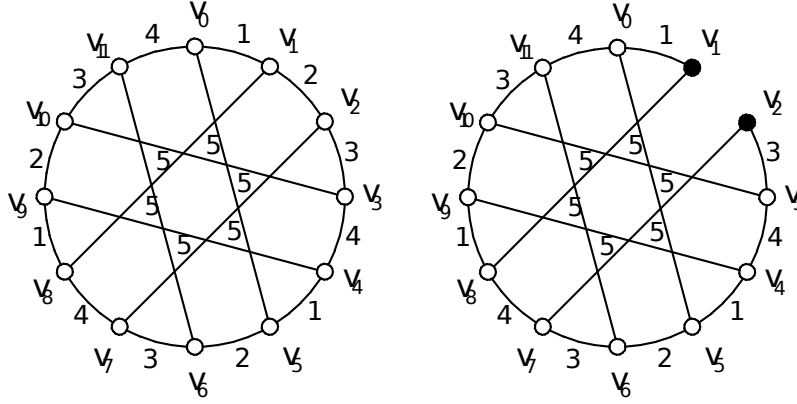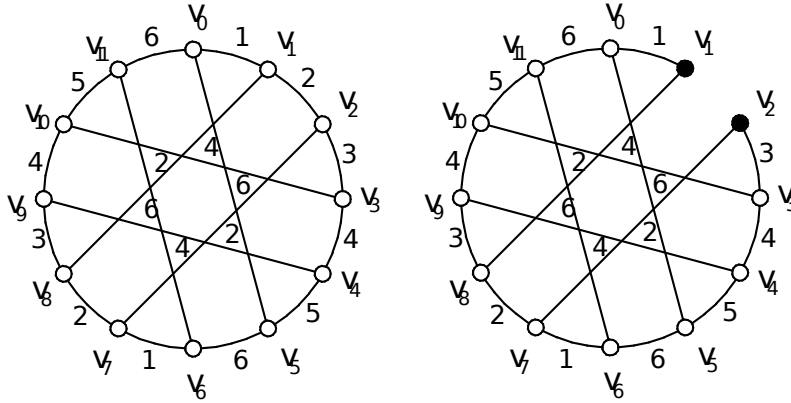
Figure 6 Assignment of colors in $C(12, 1, 5)$.

Accordingly, we assign a $(\frac{n}{2} + \frac{r+1}{2}) = (\frac{n}{2} + \frac{n}{2}) = n$ rainbow coloring $c : E(G) \to \{1, 2, \cdots, n\}$ to the edges of G and since the cycle C_{2n} is a sub graph of G, assign the colors to the edges of cycle as in Theorem 3.1. That is,

$$c(e_i) = \begin{cases} i & \text{if} \quad 1 \leq i \leq n \\ i - n & \text{if} \quad n + 1 \leq i \leq 2n \end{cases}$$

Next, we will consider the assignment of colors to the brick edge as

$$c(e_i') = \begin{cases} (2i + 4) \bmod n & \text{if} \quad 1 \leq i \leq \frac{n}{2} \\ [(2i + 4) \bmod n] - \frac{n}{2} & \text{if} \quad \frac{n}{2} + 1 \leq i \leq n \end{cases}$$

From the above assignment, it is clear that for any two vertices $x, y \in V(G)$, there exists a rainbow $x - y$ path in $G - e$ and hence $rc(G - e) \leq n$. This proves $rc(G - e) = n$. An illustration for the assignment of colors in $C(12, 1, 5)$ is provided in Figure 7.

Figure 7 Assignment of colors in $C(12, 1, 5)$.

Case 2. $n = r + 2$.

The proof for $n = r + 2$ is the same as Case 1.

As in Case 1, in this case also we fail to get the $x - y$ rainbow path for different combinations of assignment of $n - 1$ colors to the edges of $G - e$.

Accordingly, we construct a rainbow coloring $c : E(G) \rightarrow \{1, 2, \cdots, n\}$ to the edges of G as

$$c(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq n \\ i - n & \text{if } n + 1 \leq i \leq 2n \end{cases}$$

and

$$c(e_i') = \begin{cases} n + 1 - i & \text{if } 1 \leq i \leq 2 \\ i & \text{if } i = 3 \\ i + 1 & \text{if } 4 \leq i \leq n - 3 \\ i - (n - 3) & \text{if } n - 2 \leq i \leq n - 1 \\ 4 & \text{if } i = n \end{cases}$$

From the above assignment for any two vertices $x, y \in V(G)$, we obtain a rainbow $x - y$ path in $G - e$. This proves $rc(G - e) = n$. Hence the proof. An illustration for the assignment of colors in $C(14, 1, 5)$ is provided in Figure 8.
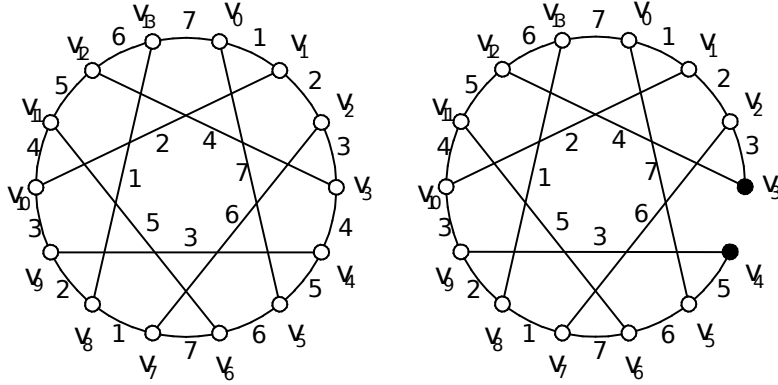
Figure 8 Assignment of colors in $C(14, 1, 5)$.

For $r \geq 5$ and $n \geq r + 3$, we have the following result.

Theorem 3.3 *Let $G = C(2n, m, r)$. Then for $m = 1$, odd $r \geq 5$ and $n \geq r + 3$,*

$$rc(G) = \begin{cases} \frac{n}{2} + 1 & \text{for } n = r + 3 \\ \lceil \frac{n}{2} \rceil + 1 & \text{for } r + 4 \leq n \leq 2r \\ (n - r + 1) - x\lfloor \frac{r}{2} \rfloor & \text{for } (2r + 1) + xr \leq n \leq 3r + xr \\ (2r + 2) + x(\lfloor \frac{r}{2} \rfloor + 1) & \text{for } (3r + 1) + xr \leq n \leq 4r + xr \\ & \text{where } x = 0, 2, 4 \cdots \end{cases}$$

Proof We consider the vertex set $V(G)$ and the edge set $E(G)$ defined in Theorem 3.1. We prove this result in different cases as follows.

Case 1. $n = r + 3$.

We have two subcases.
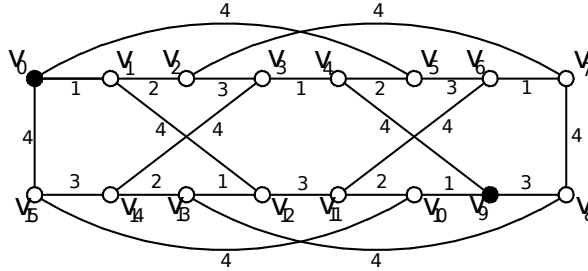
Subcase 1.1 $n = 4k$, where $k = 2, 3, \cdots$.

Here $diam(G) = \frac{n+8}{4}$. Clearly $rc(G) \geq diam(G)$. If we assign $diam(G)$-colors to the edges of G, we fail to obtain a rainbow path between a pair of vertices in G. This holds up to $\frac{n}{2}$ colors. Hence, to prove this we consider an edge coloring $c : E(G) \rightarrow \{1, 2, \cdots, \frac{n}{2}\}$ to the edges of G as

$$c(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq \frac{n}{2} \\ i - \frac{n}{2} & \text{if } \frac{n}{2} + 1 \leq i \leq n \\ i - n & \text{if } n + 1 \leq i \leq \frac{3n}{2} \\ i - \frac{3n}{2} & \text{if } \frac{3n}{2} + 1 \leq i \leq 2n \end{cases}$$

and

$$c(e_i') = \frac{n}{2} \text{ if } 1 \leq i \leq n$$

This coloring will not yield a rainbow $v_0 - v_{n+1}$ path $\forall n$, which is illustrated in Figure 9.

Figure 9 Assignment of colors in $C(16, 1, 5)$.

Subcase 1.2 $n = 5k$, where $k = 2, 3, \cdots$.

Here $diam(G) = \frac{n+6}{4}$. Clearly $rc(G) \geq diam(G)$. Similar to subcase 1 we fail to obtain a rainbow path between a pair of vertices in G up to $\frac{n}{2}$-colors. From Subcase 1 and Subcase 2, we have $rc(G) \geq \frac{n}{2} + 1$. Accordingly, we construct an edge coloring $c : E(G) \rightarrow \{1, 2, \cdots, \frac{n}{2} + 1\}$ and assign the colors to the edges of G as

$$c(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq \frac{n}{2} \\ i - \frac{n}{2} & \text{if } \frac{n}{2} + 1 \leq i \leq n \\ i - n & \text{if } n + 1 \leq i \leq \frac{3n}{2} \\ i - \frac{3n}{2} & \text{if } \frac{3n}{2} + 1 \leq i \leq 2n \end{cases}$$

and

$$c(e_i') = \frac{n}{2} + 1 \quad \text{if} \quad 1 \leq i \leq n$$

From the above assignment, it is clear that for any two vertices $x, y \in V(G)$, there exists a rainbow $x - y$ path in G and hence $rc(G) \leq \frac{n}{2} + 1$. This proves $rc(G) = \frac{n}{2} + 1$. An illustration for the assignment of colors in $C(20, 1, 5)$ is provided in Figure 10.
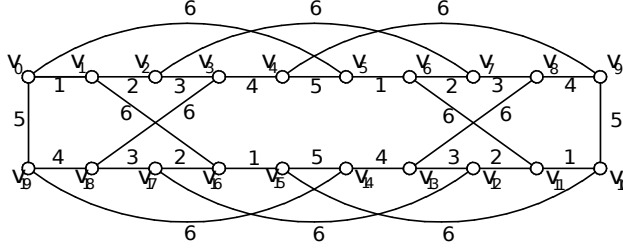
Figure 10 Assignment of colors in $C(20, 1, 5)$.

Case 2. $r + 4 \leq n \leq 2r$.

We consider the following subcases.

Subcase 2.1 n is even.

The proof is similar to the proof provided in Subcase 1.1 of Case 1.

If we consider assignment of $\frac{n}{2}$ colors in any combination to the edges of G, we fail to obtain a rainbow $v_0 - v_{n+1}$ path $\forall n$ (as illustrated in Figure 9).

This shows $rc(G) \geq \frac{n}{2} + 1$. It remains to show that $rc(G) \leq \frac{n}{2} + 1$. Define a coloring $c : E(G) \rightarrow \{1, 2, \cdots \frac{n}{2} + 1\}$ and consider the assignment of colors to the edges of G as

$$c(e_i) = \begin{cases} i & \text{if} \quad 1 \leq i \leq \frac{n}{2} \\ i - \frac{n}{2} & \text{if} \quad \frac{n}{2} + 1 \leq i \leq n \\ i - n & \text{if} \quad n + 1 \leq i \leq \frac{3n}{2} \\ i - \frac{3n}{2} & \text{if} \quad \frac{3n}{2} + 1 \leq i \leq 2n \end{cases}$$

and

$$c(e_i') = \frac{n}{2} + 1 \quad \text{if} \quad 1 \leq i \leq n$$

From the above assignment, it is clear that for any two vertices $x, y \in V(G)$, there exists a rainbow $x - y$ path with coloring c. Hence $rc(G) \leq \frac{n}{2} + 1$. An illustration for the assignment of colors in $C(20, 1, 5)$ is provided in Figure 10.

Subcase 2.2 n is odd.

If we assign $diam(G)$ colors to the edges of G, we fail to obtain a rainbow path between a pair of vertices in G. This holds up to $\lceil \frac{n}{2} \rceil$ colors. Hence, to prove this we consider an edge
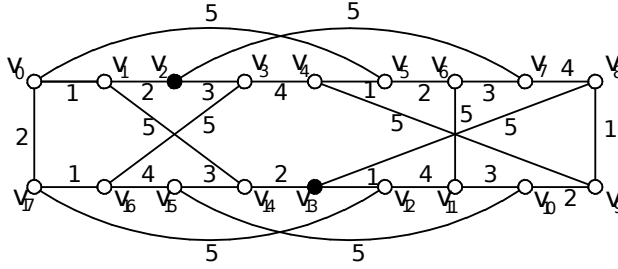
coloring $c : E(G) \rightarrow \{1, 2, \cdots, \lceil \frac{n}{2} \rceil\}$ to the edges of G as

$$c(e_i) = \begin{cases} i & \text{if} \quad 1 \le i \le \lfloor \frac{n}{2} \rfloor \\ i - \lfloor \frac{n}{2} \rfloor & \text{if} \quad \lceil \frac{n}{2} \rceil \le i \le n - 1 \\ i - (n-1) & \text{if} \quad n \le i \le \lfloor \frac{3n}{2} \rfloor - 1 \\ i - (\lfloor \frac{3n}{2} \rfloor - 1) & \text{if} \quad \lceil \frac{3n}{2} \rceil \le i \le 2n - 2 \\ i - (2n-2) & \text{if} \quad 2n - 1 \le i \le 2n \end{cases}$$

and

$$c(e_i') = \lceil \tfrac{n}{2} \rceil \quad \text{if} \quad 1 \le i \le n$$

This coloring will not yield a rainbow $v_2 - v_{\frac{3n-1}{2}}$ path $\forall n$, which is illustrated in Figure 11.

Figure 11 Assignment of colors in $C(18, 1, 5)$.

This shows $rc(G) \ge \lceil \frac{n}{2} \rceil + 1$. It remains to show that $rc(G) \le \lceil \frac{n}{2} \rceil + 1$. We construct an edge coloring $c : E(G) \rightarrow \{1, 2, \cdots, \lceil \frac{n}{2} \rceil + 1\}$ and assign the colors to the edges of G as

$$c(e_i) = \begin{cases} i & \text{if} \quad 1 \le i \le \lfloor \frac{n}{2} \rfloor \\ i - \lfloor \frac{n}{2} \rfloor & \text{if} \quad \lfloor \frac{n}{2} \rfloor + 1 \le i \le n - 1 \\ \lceil \frac{n}{2} \rceil & \text{if} \quad i = n \quad and \quad i = 2n \\ i - n & \text{if} \quad n + 1 \le i \le \lfloor \frac{3n}{2} \rfloor \\ i - \lfloor \frac{3n}{2} \rfloor & \text{if} \quad \lfloor \frac{3n}{2} \rfloor + 1 \le i \le 2n - 1 \end{cases}$$
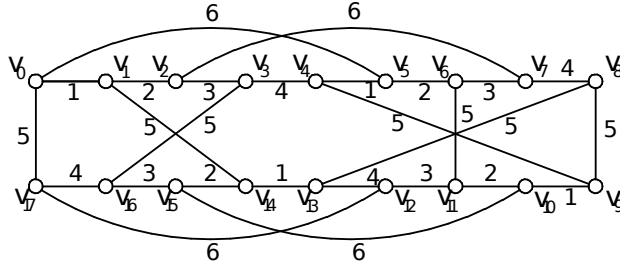
and

$$c(e_i') = \begin{cases} \lceil \frac{n}{2} \rceil + 1 & \text{if} \quad 1 \le i \le \frac{n-r}{2} \quad and \quad \frac{n+3}{2} \le i \le \frac{2n-r+1}{2} \\ \lceil \frac{n}{2} \rceil & \text{if} \quad \frac{n-r+2}{2} \le i \le \frac{n+1}{2} \quad and \quad \frac{2n-r+3}{2} \le i \le n \end{cases}$$

From the above assignment, it is clear that for any two vertices $x, y \in V(G)$, there exists a rainbow $x - y$ path. Hence $rc(G) \le \lceil \frac{n}{2} \rceil + 1$.

Combining both the Subcases, we have $rc(G) \le \lceil \frac{n}{2} \rceil + 1$, $\forall n$ such that $r + 4 \le n \le 2r$.

This proves $rc(G) = \lceil \frac{n}{2} \rceil + 1$. An illustration for the assignment of colors in $C(18, 1, 5)$ is provided in Figure 12.

Figure 12 Assignment of colors in $C(18, 1, 5)$.

Case 3. For $(2r+1) + xr \le n \le 3r + xr$ and $(3r+1) + xr \le n \le 4r + xr$ where $x = 0, 2, 4, \cdots$.

Let

$$
h = \begin{cases}
(n - r + 1) - x\lfloor \frac{r}{2} \rfloor & \text{for} \quad (2r+1) + xr \le n \le 3r + xr \\
(2r + 2) + x(\lfloor \frac{r}{2} \rfloor + 1) & \text{for} \quad (3r+1) + xr \le n \le 4r + xr \\
& \text{where} \quad x = 0, 2, 4, \cdots
\end{cases}
$$

As in Case 1 and Case 2 we again fail to obtain a rainbow path between a pair of vertices in G up to $h - 1$ colors. This shows $rc(G) \ge h$.

To show that $rc(G) \le h$, we construct a rainbow coloring $c : E(G) \to \{1, 2, \cdots, (n - r + 1) - x\lfloor \frac{r}{2} \rfloor\}$ for $(2r + 1) + xr \le n \le 3r + xr$ and $c : E(G) \to \{1, 2, \cdots, (2r + 2) + x(\lfloor \frac{r}{2} \rfloor + 1)\}$ for $(3r + 1) + xr \le n \le 4r + xr$ and assign the colors to the edges of G. Before assignment of colors, we split $E(G)$ as, e_i where $1 \le i \le n - 1$, $i = n$, $i = 2n$ and $n + 1 \le i \le 2n - 1$ (cycle edges) and e_i' where $1 \le i \le n$ (brick edges). we assign the colors to the cycle edges from e_1 to e_{n-1} as

$$
c(e_i) = \begin{cases}
i & \text{for} \quad 1 \le i \le r \\
i - kr & \text{for} \quad (2k - 1)r + 1 \le i \le l
\end{cases}
$$

where

$$
l = \begin{cases}
(2k + 1)r & \text{for each} \quad k = 1, 2, \cdots, \lceil \frac{n-r}{2r} \rceil - 1 \\
n - 1 & \text{for} \quad k = \lceil \frac{n-r}{2r} \rceil
\end{cases}
$$

and assign the colors to the cycle edge e_n and e_{2n} as

$$
c(e_n) = c(e_{2n}) = \begin{cases}
[(n - r) - x\lfloor \frac{r}{2} \rfloor] - \frac{x}{2} & \text{for} \quad (2r + 1) + xr \le n \le 3r + xr \\
2r + x[\lfloor \frac{r}{2} \rfloor + 1] - \frac{x}{2} & \text{for} \quad (3r + 1) + xr \le n \le 4r + xr \\
& \text{where} \quad x = 0, 2, \cdots
\end{cases}
$$

Also, we assign the colors to the cycle edges from e_{n+1} to e_{2n-1} as

$$
c(e_i) = \begin{cases}
i - n & \text{for} \quad n + 1 \le i \le n + r \\
i - (n + kr) & \text{for} \quad n + (2k - 1)r + 1 \le i \le m
\end{cases}
$$

where

$$m = \begin{cases} n + (2k+1)r & \text{for each } k = 1, 2, \cdots, \lceil \frac{n-r}{2r} \rceil - 1 \\ 2n - 1 & \text{for } k = \lceil \frac{n-r}{2r} \rceil \end{cases}$$

and assign the colors to the brick edges from e_1' to e_n' as

$$c(e_i') = \begin{cases} c(e_n) + k & \text{for } (k-1)r + 1 \leq i \leq l \\ c(e_n) & \text{for } \lfloor \frac{n-4}{2} \rfloor + 1 \leq i \leq \lceil \frac{n}{2} \rceil \\ c(e_n) + k & \text{for } \lceil \frac{n}{2} \rceil + (k-1)r + 1 \leq i \leq m \\ c(e_n) & \text{for } (n-1) \leq i \leq n \end{cases}$$

where

$$l = \begin{cases} kr & \text{for each } k = 1, 2, \cdots, \lceil \frac{n-r}{2r} \rceil - 1 \\ \lfloor \frac{n-4}{2} \rfloor & \text{for } k = \lceil \frac{n-r}{2r} \rceil \end{cases}$$

and

$$m = \begin{cases} \lceil \frac{n}{2} \rceil + kr & \text{for each } k = 1, 2, \cdots, \lceil \frac{n-r}{2r} \rceil - 1 \\ n - 2 & \text{for } k = \lceil \frac{n-r}{2r} \rceil \end{cases}$$

From the above assignment, it is clear that for any two vertices $x, y \in V(G)$, there exists a rainbow $x - y$ path and hence

$$rc(G) = \begin{cases} (n - r + 1) - x\lfloor \frac{r}{2} \rfloor & \text{for } (2r+1) + xr \leq n \leq 3r + xr \\ (2r + 2) + x(\lfloor \frac{r}{2} \rfloor + 1) & \text{for } (3r+1) + xr \leq n \leq 4r + xr \\ & \text{where } x = 0, 2, 4, \cdots \end{cases}$$

Hence the proof. An illustration for the assignment of colors in $C(38, 1, 5)$ is provided in Figure 13.   □
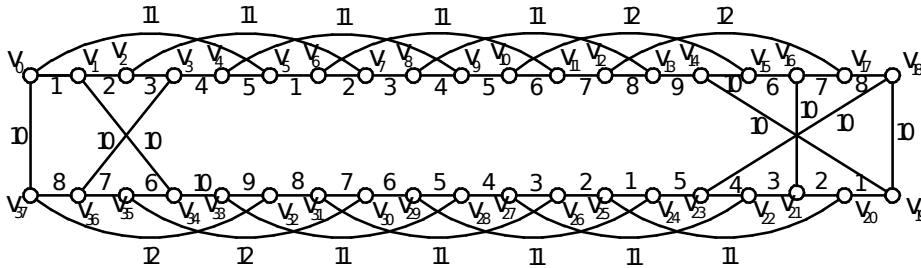
Figure 13 Assignment of colors in $C(38, 1, 5)$.

References

[1] B. Alspach, B. Alspach, C.C.Chen, Kevin McAvaney, On a class of Hamiltonian laceable 3-regular graphs, *Discrete Mathematics*, 151, (1996), pp. 19-38.

[2] Chartrand G, Johns GL, McKeon KA, Zhang P., Rainbow connection in graphs, *Math Bohem*, 133, (2008), pp. 85-98.

[3] Li X, Shi Y, Sun Y., Rainbow connection of graphs: a survey, *Graphs and Combin.*, 29(1), (2013), pp. 1-38.

[4] Li X, Sun Y., *Rainbow Connection of Graphs*, New York: Springer-Verlag, 2012.

[5] K.Srinivasa Rao, R.Murali., Rainbow critical graphs, *Int. J. of Comp. Application*, 4(4), (2014), pp. 252-259.

[6] K.Srinivasa Rao, R.Murali, S.K.Rajendra, Rainbow and strong rainbow criticalness of some standard graphs, *Int. J. of Mathematics and Computer Research*, 3(1), (2015), pp. 829-836

[7] K. Srinivasa Rao, R. Murali and S.K.Rajendra, Rainbow connection number in brick product graphs, *Bulletin of International Mathematical Virtual Institute*, 8(1)(2018), 55-66.

Mannheim Partner Curve a Different View

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Abstract: In this study, we investigated special Smarandache curves belonging to Sabban frame drawn on the surface of the sphere by Darboux vector of Mannheim partner curve . We created Sabban frame belonging to this curve. It were explained Smarandache curves position vector is consisted by Sabban vectors belonging to this curve. Then, we calculated geodesic curvatures of this Smarandache curves. Found results were expressed depending on the Mannheim curve.

Key Words: Mannheim curve pair, Darboux vector, Smarandache curves, Sabban frame, geodesic curvature.

AMS(2010): 53A04.

§1. Introduction and Preliminaries

Let $\alpha : I \to E^3$ be a unit speed curve denote by $\{T, N, B, p, q\}$ the moving Frenet apparatus. The Frenet formulae is given by ([5])

$$\begin{cases} T'(s) = p(s)N(s) \\ N'(s) = -p(s)T(s) + q(s)B(s) \\ B'(s) = -q(s)N(s). \end{cases} \qquad (1.1)$$

The Darboux vector defined by
$$W = qT + pB.$$

By the unit Darboux vector, we have
$$\sin\varphi = \frac{q}{\sqrt{p^2+q^2}}, \quad \cos\varphi = \frac{p}{\sqrt{p^2+q^2}}$$
including
$$C = \sin\varphi T + \cos\varphi B$$

where $\angle(W, B) = \varphi$, ([4]). Let α and α_1 be the C^2-class differentiable two curves and $T_1(s)$, $N_1(s)$, $B_1(s)$ be the Frenet vectors of α_1. If the binormal vector of the curve α_1 is linearly dependent on the principal normal vector of the curve α, then (α) is defined a Mannheim curve

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[1]Received July 12, 2016, Accepted May 23, 2017.

and (α_1) a Mannheim partner curve of (α). The relations between the Frenet vectors we can write [3, 6, 7]

$$\begin{cases} T_1 = \cos\sigma T - \sin\sigma B \\ N_1 = \sin\sigma T + \cos\sigma B \\ B_1 = N \end{cases} \tag{1.2}$$

and for the curvatures we get

$$\begin{cases} \overline{p} = \dfrac{p\sigma'}{\lambda q\sqrt{p^2+q^2}} \\ \overline{q} = \dfrac{p}{\lambda q}. \end{cases} \tag{1.3}$$

Let (α, α_1) be a curve pair in $\mathbb{E}^3$. For the vector C_1 is the direction of the Mannheim partner curve α_1 we have [3,8]

$$C_1 = \frac{\sqrt{p^2+q^2}}{\sqrt{p^2+q^2+\sigma'^2}}C + \frac{\sigma'}{\sqrt{p^2+q^2+\sigma'^2}}N. \tag{1.4}$$

Let $\omega : I \to S^2$ be a unit speed spherical curve. We can write ([10])

$$\omega(s) = \omega(s), \ \ t(s) = \omega'(s), \ \ d(s) = \omega(s) \wedge t(s), \tag{1.5}$$

where, $\{\omega(s), t(s), d(s)\}$ frame is expressed the Sabban frame of ω on S^2. Then we have equations ([10])

$$\omega'(s) = t(s), \ \ t'(s) = -\omega(s) + \kappa_g(s)d(s), \ \ d'(s) = -\kappa_g(s)t(s), \tag{1.6}$$

where κ_g is expressed the geodesic curvature of the curve ω on S^2 which is ([10])

$$\kappa_g(s) = \langle t'(s), d(s)\rangle. \tag{1.7}$$

§2. Mannheim Partner Curve a Different View

Let (C_1) be a unit speed spherical curve on S^2. Then we can write

$$\begin{cases} C_1 = \sin\overline{\varphi}T_1 + \cos\overline{\varphi}B_1, \\ T_{C_1} = \cos\overline{\varphi}T_1 - \sin\overline{\varphi}B_1, \\ C_1 \wedge T_{C_1} = N_1, \end{cases} \tag{2.1}$$

where $\angle(C_1, B_1) = \overline{\varphi}$. Then from the equation (1.6) we have the following equations of (C_1) are

$$\begin{cases} C_1' = T_{C_1} \\ T_{C_1}' = -C_1 + \dfrac{\sqrt{\overline{p}^2+\overline{q}^2}}{\overline{\varphi}'}C_1 \wedge T_{C_1} \\ (C_1 \wedge T_{C_1})' = -\dfrac{\sqrt{\overline{p}^2+\overline{q}^2}}{\overline{\varphi}'}T_{C_1}. \end{cases} \tag{2.2}$$

From the equation (1.7), we have the following geodesic curvature of (C_1) is

$$\kappa_g = \langle T'_{C_1}, C_1 \wedge T_{C_1} \rangle \Longrightarrow \kappa_g = \frac{\sqrt{\overline{p}^2 + \overline{q}^2}}{\overline{\varphi}'}. \tag{2.3}$$

2.1 The β_1-*Smarandache curve* can be defined by

$$\beta_1(s) = \frac{1}{\sqrt{2}}(C_1 + T_{C_1}) \tag{2.4}$$

or substituting the equation (2.1) into equation (2.4) we obtain

$$\beta_1(s) = \frac{1}{\sqrt{2}}\Big((\sin\overline{\varphi} + \cos\overline{\varphi})T_1 + (\cos\overline{\varphi} - \sin\overline{\varphi})B_1\Big). \tag{2.5}$$

Differentiating (2.4) we can write

$$T_{\beta_1} = \frac{\overline{\varphi}'(\cos\overline{\varphi} - \sin\overline{\varphi})}{\sqrt{2\overline{\varphi}'^2 + \overline{p}^2 + \overline{q}^2}}T_1 + \frac{\sqrt{\overline{p}^2 + \overline{q}^2}}{\sqrt{2\overline{\varphi}'^2 + \overline{p}^2 + \overline{q}^2}}N_1 - \frac{\overline{\varphi}'(\cos\overline{\varphi} + \sin\overline{\varphi})}{\sqrt{2\overline{\varphi}'^2 + \overline{p}^2 + \overline{q}^2}}B_1. \tag{2.6}$$

Considering the equations (2.5) and (2.6) we get

$$\beta_1 \wedge T_{\beta_1} = \frac{\sqrt{\overline{p}^2 + \overline{q}^2}(\cos\overline{\varphi} + \sin\overline{\varphi})}{\sqrt{2\overline{p}^2 + 2\overline{q}^2 + 4\overline{\varphi}'^2}}T_1 - \frac{\overline{\varphi}'}{\sqrt{2\overline{p}^2 + 2\overline{q}^2 + 4\overline{\varphi}'^2}}N_1 + \frac{\sqrt{\overline{p}^2 + \overline{q}^2}(\cos\overline{\varphi} + \sin\overline{\varphi})}{\sqrt{2\overline{p}^2 + 2\overline{q}^2 + 4\overline{\varphi}'^2}}B_1. \tag{2.7}$$

Differentiating (2.6) where coefficients

$$
\begin{aligned}
\chi_1 &= -2 - \Big(\frac{\sqrt{\overline{p}^2 + \overline{q}^2}}{\overline{\varphi}'}\Big)^2 + \Big(\frac{\sqrt{\overline{p}^2 + \overline{q}^2}}{\overline{\varphi}'}\Big)'\Big(\frac{\sqrt{\overline{p}^2 + \overline{q}^2}}{\overline{\varphi}'}\Big) \\
\chi_2 &= -2 - 3\Big(\frac{\sqrt{\overline{p}^2 + \overline{q}^2}}{\overline{\varphi}'}\Big)^2 - \Big(\frac{\sqrt{\overline{p}^2 + \overline{q}^2}}{\overline{\varphi}'}\Big)^4 - \Big(\frac{\sqrt{\overline{p}^2 + \overline{q}^2}}{\overline{\varphi}'}\Big)'\Big(\frac{\sqrt{\overline{p}^2 + \overline{q}^2}}{\overline{\varphi}'}\Big) \\
\chi_3 &= 2\Big(\frac{\sqrt{\overline{p}^2 + \overline{q}^2}}{\overline{\varphi}'}\Big) + \Big(\frac{\sqrt{\overline{p}^2 + \overline{q}^2}}{\overline{\varphi}'}\Big)^3 + \Big(\frac{\sqrt{\overline{p}^2 + \overline{q}^2}}{\overline{\varphi}'}\Big)'
\end{aligned} \tag{2.8}
$$

including we can reach,

$$T'_{\beta_1} = \frac{\overline{\varphi}'^4\sqrt{2}(\chi_1\sin\overline{\varphi} + \chi_2\cos\overline{\varphi})}{\Big(\overline{p}^2 + \overline{q}^2 + \overline{\varphi}'^2\Big)^2}T_1 + \frac{\chi_3\overline{\varphi}'^4\sqrt{2}}{\Big(\overline{p}^2 + \overline{q}^2 + \overline{\varphi}'^2\Big)^2}N_1 + \frac{\overline{\varphi}'^4\sqrt{2}(\chi_1\cos\overline{\varphi} - \chi_2\sin\overline{\varphi})}{\Big(\overline{p}^2 + \overline{q}^2 + \overline{\varphi}'^2\Big)^2}B_1. \tag{2.9}$$

From the equation (2.7) and (2.9) $\kappa_g^{\beta_1}$ geodesic curvature for Mannheim partner curve β_1 is

$$
\begin{aligned}
\kappa_g^{\beta_1} &= \langle T'_{\beta_1}, (C_1 \wedge T_{C_1})_{\beta_1} \rangle \\
&= \frac{1}{\Big(2 + \big(\frac{\sqrt{\overline{p}^2 + \overline{q}^2}}{\overline{\varphi}'}\big)^2\Big)^{\frac{5}{2}}}\Big(\frac{\sqrt{\overline{p}^2 + \overline{q}^2}}{\overline{\varphi}'}\chi_1 - \frac{\sqrt{\overline{p}^2 + \overline{q}^2}}{\overline{\varphi}'}\chi_2 + 2\chi_3\Big).
\end{aligned} \tag{2.10}
$$

From the equation (1.2) and (1.3) Sabban apparatus of the β_1-Smarandache curve for

Mannheim curve are

$$\beta_1(s) = \frac{(\sigma' + \sqrt{p^2 + q^2})\cos\sigma}{\sqrt{2\sigma'^2 + 2p^2 + q^2}}T - \frac{\sigma' - \sqrt{p^2 + q^2}}{\sqrt{2\sigma'^2 + 2p^2 + q^2}}N - \frac{(\sigma' + \sqrt{p^2 + q^2})\sin\sigma}{\sqrt{2\sigma'^2 + 2p^2 + q^2}}B,$$

$$T_{\beta_1} = \frac{(\sigma' - \sqrt{p^2 + q^2})\eta\cos\sigma - \sqrt{p^2 + q^2 + \sigma'^2}\cos\sigma}{\sqrt{p^2 + q^2 + \sigma'^2}\sqrt{1 + 2\eta^2}}T + \frac{\eta(\sigma' + \sqrt{p^2 + q^2})}{\sqrt{p^2 + q^2 + \sigma'^2}\sqrt{1 + 2\eta^2}}N$$
$$+ \frac{(\sqrt{p^2 + q^2} - \sigma')\eta\sin\sigma + \sqrt{p^2 + q^2 + \sigma'^2}\cos\sigma}{\sqrt{p^2 + q^2 + \sigma'^2}\sqrt{1 + 2\eta^2}}B,$$

$$\beta_1 \wedge T_{\beta_1} = \frac{(\sqrt{p^2 + q^2} - \sigma')\cos\sigma + 2\eta\sqrt{p^2 + q^2 + \sigma'^2}\sin\sigma}{\sqrt{2 + 4\eta^2}\sqrt{p^2 + q^2 + \sigma'^2}}T + \frac{\sigma' + \eta\sqrt{p^2 + q^2}}{\sqrt{2 + 4\eta^2}\sqrt{p^2 + q^2 + \sigma'^2}}N$$
$$- \frac{(\sqrt{p^2 + q^2} + \sigma')\sin\sigma - 2\eta\sqrt{p^2 + q^2 + \sigma'^2}\cos\sigma}{\sqrt{2 + 4\eta^2}\sqrt{p^2 + q^2 + \sigma'^2}}B,$$

$$T'_{\beta_1} = \frac{(\overline{\chi}_1\sqrt{p^2 + q^2} + \overline{\chi}_2\sigma')\eta^4\sqrt{2}\cos\sigma - \overline{\chi}_3\eta^4\sqrt{2p^2 + q^2 + 2\sigma'^2}\sin\sigma}{(1 + 2\eta^2)^2\sqrt{p^2 + q^2 + \sigma'^2}}T$$
$$+ \frac{(\overline{\chi}_1\sigma' - \overline{\chi}_2\sqrt{p^2 + q^2})\eta^4\sqrt{2}}{(1 + 2\eta^2)^2\sqrt{p^2 + q^2 + \sigma'^2}}N$$
$$+ \frac{(\overline{\chi}_2\sigma' - \overline{\chi}_1\sqrt{p^2 + q^2})\eta^4\sqrt{2}\sin\sigma + \overline{\chi}_3\eta^4\sqrt{2p^2 + q^2 + 2\sigma'^2}\cos\sigma}{(1 + 2\eta^2)^2\sqrt{p^2 + q^2 + \sigma'^2}}B$$

and

$$\kappa_g^{\beta_1} = \frac{1}{\left(2 + \frac{1}{\eta^2}\right)^{\frac{5}{2}}}\left(\frac{1}{\eta}\overline{\chi}_1 - \frac{1}{\eta}\overline{\chi}_2 + 2\overline{\chi}_3\right), \tag{2.11}$$

where

$$\frac{1}{\eta} = \frac{(\overline{\varphi})'}{\sqrt{p^2 + q^2}} = \left(\frac{\sqrt{p^2 + q^2}}{\sqrt{\sigma'^2 + p^2 + q^2}}\right)'\frac{\lambda q\sqrt{p^2 + q^2}}{\sigma'} \tag{2.12}$$

and

$$\begin{cases} \overline{\chi}_1 = -2 - \frac{1}{\eta^2} + \frac{1}{\eta'}\frac{1}{\eta} \\ \overline{\chi}_2 = -2 - 3\frac{1}{\eta^2} - \frac{1}{\eta^4} - \frac{1}{\eta'}\frac{1}{\eta} \\ \overline{\chi}_3 = 2\frac{1}{\eta} + \frac{1}{\eta^3} + 2\frac{1}{\eta'} \end{cases} \tag{2.13}$$

2.2 The β_2-*Smarandache curve* can be defined by

$$\beta_2(s) = \frac{1}{\sqrt{2}}(C_1 + C_1 \wedge T_{C_1}) \tag{2.14}$$

or from the equation (1.2), (1.4) and (2.1) we can write

$$
\begin{aligned}
\beta_2(s) \quad = \quad & \frac{\sqrt{p^2+q^2}\cos\sigma + \sqrt{\sigma'^2+p^2+q^2}\sin\sigma}{\sqrt{2\sigma'^2+2p^2+q^2}}T + \frac{\sigma'}{\sqrt{2\sigma'^2+2p^2+q^2}}N \\
& + \frac{\sqrt{\sigma'^2+p^2+q^2}\cos\sigma - \sqrt{p^2+q^2}\sin\sigma}{\sqrt{2\sigma'^2+2p^2+q^2}}B.
\end{aligned}
\tag{2.15}
$$

Differentiating (2.15) we can write

$$
T_{\beta_2} = \frac{\sigma'\cos\sigma}{\sqrt{p^2+q^2+\sigma'^2}}T - \frac{\sqrt{p^2+q^2}}{\sqrt{p^2+q^2+\sigma'^2}}N - \frac{\sigma'\sin\sigma}{\sqrt{p^2+q^2+\sigma'^2}}B.
\tag{2.16}
$$

Considering the equations (2.15) and (2.16) it is easily seen

$$
\begin{aligned}
\beta_2 \wedge T_{\beta_2} \quad = \quad & \frac{\sqrt{p^2+q^2+\sigma'^2}\sin\sigma - \sqrt{p^2+q^2}\cos\sigma}{\sqrt{2p^2+q^2+2\sigma'^2}}T - \frac{\sigma'}{\sqrt{2p^2+q^2+2\sigma'^2}}N \\
& + \frac{\sqrt{p^2+q^2}\sin\sigma + \sqrt{p^2+q^2+\sigma'^2}\cos\sigma}{\sqrt{2p^2+q^2+2\sigma'^2}}B.
\end{aligned}
\tag{2.17}
$$

Differentiating (2.16) we can write

$$
\begin{aligned}
T'_{\beta_2} \quad = \quad & \frac{\sqrt{2p^2+q^2+2\sigma'^2}\sin\sigma - \eta\sqrt{2}\sqrt{p^2+q^2}\cos\sigma}{(\eta-1)\sqrt{p^2+q^2+\sigma'^2}}T + \frac{\eta\sqrt{2}\sigma'}{(\eta-1)\sqrt{p^2+q^2+\sigma'^2}}N \\
& + \frac{\eta\sqrt{2}\sqrt{p^2+q^2}\sin\sigma + \sqrt{2p^2+q^2+2\sigma'^2}\cos\sigma}{(\eta-1)\sqrt{p^2+q^2+\sigma'^2}}B,
\end{aligned}
\tag{2.18}
$$

where, $\kappa_g^{\beta_2}$ geodesic curvature for β_2 is

$$
\kappa_g^{\beta_2} = \frac{1+\eta}{\eta-1}.
\tag{2.19}
$$

2.3 The β_3-*Smarandache curves* can be defined by

$$
\beta_3(s) = \frac{1}{\sqrt{2}}(T_{C_1} + C_1 \wedge T_{C_1})
\tag{2.20}
$$

or from the equation (2.1), (1.4) and (1.2) we can write

$$
\begin{aligned}
\beta_3(s) \quad = \quad & \frac{\sigma'\cos\sigma + \sqrt{p^2+q^2+\sigma'^2}\sin\sigma}{\sqrt{2\sigma'^2+2p^2+q^2}}T + \frac{\sqrt{p^2+q^2}}{\sqrt{2\sigma'^2+2p^2+q^2}}N \\
& + \frac{\sqrt{p^2+q^2+\sigma'^2}\cos\sigma - \sigma'\sin\sigma}{\sqrt{2\sigma'^2+2p^2+q^2}}B.
\end{aligned}
\tag{2.21}
$$

Differentiating (2.21) we reach

$$
T_{\beta_3} = \frac{\sqrt{p^2 + q^2 + \sigma'^2}\sin\sigma - (\sigma' + \eta\sqrt{p^2+q^2})\cos\sigma}{\sqrt{2+\eta^2}\sqrt{p^2+q^2+\sigma'^2}}T + \frac{\sqrt{p^2+q^2}-\eta\sigma'}{\sqrt{2+\eta^2}\sqrt{p^2+q^2+\sigma'^2}}N
$$

$$
+ \frac{(\eta\sqrt{p^2+q^2}+\sigma')\sin\sigma + \sqrt{p^2+q^2+\sigma'^2}\cos\sigma}{\sqrt{2+\eta^2}\sqrt{p^2+q^2+\sigma'^2}}B. \tag{2.22}
$$

Considering the equations (2.21) and (2.22) it is easily seen

$$
\beta_3 \wedge T_{\beta_3} = \frac{(2\sqrt{p^2+q^2}-\eta\sigma')\cos\sigma + \eta\sqrt{p^2+q^2+\sigma'^2}\sin\sigma}{\sqrt{4+2\eta^2}\sqrt{p^2+q^2+\sigma'^2}}T
$$

$$
+ \frac{2\sigma' - \eta\sqrt{p^2+q^2}}{\sqrt{4+2\eta^2}\sqrt{p^2+q^2+\sigma'^2}}N
$$

$$
+ \frac{(\eta\sigma' - 2\sqrt{p^2+q^2})\sin\sigma + \eta\sqrt{p^2+q^2+\sigma'^2}\cos\sigma}{\sqrt{4+2\eta^2}\sqrt{p^2+q^2+\sigma'^2}}B. \tag{2.23}
$$

Differentiating (2.22) where

$$
\begin{cases}
\overline{\delta}_1 = \frac{1}{\eta} + 2\frac{1}{\eta^3} + 2\frac{1}{\eta'}\frac{1}{\eta} \\
\overline{\delta}_2 = -1 - 3\frac{1}{\eta^2} - 2\frac{1}{\eta^4} - \frac{1}{\eta'} \\
\overline{\delta}_3 = -\frac{1}{\eta^2} - 2\frac{1}{\eta^4} + \frac{1}{\eta'}
\end{cases} \tag{2.24}
$$

including we have

$$
T'_{\beta_3} = \frac{(\overline{\delta}_2\sqrt{p^2+q^2}-\overline{\delta}_1\sigma')\eta^4\sqrt{2}\cos\sigma + \overline{\delta}_3\eta^4\sqrt{2p^2+q^2+2\sigma'^2}\sin\sigma}{(2+\eta^2)^2\sqrt{\sigma'^2+p^2+q^2}}T
$$

$$
+ \frac{\eta^4\sqrt{2}(\overline{\delta}_2\sigma' + \overline{\delta}_1\sqrt{p^2+q^2})}{(2+\eta^2)^2\sqrt{\sigma'^2+p^2+q^2}}N
$$

$$
+ \frac{(\overline{\delta}_1\sigma' - \overline{\delta}_2\sqrt{p^2+q^2})\eta^4\sqrt{2}\sin\sigma + \overline{\delta}_3\eta^4\sqrt{2p^2+q^2+2\sigma'^2}\cos\sigma}{(2+\eta^2)^2\sqrt{\sigma'^2+p^2+q^2}}B, \tag{2.25}
$$

where, $\kappa_g^{\beta_3}$ geodesic curvature for Mannheim curve β_3 is

$$
\kappa_g^{\beta_3} = \frac{1}{(2+\eta^2)^{\frac{5}{2}}}\left(2\eta^5\overline{\delta}_1 - \eta^4\overline{\delta}_2 + \eta^4\overline{\delta}_3\right). \tag{2.26}
$$

2.4 The β_4-*Smarandache curves* can be defined by

$$
\beta_4(s) = \frac{1}{\sqrt{3}}(C_1 + T_{C_1} + C_1 \wedge T_{C_1}) \tag{2.27}
$$

or from the equation (1.2), (1.4) and (2.1) we can write

$$
\begin{aligned}
\beta_4(s) &= \frac{\left(\sigma' + \sqrt{p^2 + q^2}\right)\cos\sigma + \sqrt{\sigma'^2 + p^2 + q^2}\sin\sigma}{\sqrt{3\sigma'^2 + 3p^2 + q^2}}T + \frac{\sigma' - \sqrt{p^2 + q^2}}{\sqrt{3\sigma'^2 + 3p^2 + q^2}}N \\
&\quad + \frac{\sqrt{\sigma'^2 + p^2 + q^2}\cos\sigma - \left(\sigma' + \sqrt{p^2 + q^2}\right)\sin\sigma}{\sqrt{3\sigma'^2 + 3p^2 + q^2}}B.
\end{aligned}
\tag{2.28}
$$

Differentiating (2.28), we reach

$$
\begin{aligned}
T_{\beta_4} &= \frac{\left((\eta - 1)\sigma' - \eta\sqrt{p^2 + q^2}\right)\cos\sigma + \sqrt{p^2 + q^2 + \sigma'^2}\sin\sigma}{\sqrt{2(1 - \eta + \eta^2)}\sqrt{p^2 + q^2 + \sigma'^2}}T \\
&\quad + \frac{\eta\sigma' - (1 - \eta)\sqrt{p^2 + q^2}}{\sqrt{2(1 - \eta + \eta^2)}\sqrt{p^2 + q^2 + \sigma'^2}}N \\
&\quad + \frac{\left(\eta\sqrt{p^2 + q^2} + (1 - \eta)\sigma'\right)\sin\sigma + \sqrt{p^2 + q^2 + \sigma'^2}\cos\sigma}{\sqrt{2(1 - \eta + \eta^2)}\sqrt{p^2 + q^2 + \sigma'^2}}B.
\end{aligned}
\tag{2.29}
$$

Considering the equations (2.28) and (2.29) it is easily seen

$$
\begin{aligned}
\beta_4 \wedge T_{\beta_4} &= \frac{\left((2 - \eta)\sqrt{p^2 + q^2} - (1 + \eta)\sigma'\right)\cos\sigma + (2\eta 1)\sqrt{p^2 + q^2 + \sigma'^2}\sin\sigma}{\sqrt{6 - 6\eta + 6\eta^2}\sqrt{p^2 + q^2 + \sigma'^2}}T \\
&\quad + \frac{(2 - \eta)\sigma' + (1 + \eta)\sqrt{p^2 + q^2}}{\sqrt{6 - 6\eta + 6\eta^2}\sqrt{p^2 + q^2 + \sigma'^2}}N \\
&\quad + \frac{\left((\eta - 2)\sqrt{p^2 + q^2} - (1 + \eta)\sigma'\right)\sin\sigma + (2\eta - 1)\sqrt{p^2 + q^2 + \sigma'^2}\cos\sigma}{\sqrt{6 - 6\eta + 6\eta^2}\sqrt{p^2 + q^2 + \sigma'^2}}B.
\end{aligned}
\tag{2.30}
$$

Differentiating (2.29) where

$$
\begin{cases}
\overline{\rho}_1 = -2 + 4\frac{1}{\eta} - 4\frac{1}{\eta^2} + 2\frac{1}{\eta^3} + 2\frac{1}{\eta'}\left(2\frac{1}{\eta} - 1\right) \\
\overline{\rho}_2 = -2 + 2\frac{1}{\eta} - 4\frac{1}{\eta^2} + 2\frac{1}{\eta^3} - 2\frac{1}{\eta^4} - \frac{1}{\eta'}\left(1 + \frac{1}{\eta}\right) \\
\overline{\rho}_3 = 2\frac{1}{\eta} - 4\frac{1}{\eta^2} + 4\frac{1}{\eta^3} - 2\frac{1}{\eta^4} + \frac{1}{\eta'}\left(2 - \frac{1}{\eta}\right)
\end{cases}
\tag{2.31}
$$

including we can write

$$
\begin{aligned}
T'_{\beta_4} &= \frac{\left(\overline{\rho}_1\sqrt{p^2 + q^2} + \overline{\rho}_2\sigma'\right)\eta^4\sqrt{3}\cos\sigma + \overline{\rho}_3\eta^4\sqrt{3p^2 + q^2 + 3\sigma'^2}\sin\sigma}{4(1 - \eta + \eta^2)^2\sqrt{\sigma'^2 + p^2 + q^2}}T \\
&\quad + \frac{\left(\overline{\rho}_1\sigma' - \overline{\rho}_2\sqrt{p^2 + q^2}\right)\eta^4\sqrt{3}}{4(1 - \eta + \eta^2)^2\sqrt{\sigma'^2 + p^2 + q^2}}N \\
&\quad + \frac{\overline{\rho}_3\eta^4\sqrt{3p^2 + q^2 + 3\sigma'^2}\cos\sigma - \left(\overline{\rho}_1\sqrt{p^2 + q^2} + \overline{\rho}_2\sigma'\right)\eta^4\sqrt{3}\sin\sigma}{4(1 - \eta + \eta^2)^2\sqrt{\sigma'^2 + p^2 + q^2}}B,
\end{aligned}
\tag{2.32}
$$

where, $\kappa_g^{\beta_4}$ geodesic curvature for Mannheim curve $\beta_4(s_{\beta_4})$ is

$$\kappa_g^{\beta_4} = \frac{(2\eta^4 - \eta^5)\overline{\rho}_1 - (\eta^5 + \eta^4)\overline{\rho}_2 + (2\eta^5 - \eta^4)\overline{\rho}_3}{4\sqrt{2}(1 - \eta + \eta^2)^{\frac{5}{2}}}. \tag{2.33}$$

References

[1] Ali A.T., Special Smarandache curves in the Euclidian space, *International Journal of Mathematical Combinatorics*, 2, 30-36, 2010.

[2] Çalışkan A. and Şenyurt S., Smarandache curves in terms of Sabban frame of spherical indicatrix curves, *Gen. Math. Notes*, 31(2), 1-15, 2015.

[3] Çalışkan A. and Şenyurt S., N^*C^*-Smarandache curves of Mannheim curve couple according to Frenet frame, *International Journal of Mathematical Combinatorics*, 1, 1-13, 2015.

[4] Fenchel, W.,*On the Differential Geometry of closed space curves*, Bulletin of the American Mathematical Society, 57, 44-54, 1951.

[5] Hacısalihoğlu H.H., *Differential Geometry(in Turkish)*, Academic Press Inc. Ankara, 1994.

[6] Liu H. and Wang, F., Mannheim partner curves in 3-space, *Journal of Geometry*, 88(1-2), 120-126, 2008.

[7] Orbay K. and Kasap E., On Mannheim partner curves E^3, *International Journal of Physical Sciences*, 4(5), 261-264, 2009.

[8] Şenyurt S., Natural lifts and the geodesic sprays for the spherical indicatrices of the Mannheim partner curves in E^3, *International Journal of the Physical Sciences*, 7(16), 2414-2421, 2012.

[9] Turgut M. and Yılmaz S., Smarandache curves in Minkowski space-time, *International Journal of Mathematical Combinatorics*, 3, 51-55, 2008.

[10] Taşköprü K. and Tosun M., Smarandache curves on S^2, *Boletim da Sociedade Paranaense de Matematica 3 Srie.*, 32(1), 51-59, 2014.

F-Root Square Mean Labeling of Graphs Obtained From Paths

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Abstract: A function f is called a F-root square mean labeling of a graph $G(V, E)$ with p vertices and q edges if $f : V(G) \rightarrow \{1, 2, 3, \ldots, q+1\}$ is injective and the induced function f^* defined as

$$f^*(uv) = \left\lfloor \sqrt{\frac{f(u)^2 + f(v)^2}{2}} \right\rfloor$$

for all $uv \in E(G)$, is bijective. A graph that admits a F-root square mean labeling is called a F-root square mean graph. In this paper, we study the F-root square meanness of the path P_n, the graph $P_n \circ S_m$, the graph $P_n \circ K_2$, the graph $TW(P_n)$, the graph $[P_n; S_m]$, the graph $S(P_n \circ K_1)$, the graph $M(P_n)$, the graph $T(P_n)$, the graph P_n^2, the ladder graph L_n and the slanting ladder graph SL_n.

Key Words: Labeling, Smarandache power root mean labeling, F-root square mean labeling, F-root square mean graph.

AMS(2010): 05C12.

§1. Introduction

Throughout this paper, by a graph we mean a finite, undirected and simple graph. Let $G(V, E)$ be a graph with p vertices and q edges. For notations and terminology, we follow [3]. For a detailed survey on graph labeling, we refer [2].

Path on n vertices is denoted by P_n. A star graph S_n is the complete bipartite graph $K_{1,n}$. The graph $G \circ S_m$ is obtained from G by attaching m pendant vertices to each vertex of G. A Twig $TW(P_n), n \geq 4$ is a graph obtained from a path by attaching exactly two pendant vertices to each internal vertices of the path P_n. If $v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, \cdots, v_{m+1}^{(i)}$ and $u_1, u_2, u_3, \cdots, u_n$ be the vertices of ith copy of the star graph S_m and the path P_n respectively, then the graph $[P_n; S_m]$ is obtained from n copies of S_m and the path P_n by joining u_i with the central vertex $v_1^{(i)}$ of the ith copy of S_m by means of an edge, for $1 \leq i \leq n$.

A subdivision of a graph G, denoted by $S(G)$, is a graph obtained by subdividing edge of

---

G by a vertex.

The middle graph $M(G)$ of a graph G is the graph whose vertex set is $\{v : v \in V(G)\} \cup \{e : e \in E(G)\}$ and the edge set is $\{e_1 e_2 : e_1, e_2 \in E(G)$ and e_1 and e_2 are adjacent edges of $G\} \cup \{ve : v \in V(G), e \in E(G)$ and e is incident with $v\}$. The total graph $T(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent if and only if either they are adjacent vertices of G or adjacent edges of G or one is a vertex of G and the other one is an edge incident on it. Square of a graph G, denoted by G^2, has the vertex set as in G and two vertices are adjacent in G^2 if they are at a distance either 1 or 2 apart in G. Let G_1 and G_2 be any two graphs with p_1 and p_2 vertices respectively. Then the *Cartesian product* $G_1 \times G_2$ has $p_1 p_2$ vertices which are $\{(u, v) : u \in G_1, v \in G_2\}$ and the edges are obtained as follows: (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \times G_2$ if either $u_1 = u_2$ and v_1 and v_2 are adjacent in G_2 or u_1 and u_2 are adjacent in G_1 and $v_1 = v_2$. A ladder graph L_n is the graph $P_2 \times P_n$.

The slanting ladder SL_n is a graph obtained from two paths $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ by joining each v_i, with $u_{i+1}, 1 \leq i \leq n - 1$.

The concept of root square mean labeling was introduced and studied by S.S. Sandhya et al. in [4,5]. Motivated by the works of so many authors in the area of graph labeling, we introduce a new type of labeling called F-root square mean labeling.

A labeling f on a graph $G(V, E)$ with p vertices and q edges is called a *Smarandache power root mean labeling* for an integer $m \geq 1$ if $f : V(G) \rightarrow \{1, 2, 3, \cdots, q + 1\}$ is injective and the induced function f^* defined by

$$f^*(uv) = \left\lfloor \sqrt[m]{\frac{f(u)^m + f(v)^m}{2}} \right\rfloor$$

is bijective for all $uv \in E(G)$. Particularly, if $m = 1$, such a Smarandache power root mean labeling is nothing else but the mean labeling and $m = 2$ is called the *F-root square mean labeling* on graph $G(V, E)$ with an injective $f : V(G) \rightarrow \{1, 2, 3, \cdots, q + 1\}$ and an induced bijective

$$f^*(uv) = \left\lfloor \sqrt{\frac{f(u)^2 + f(v)^2}{2}} \right\rfloor$$

for all $uv \in E(G)$. A graph that admits a F-root square mean labeling is called a F-root square mean graph.

In [4], S.S. Sandhya et al. defined the root square mean labeling as follows:

A graph $G(V, E)$ with p vertices and q edges is said to be Root Square Mean graph if it is possible to label the vertices $x \in V$ with distinct labels $f(x)$ from $1, 2, 3, \cdots, q + 1$ in such a way that when each edge $e = uv$ is labeled with

$$\left\lceil \sqrt{\frac{f(u)^2 + f(v)^2}{2}} \right\rceil \quad \text{or} \quad \left\lfloor \sqrt{\frac{f(u)^2 + f(v)^2}{2}} \right\rfloor,$$

then the edge labels are distinct. In this case f is called a root square mean labeling of G.

In the above definition, the readers will get some confusion in finding the edge labels which

edge is assigned by flooring function and which edge is assigned by ceiling function. To avoid the confusion of assigning the edge labels in their definition, we just consider the flooring function

$$\left\lfloor \sqrt{\frac{f(u)^2 + f(v)^2}{2}} \right\rfloor$$

for our discussion. Based on our definition, a F-root square mean labeling of the graph $K_5 - e$ is given in Figure 1.
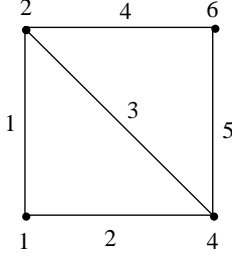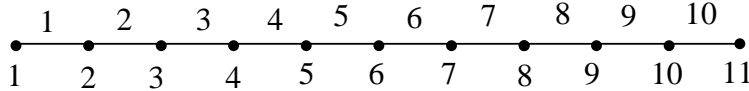

Figure 1. A F-root square mean labeling of $K_4 - e$.

In this paper, we study the F-root square meanness of the path P_n, the graph $P_n \circ S_m$, the graph $P_n \circ K_2$, the twig graph $TW(P_n)$, the graph $[P_n; S_m]$, the graph $S(P_n \circ K_1)$, the middle graph $M(P_n)$, the total graph $T(P_n)$, the square graph P_n^2, the ladder graph L_n and the slanting ladder graph SL_n.

§2. **Main Results**

Theorem 2.1 *Every path P_n is a F-root square mean graph.*

Proof Let $v_1, v_2, v_3, \cdots, v_n$ be the vertices of the path P_n. Define $f : V(G) \to \{1, 2, 3, \cdots, n\}$ as $f(v_i) = i$, for $1 \le i \le n$. Then the induced edge labeling is $f^*(v_i v_{i+1}) = i$, for $1 \le i \le n - 1$. Hence f is a F-Root Square Mean labeling of path P_n. Thus the path P_n is a F-root square mean graph. □

Figure 2. A F-root square mean labeling of P_{11}.

Theorem 2.2 *The graph $P_n \circ S_m$ is a F-root square mean graph for $n \ge 1$ and $m \le 4$.*

Proof Let $v_1, v_2, v_3, \cdots, v_n$ be the vertices of the path P_n and $u_1^{(i)}, u_2^{(i)}, u_3^{(i)}, \cdots, u_m^{(i)}$ be the pendant vertices at each v_i, for $1 \le i \le n$.

Case 1. $m = 1$.

Define $f : V(P_n \circ S_1) \to \{1, 2, 3, \cdots, 2n\}$ as follows

$$f(v_i) = 2i - 1, \text{ for } 1 \le i \le n,$$
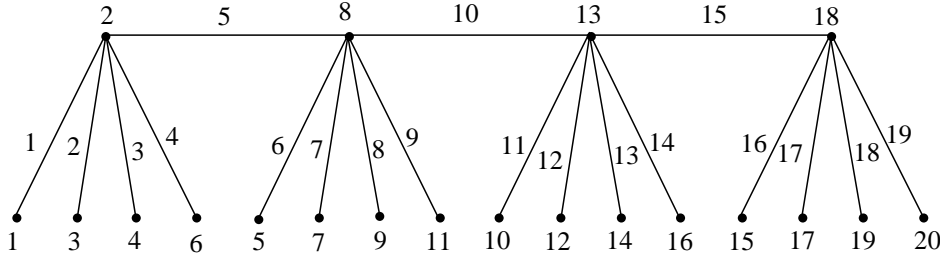
$$f\left(u_1^{(i)}\right) = 2i, \text{ for } 1 \le i \le n.$$

Then the induced edge labeling is obtained as follows

$$f^*(v_i v_{i+1}) = 2i, \text{ for } 1 \le i \le n - 1,$$

$$f^*\left(v_i u_1^{(i)}\right) = 2i - 1, \text{ for } 1 \le i \le n.$$

Case 2.　$m = 2$.

Define $f : V(P_n \circ S_2) \to \{1, 2, 3, \cdots, 3n\}$ as follows

$$f(v_i) = 3i - 1, \text{ for } 1 \le i \le n,$$

$$f\left(u_1^{(i)}\right) = 3i - 2, \text{ for } 1 \le i \le n,$$

$$f\left(u_2^{(i)}\right) = 3i, \text{ for } 1 \le i \le n.$$

Then the induced edge labeling is obtained as follows

$$f^*(v_i v_{i+1}) = 3i, \text{ for } 1 \le i \le n - 1,$$

$$f^*\left(v_i u_1^{(i)}\right) = 3i - 2, \text{ for } 1 \le i \le n,$$

$$f^*\left(v_i u_2^{(i)}\right) = 3i - 1, \text{ for } 1 \le i \le n.$$

Case 3.　$m = 3$.

Define $f : V(P_n \circ S_3) \to \{1, 2, 3, \cdots, 4n\}$ as follows

$$f(v_i) = 4i - 2, \text{ for } 1 \le i \le n,$$

$$f\left(u_1^{(i)}\right) = 4i - 3, \text{ for } 1 \le i \le n,$$

$$f\left(u_2^{(i)}\right) = 4i - 1, \text{ for } 1 \le i \le n,$$

$$f\left(u_3^{(i)}\right) = 4i, \text{ for } 1 \le i \le n.$$

Then the induced edge labeling is obtained as follows

$$f^*(v_i v_{i+1}) = 4i, \text{ for } 1 \le i \le n - 1,$$

$$f^*\left(v_i u_1^{(i)}\right) = 4i - 3, \text{ for } 1 \le i \le n,$$

$$f^*\left(v_i u_2^{(i)}\right) = 4i - 2, \text{ for } 1 \le i \le n,$$

$$f^*\left(v_i u_3^{(i)}\right) = 4i - 1, \text{ for } 1 \le i \le n.$$

Case 4. $m = 4$.

Define $f : V(P_n \circ S_4) \to \{1, 2, 3, \cdots, 5n\}$ as follows

$$f(v_1) = 2, \quad f(v_i) = 5i - 2, \text{ for } 1 \leq i \leq n, \quad f\left(u_1^{(1)}\right) = 1,$$

$$f\left(u_1^{(i)}\right) = 5i - 5, \text{ for } 2 \leq i \leq n, \quad f\left(u_2^{(1)}\right) = 3,$$

$$f\left(u_2^{(i)}\right) = 5i - 3, \text{ for } 2 \leq i \leq n,$$

$$f\left(u_3^{(i)}\right) = 5i - 1, \text{ for } 1 \leq i \leq n,$$

$$f\left(u_4^{(i)}\right) = 5i + 1, \text{ for } 1 \leq i \leq n - 1, \quad f\left(u_4^{(n)}\right) = 5n.$$

Then the induced edge labeling is obtained as follows

$$f^*(v_i v_{i+1}) = 5i, \text{ for } 1 \leq i \leq n - 1, \quad f^*\left(v_i u_1^{(i)}\right) = 5i - 4, \text{ for } 1 \leq i \leq n,$$

$$f^*\left(v_i u_2^{(i)}\right) = 5i - 3, \text{ for } 1 \leq i \leq n, \quad f^*\left(v_i u_3^{(i)}\right) = 5i - 2, \text{ for } 1 \leq i \leq n,$$

$$f^*\left(v_i u_4^{(i)}\right) = 5i - 1, \text{ for } 1 \leq i \leq n.$$

Hence, f is a F-root square mean labeling of the graph $P_n \circ S_m$. Thus the graph $P_n \circ S_m$ is a F-root square mean graph for $n \geq 1$ and $m \leq 4$.   □
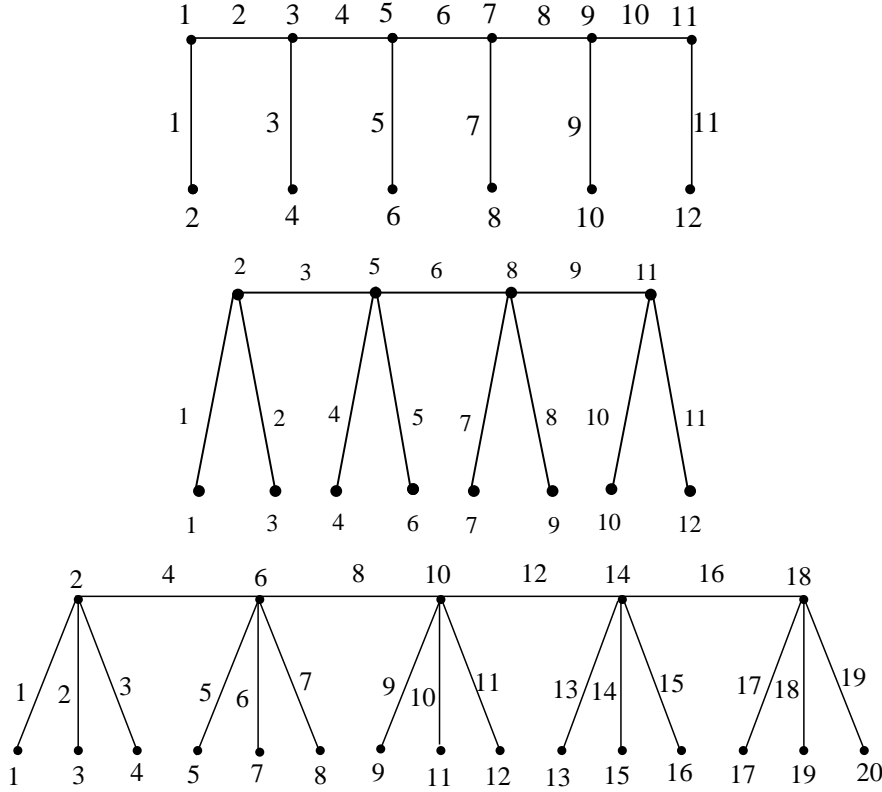
Figure 3. A F-root square mean labeling of $P_6 \circ S_1$, $P_4 \circ S_2$ and $P_5 \circ S_3$.

Figure 4. A F-root square mean labeling of $P_4 \circ S_4$.

Theorem 2.3 *The graph $P_n \circ K_2$ is a F-root square mean graph for $n \geq 1$.*

Proof Let $v_1, v_2, v_3, \cdots, v_n$ be the vertices of the path P_n and $u_i^{(1)}, u_i^{(2)}$ be the vertices of ith copy of K_2 attached with v_i, for $1 \leq i \leq n$.

Define $f : V(P_n \circ K_2) \to \{1, 2, 3, \cdots, 4n\}$ as follows

$$f(v_i) = 4i - 2, \text{ for } 1 \leq i \leq n,$$
$$f\left(u_i^{(1)}\right) = 4i - 3, \text{ for } 1 \leq i \leq n,$$
$$f\left(u_i^{(2)}\right) = 4i, \text{ for } 1 \leq i \leq n.$$

Then the induced edge labeling is obtained as follows

$$f^*(v_i v_{i+1}) = 4i, \text{ for } 1 \leq i \leq n - 1,$$
$$f^*\left(u_i^{(1)} u_i^{(2)}\right) = 4i - 2, \text{ for } 1 \leq i \leq n,$$
$$f^*\left(u_i^{(1)} v_1\right) = 4i - 3, \text{ for } 1 \leq i \leq n,$$
$$f^*\left(u_i^{(2)} v_i\right) = 4i - 1, \text{ for } 1 \leq i \leq n.$$

Hence f is a F-root square mean labeling of the graph $P_n \circ K_2$. Thus the graph $P_n \circ K_2$ is a F-root square mean graph for $n \geq 1$. $\square$
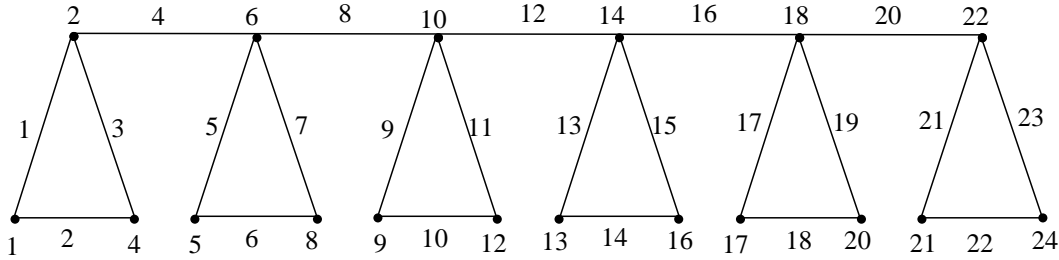
Figure 5. A F-root square mean labeling of $P_6 \circ K_2$.

Theorem 2.4 *The twig graph $TW(P_n)$ of the path P_n is a F-root square mean graph for $n \geq 4$.*

Proof Let $v_1, v_2, v_3, \cdots, v_n$ be the vertices of the path P_n and $u_1^{(i)}, u_2^{(i)}$ be the pendant vertices at each vertex v_i, for $2 \leq i \leq n - 1$. Define $f : V(TW(P_n)) \to \{1, 2, 3, \cdots, 3n - 4\}$ as follows

$$f(v_1) = 1, \quad f(v_2) = 2, \quad f(v_i) = 3i - 3, \text{ for } 3 \leq i \leq n - 1,$$
$$f(v_n) = 3n - 4, \quad f\left(u_1^{(2)}\right) = 3, \quad f\left(u_1^{(i)}\right) = 3i - 4, \text{ for } 3 \leq i \leq n - 1,$$
$$f\left(u_2^{(i)}\right) = 3i - 2, \text{ for } 2 \leq i \leq n - 1.$$

Then the induced edge labeling is obtained as follows

$$f^*(v_i v_{i+1}) = 3i - 2, \text{ for } 1 \leq i \leq n - 1,$$
$$f^*\left(v_i u_1^{(i)}\right) = 3i - 4, \text{ for } 2 \leq i \leq n - 1$$
$$f^*\left(v_i u_2^{(i)}\right) = 3i - 3, \text{ for } 2 \leq i \leq n - 1.$$

Hence f is a F-root square mean labeling of the graph $TW(P_n)$. Thus the graph $TW(P_n)$ is a F-root square mean graph for $n \geq 1$.   □
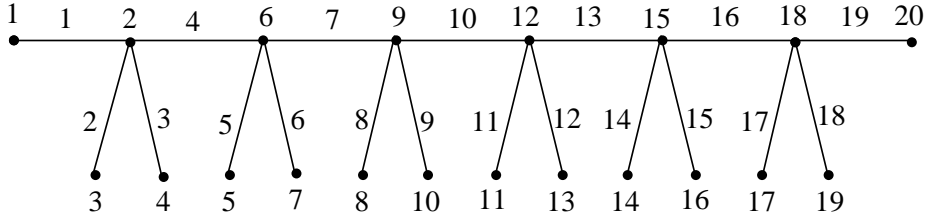
Figure 6. A F-root square mean labeling of $TW(P_8)$.

Theorem 2.5 *The graph* $[P_n; S_m]$ *is a F-root square mean graph for* $m \leq 2$ *and* $n \geq 1$.

Proof Let $u_1, u_2, u_3, \ldots, u_n$ be the vertices of the path P_n and $v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, \ldots, v_{m+1}^{(i)}$ be the vertices of the star graph S_m such that $v_1^{(i)}$ is the central vertex of the star graph $S_m, 1 \leq i \leq n$.

Case 1. $m = 1$

Define $f : V([P_n; S_1]) \to \{1, 2, 3, \ldots, 3n\}$ as follows

$$f(u_i) = \begin{cases} 3i, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 3i - 2, & 1 \leq i \leq n \text{ and } i \text{ is even,} \end{cases}$$

$$f\left(v_1^{(i)}\right) = 3i - 1, \text{ for } 1 \leq i \leq n,$$

$$f\left(v_2^{(i)}\right) = \begin{cases} 3i - 2, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ 3i, & 1 \leq i \leq n \text{ and } i \text{ is even.} \end{cases}$$

Then the induced edge labeling is obtained as follows

$$f^*(u_i u_{i+1}) = 3i, \text{ for } 1 \le i \le n-1,$$

$$f^*\left(u_i v_1^{(i)}\right) = \begin{cases} 3i-1, & 1 \le i \le n \text{ and } i \text{ is odd} \\ 3i-2, & 1 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$f^*\left(v_1^{(i)} v_2^{(i)}\right) = \begin{cases} 3i-2, & 1 \le i \le n \text{ and } i \text{ is odd} \\ 3i-1, & 1 \le i \le n \text{ and } i \text{ is even.} \end{cases}$$

Case 2. $m = 2$

Define $f : V([P_n; S_2]) \rightarrow \{1, 2, 3, \ldots, 4n\}$ as follows

$$f(u_i) = \begin{cases} 4i-1, & 1 \le i \le n \text{ and } i \text{ is odd} \\ 4i-3, & 1 \le i \le n \text{ and } i \text{ is even,} \end{cases}$$

$$f\left(v_1^{(i)}\right) = 4i-2, \text{ for } 1 \le i \le n,$$

$$f\left(v_2^{(i)}\right) = \begin{cases} 4i-3, & 1 \le i \le n \text{ and } i \text{ is odd} \\ 4i-1, & 1 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$f\left(v_3^{(i)}\right) = 4i, \text{ for } 1 \le i \le n.$$

Then the induced edge labeling is obtained as follows

$$f^*(u_i u_{i+1}) = 4i, \text{ for } 1 \le i \le n-1,$$

$$f^*\left(u_i v_1^{(i)}\right) = \begin{cases} 4i-2, & 1 \le i \le n \text{ and } i \text{ is odd} \\ 4i-3, & 1 \le i \le n \text{ and } i \text{ is even,} \end{cases}$$

$$f^*\left(v_1^{(i)} v_2^{(i)}\right) = \begin{cases} 4i-3, & 1 \le i \le n \text{ and } i \text{ is odd} \\ 4i-2, & 1 \le i \le n \text{ and } i \text{ is even} \end{cases}$$

$$f^*\left(v_1^{(i)} v_3^{(i)}\right) = 4i-1, \text{ for } 1 \le i \le n.$$

Hence f is a F-root square mean labeling of the graph $[P_n; S_m]$. Thus the graph $[P_n; S_m]$ is a F-root square mean graph for $n \ge 1$. $\square$
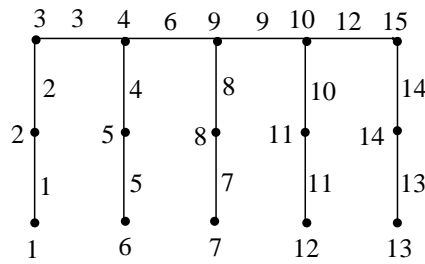
Figure 7. A F-root square mean labeling of $[P_5; S_1]$.
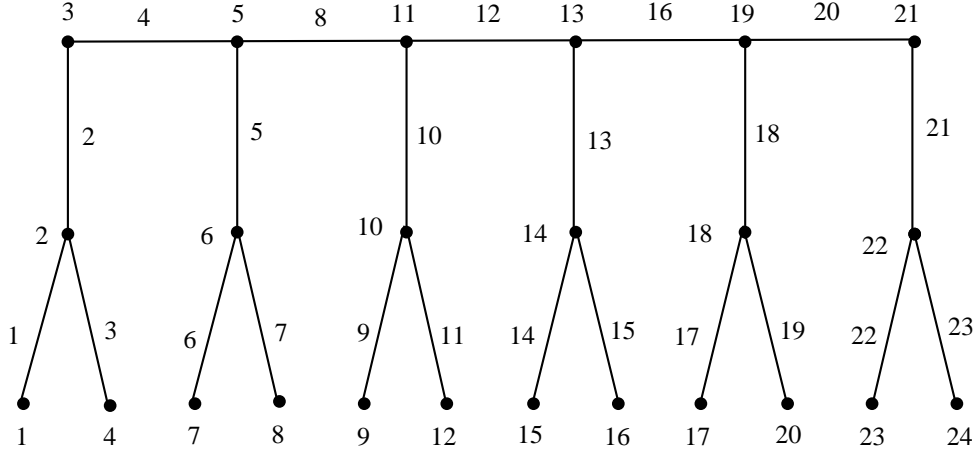
Figure 8. A F-root square mean labeling of $[P_6; S_2]$.

Theorem 2.6 *The graph $S(P_n \circ K_1)$ is a F-root square mean graph for $n \geq 1$.*

Proof In $P_n \circ K_1$, let $u_i, 1 \leq i \leq n$, be the vertices on the path P_n and v_i be the vertex attached at each vertex $u_i, 1 \leq i \leq n$.

Let x_i be the vertex which divides the edge $u_i v_i$, for $1 \leq i \leq n$ and y_i be the vertex which divides the edge $u_i u_{i+1}$, for $1 \leq i \leq n - 1$. Then,

$$V(S(P_n \circ K_1)) = \{u_i, v_i, x_i, y_j; 1 \leq i \leq n \text{ and } 1 \leq j \leq n - 1\},$$
$$E(S(P_n \circ K_1)) = \{u_i x_i, v_i y_i; 1 \leq i \leq n\} \bigcup \{u_i y_i, y_i u_{i+1}; 1 \leq i \leq n - 1\}.$$

Define $f : V(S(P_n \circ K_1)) \to \{1, 2, 3, \ldots, 4n - 1\}$ as follows

$$f(u_i) = 4i - 3, \text{ for } 1 \leq i \leq n,$$
$$f(y_i) = 4i - 1, \text{ for } 1 \leq i \leq n - 1,$$
$$f(x_i) = 4i - 2, \text{ for } 1 \leq i \leq n,$$
$$f(v_i) = 4i, \text{ for } 1 \leq i \leq n - 1,$$
$$f(v_n) = 4n - 1.$$

Then the induced edge labeling is obtained as follows

$$f^*(u_i y_i) = 4i - 2, \text{ for } 1 \leq i \leq n - 1,$$
$$f^*(y_i u_{i+1}) = 4i, \text{ for } 1 \leq i \leq n - 1,$$
$$f^*(u_i x_i) = 4i - 3, \text{ for } 1 \leq i \leq n,$$
$$f^*(x_i v_i) = 4i - 1, \text{ for } 1 \leq i \leq n - 1,$$
$$f^*(x_n v_n) = 4n - 2.$$

Hence f is a F-root square mean labeling of $S(P_n \circ K_1)$. Thus the graph $S(P_n \circ K_1)$ is a

F-root square mean graph for $n \geq 1$. $\square$
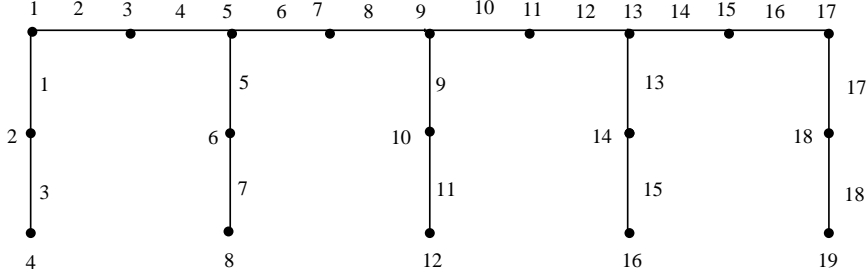
Figure 9. A *F*-root square mean labeling of $S(P_5 \circ K_1)$.

Theorem 2.7 *The middle graph $M(P_n)$ of a path P_n is a F-root square mean graph.*

Proof Let $V(P_n) = \{v_1, v_2, v_3, \cdots, v_n\}$ and $E(P_n) = \{e_i = v_i v_{i+1}; 1 \leq i \leq n-1\}$ be the vertex set and edge set of the path P_n. Then,

$$V(M(P_n)) = \{v_1, v_2, v_3, \cdots, v_n, e_1, e_2, e_3, \cdots, e_{n-1}\},$$
$$E(M(P_n)) = \{v_i e_i, e_i v_{i+1}; 1 \leq i \leq n-1\} \bigcup \{e_i e_{i+1}; 1 \leq i \leq n-2\}.$$

Define $f : V(M(P_n)) \rightarrow \{1, 2, 3, \ldots, 3n-3\}$ as follows

$$f(v_i) = \begin{cases} 1, & i = 1 \\ 3i - 3, & 2 \leq i \leq n, \end{cases}$$
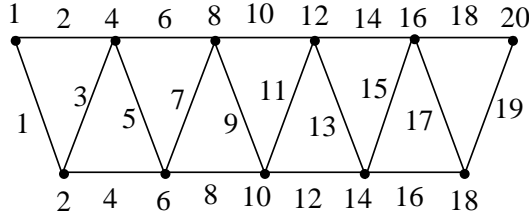$$f(e_i) = 3i - 1, \text{ for } 1 \leq i \leq n-1.$$

Then the induced edge labeling is obtained as follows

$$f^*(v_i e_i) = 3i - 2, \text{ for } 1 \leq i \leq n-1,$$
$$f^*(e_i v_{i+1}) = 3i - 1, \text{ for } 1 \leq i \leq n-1,$$
$$f^*(e_i e_{i+1}) = 3i, \text{ for } 1 \leq i \leq n-2.$$

Hence f is a *F*-root square mean labeling of $M(P_n)$. Thus the graph $M(P_n)$ is a *F*-root square mean graph. $\square$
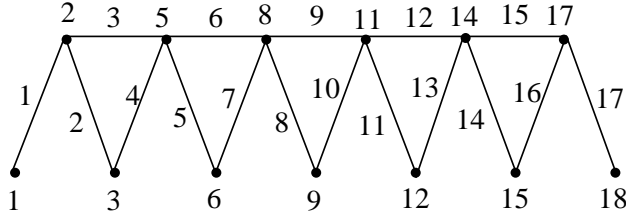
Figure 10. A *F*-root square mean labeling of $M(P_7)$.

Theorem 2.8 *The total graph $T(P_n)$ of a path P_n is a F-root square mean graph for $n \geq 1$.*

Proof Let $V(P_n) = \{v_1, v_2, v_3, \cdots, v_n\}$ and $E(P_n) = \{e_i = v_i v_{i+1}; 1 \leq i \leq n-1\}$ be the vertex set and edge set of the path P_n. Then,

$$V(T(P_n)) = \{v_1, v_2, v_3, \cdots, v_n, e_1, e_2, e_3, \cdots, e_{n-1}\}$$
$$E(T(P_n)) = \{v_i v_{i+1}, e_i v_i, e_i v_{i+1}; 1 \leq i \leq n-1\} \cup \{e_i e_{i+1}; 1 \leq i \leq n-2\}.$$

Define $f : V(T(P_n)) \rightarrow \{1, 2, 3, \ldots, 4(n-1)\}$ as follows

$$f(v_1) = 1, \quad f(v_i) = 4i - 4, \text{ for } 2 \leq i \leq n$$
$$f(e_i) = 4i - 2, \text{ for } 1 \leq i \leq n-1.$$

Then the induced edge labeling is obtained as follows

$$f^*(v_i v_{i+1}) = 4i - 2, \text{ for } 1 \leq i \leq n-1, \quad f^*(e_i e_{i+1}) = 4i, \text{ for } 1 \leq i \leq n-2,$$
$$f^*(v_i e_i) = 4i - 3, \text{ for } 1 \leq i \leq n-1$$
$$f^*(e_i v_{i+1}) = 4i - 1, \text{ for } 1 \leq i \leq n-1.$$

Hence f is a F-root square mean labeling of $T(P_n)$. Thus the graph $T(P_n)$ is a F-root square mean graph. □

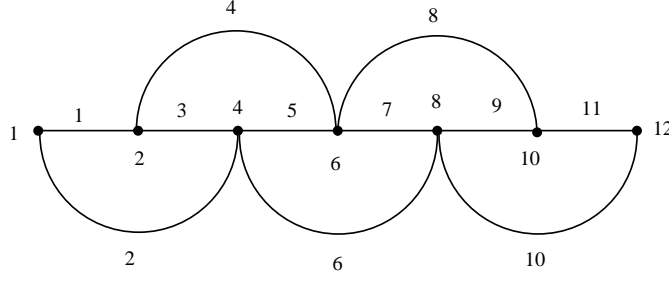**Figure 11.** A F-root square mean labeling of $T(P_6)$.

Theorem 2.9 *The square graph P_n^2 of the path P_n is a F-root square mean graph for $n \geq 1$.*

Proof Let $v_1, v_2, v_3, \cdots, v_n$ be the vertices of the path P_n. Define $f : V\left(P_n^2\right) \rightarrow \{1, 2, 3, \cdots, 2(n-1)\}$ as $f(v_1) = 1$ and $f(v_i) = 2i - 2$ for $2 \leq i \leq n$. Then the induced edge labeling is obtained as follows

$$f^*(v_i v_{i+1}) = 2i - 1, \text{ for } 1 \leq i \leq n-1,$$
$$f^*(v_i v_{i+2}) = 2i, \text{ for } 1 \leq i \leq n-2.$$

Hence f is a F-root square mean labeling of P_n^2. Thus the graph P_n^2 is a F-root square mean graph. □

Figure 12. A F-root square mean labeling of P_7^2.

Theorem 2.10 *The ladder graph L_n is a F-root square mean graph for $n \geq 1$.*

Proof Let $G = P_2 \times P_n$ be the ladder graph for any positive integer n, having $2n$ vertices and $3n - 2$ edges. Let $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$ be the vertices of G. Then the edge set of G is $\{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n - 1\} \bigcup \{u_i v_i : 1 \leq i \leq n\}$.
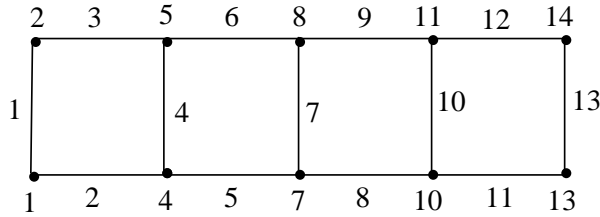
Define $f : V(G) \to \{1, 2, 3, \ldots, 3n - 1\}$ as follows

$$f(u_i) = 3i - 1, \text{ for } 1 \leq i \leq n,$$
$$f(v_i) = 3i - 2, \text{ for } 1 \leq i \leq n.$$

Then the induced edge labeling is obtained as follows

$$f^*(u_i u_{i+1}) = 3i, \text{ for } 1 \leq i \leq n - 1,$$
$$f^*(v_i v_{i+1}) = 3i - 1, \text{ for } 1 \leq i \leq n - 1$$
$$f^*(u_i v_i) = 3i - 2, \text{ for } 1 \leq i \leq n.$$

Hence f is a F-root square mean labeling of L_n. Thus the graph L_n is a F-root square mean graph. □

Figure 13. A F-root square mean labeling of L_5.

Theorem 2.11 *The slanting ladder graph SL_n is a F-root square mean graph for $n \geq 2$.*

Proof Let the vertex set of SL_n be $\{u_1, u_2, u_3, \cdots, u_n, v_1, v_2, v_3, \cdots, v_n\}$ and the edge set of SL_n be $\{u_i u_{i+1}; 1 \leq i \leq n - 1\} \bigcup \{v_i v_{i+1}; 1 \leq i \leq n - 1\} \bigcup \{v_i u_{i+1}; 1 \leq i \leq n - 1\}$. Then SL_n has $2n$ vertices and $3n - 3$ edges.
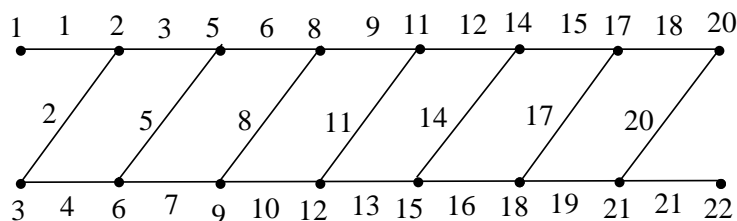
Define $f : V(SL_n) \rightarrow \{1, 2, 3, \cdots, 3n - 2\}$ as follows

$$f(u_1) = 1, \quad f(u_i) = 3i - 4, \text{ for } 2 \leq i \leq n,$$
$$f(v_i) = 3i, \text{ for } 1 \leq i \leq n - 1 \quad \text{and } f(v_n) = 3n - 2.$$

Then the induced edge labeling is obtained as follows

$$f^*(u_i u_{i+1}) = \begin{cases} 1, & i = 1 \\ 3i - 3, & 2 \leq i \leq n - 1, \end{cases}$$
$$f^*(v_i v_{i+1}) = 3i + 1, \text{ for } 1 \leq i \leq n - 2,$$
$$f^*(v_{n-1} v_n) = 3n - 3$$
$$f^*(v_i u_{i+1}) = 3i - 1, \text{ for } 1 \leq i \leq n - 1.$$

Hence f is a F-root square mean labeling of SL_n. Thus the graph SL_n is a F-root square mean graph. $\qquad\square$

Figure 14. A F-root square mean labeling of SL_8.

References

[1] A. Durai Baskar and S. Arockiaraj, F-harmonic mean graphs, *International Journal of Mathematical Archive,* Vol.5, 8(2014), 92–108.

[2] J.A. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics,* **17**(2014), #DS6.

[3] F. Harary, *Graph Theory,* Narosa Publishing House Reading, New Delhi, 1988.

[4] S.S. Sandhya, S. Somasundaram and S. Anusa, Some new Results on root square mean Labeling, *International Mathematical Archive,* Vol.5, 12(2014), 130–135.

[5] S.S. Sandhya, S. Somasundaram and S. Anusa, Root square mean labeling of some more disconnected graphs, *International Mathematical Forum,***10**(1)(2015), 25–34.

Some More 4-Prime Cordial Graphs

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Abstract: Let G be a (p,q) graph, $H \prec G$ and $f : V(G) \to \{1, 2, \cdots, k\}$ be a map. For each edge uv, assign the label $\gcd(f(u), f(v))$. Then, f is called Smarandachely k-prime cordial labeling on G to H if $\left|v_f^H(i) - v_f^H(j)\right| \le 1$, $i, j \in \{1, 2, \cdots, k\}$ and $\left|e_f^H(0) - e_f^H(1)\right| \le 1$, but there exist integers $0 \le i \ne j \le k$ such that $\left|v_f^{G \setminus H}(i) - v_f^{G \setminus H}(j)\right| \ge 2$, or $\left|e_f^{G \setminus H}(0) - e_f^{G \setminus H}(1)\right| \ge 2$, where $v_f^H(x)$, $v_f^{G \setminus H}(x)$ respectively denotes the numbers of vertices of H, $G \setminus H$ labeled with x, $e_f^H(1)$, $e_f^H(0)$ and $e_f^{G \setminus H}(1)$, $e_f^{G \setminus H}(0)$ respectively denote the number of edges labeled with 1 and not labeled with 1 in H, $G \setminus H$. Particularly, a Smarandachely k-prime cordial labeling on G to G is called k-prime cordial labeling with $v_f(x)$, $e_f(1)$ and $e_f(0)$ replacing notations $v_f^H(x)$, $e_f^H(1)$ and $e_f^H(0)$ for abbreviation. A graph with a k-prime cordial labeling is called a k-prime cordial graph. In this paper we investigate 4-prime cordial labeling behavior of lotus inside a circle, sunflower graph, $S(K_2 + mK_1)$, $S(P_n \odot K_1)$, dodecahedron, and some more graphs.

Key Words: Smarandachely k-prime labelling, k-prime labelling, cycle, wheel, join, sunflower graph, lotus inside a circle.

AMS(2010): 05C78.

§1. Introduction

In this paper graphs are finite, simple and undirected. Let G be a (p, q) graph where p refers the number of vertices of G and q refers the number of edge of G. The number of vertices of a graph G is called order of G, and the number of edges is called size of G. In 1987, Cahit introduced the concept of cordial labeling of graphs [1]. Sundaram, Ponraj, Somasundaram [5] have introduced the notion of prime cordial labeling of graphs. Also they discussed the prime cordial labeling behavior of various graphs. Recently Ponraj et al. [7], introduced k-prime cordial labeling of graphs. They have studied 3-prime cordiality of several graphs in [7, 8]. In [9, 10] Ponraj et al. studied the 4-prime cordial labeling behavior of complete graph,

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book, flower, mC_n, wheel, gear, double cone, helm, closed helm, butterfly graph, and friendship graph and some more graphs. In this paper we have studied about the 4-prime cordiality of lotus inside a circle, sunflower graph and some more graphs. Let x be any real number. Then $\lfloor x \rfloor$ stands for the largest integer less than or equal to x and $\lceil x \rceil$ stands for smallest integer greater than or equal to x. Terms not defined here follow from Harary [3] and Gallian [2].

§2. **Preliminaries**

Remark 2.1([6]) A 2-prime cordial labeling is a product cordial labeling.

Remark 2.2([5]) A p-prime cordial labeling is a prime cordial labeling.

Definition 2.3   *The join of two graphs $G_1 + G_2$ is obtained from G_1 and G_2 and whose vertex set is $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$.*

Definition 2.4 *The graph $C_n + K_1$ is called a wheel. In a wheel, the vertex of degree n is called the central vertex and the vertices on the cycle C_n are called rim vertices.*

Definition 2.5   *The sunflower graph SF_n is obtained by taking a wheel with central vertex u and the cycle $C_n : u_1 u_2 \cdots u_n u_1$ and new vertices $v_1, v_2, \ldots, v_n$ where v_i is joined by vertices $u_i, u_{i+1 \pmod n}$.*

Definition 2.6 *The lotus inside a circle LC_n is a graph obtained from the cycle $C_n : v_1 v_2 \cdots v_n v_1$ and a star $K_{1,n}$ with central vertex u and the end vertices $u_1, u_2, \cdots, u_n$ by joining each u_i to v_i and $v_{i+1 \pmod n}$.*

Definition 2.7 *The subdivision graph $S(G)$ of a graph G is obtained by replacing each edge uv by a path uwv.*

Definition 2.8   *The graph P_n^2 is obtained from the path P_n by adding edges that joins all vertices u and v with $d(u,v) = 2$.*

Definition 2.9 *Let G_1, G_2 respectively be (p_1, q_1), (p_2, q_2) graphs. The corona of G_1 with G_2, $G_1 \odot G_2$ is the graph obtained by taking one copy of G_1 and p_1 copies of G_2 and joining the ith vertex of G_1 with an edge to every vertex in the ith copy of G_2.*

Definition 2.10 *The one-point union of t copies of the cycle C_3 is called a friendship graph $C_3^{(t)}$.*

Definition 2.11   *The DH_n is a graph with vertex set $V(DH_n) = \{u_i, v_i, x_i, y_i : 1 \leq i \leq n\}$ and the edge set $E(DH_n) = \{uu_{i+1}, y_i v_{i+1}, x_i x_{i+1} : 1 \leq i \leq n-1\} \bigcup \{u_i v_i, v_i y_i, x_i y_i : 1 \leq i \leq n\} \bigcup \{u_1 u_n, y_n v_1, x_1 x_n\}$. DH_5 is called a Tetrahedron.*

Definition 2.12 *The Cartesian product graph $G_1 \square G_2$ is defined as follows: Consider any two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V = V_1 \times V_2$. Then u and v are adjacent in $G_1 \square G_2$*

whenever $[u_1 = v_1 \text{ and } u_2v_2 \in E(G_2)]$ *or* $[u_2 = v_2 \text{ and } u_1v_1 \in E(G_1)]$.

Definition 2.13 *A ladder L_n is the graph $P_n \times P_2$. Let $V(L_n) = \{u_i, v_i : 1 \le i \le n\}$ and $E(L_n) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \le i \le n-1\} \bigcup \{u_iv_i : 1 \le i \le n\} \bigcup \{u_1u_n, v_1v_n\}$.*

The graph GL_n is obtained from the ladder L_n with $V(GL_n) = V(L_n)$ and $E(GL_n) = E(L_n) \cup \{u_iv_{i+1} : 1 \le i \le n-1\}$.

Theorem 2.14([7]) *The cycle C_n, $n \ne 3$ is k-prime cordial where k is even.*

§3. Main Results

3.1 Cycle Related Graphs

Theorem 3.1 *The lotus inside a circle LC_n is 4-prime cordial if and only if $n > 4$.*

Proof Note that the order and size of LC_n are $2n+1$ and $4n$ respectively. Suppose $n = 3$ or 4 then one can easily check that there does not exists a 4-prime cordial labeling and so we assume $n > 4$. Here we divide the proof into four cases.

Case 1. $n \equiv 0 \pmod 4$.

We construct a labeling f as follows: assign the label 4 to the vertices $v_1, v_2, \cdots v_{\frac{n}{2}}$ then put the label 3 to the vertex $v_{\frac{n}{2}+1}$. The remaining vertices of the cycle, namely, $v_{\frac{n}{2}+2}, v_{\frac{n}{2}+3}, \cdots, v_n$ are labeled by 1. Now we consider the center of the star u. The vertex u is labeled by 2. For the vertices $u_1, u_2, \cdots, u_n$, first we consider the vertices u_{n-1} and u_n. Assign the labels $1, 2$ respectively to the vertices u_{n-1} and u_n. Consider the vertices $u_1, u_2, \cdots u_{\frac{n}{2}-1}$. Fix the label 2 to this vertices. Then the vertices $u_{\frac{n}{2}}, u_{\frac{n}{2}+1}, \cdots, u_{n-2}$ are labeled by 3.

Case 2. $n \equiv 1 \pmod 4$.

Assign the labels to the vertices u, u_i, $1 \le i \le n-3$, v_j, $1 \le j \le n-1$ as in case 1. The assign the labels $3, 1, 2$ to the vertices u_{n-2}, u_{n-1}, u_n respectively. Finally we assign the label 4 to the vertex v_n.

Case 3. $n \equiv 2 \pmod 4$.

Assign the label 4 to the vertices $v_1, v_2, \cdots v_{\frac{n}{2}}$. For the vertex $v_{\frac{n}{2}+1}$ we assign the number 3. The remaining vertices of the cycle from $v_{\frac{n}{2}+2}, v_{\frac{n}{2}+3}, \cdots, v_n$ receives the label 1. Put the label 2 to the vertex u. Now we consider the vertices u_i, $1 \le i \le n$. Assign the label 2 to the vertices u_n, u_i where $1 \le i \le \frac{n}{2}-1$, then assign the integer 3 to the vertices $u_{\frac{n}{2}}, u_{\frac{n}{2}+1}, \cdots, u_{n-2}$. Finally we assign the label 1 to the vertex u_{n-1}.

Case 4. $n \equiv 3 \pmod 4$.

First we consider the vertices of the cycle. The label 4 is used to the vertices v_i, $1 \le i \le \frac{n+1}{2}$. Put the label 3 to the vertex $v_{\frac{n+3}{2}}$. The unlabeled vertices of the cycle are now labeled by 1. Then put the number 2 to the vertex u. If we consider the vertices u_i, $1 \le i \le n$, we assign the label 2 to the vertices u_j where $1 \le j \le \frac{n-1}{2}$ then assign 3 to the vertices $u_{\frac{n+1}{2}}, \cdots, u_{n-1}$.

Finally put the number 1 to the vertex u_n.   □

The following Table 1 establish that the above mentioned labeling f is a 4-prime cordial labeling.

Values of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(0)$	$e_f(1)$
$n \equiv 0 \pmod 4$	$\lfloor \frac{2n+1}{4} \rfloor$	$\lceil \frac{2n+1}{4} \rceil$	$\lfloor \frac{2n+1}{4} \rfloor$	$\lfloor \frac{2n+1}{4} \rfloor$	$2n$	$2n$
$n \equiv 1 \pmod 4$	$\lfloor \frac{2n+1}{4} \rfloor$	$\lceil \frac{2n+1}{4} \rceil$	$\lceil \frac{2n+1}{4} \rceil$	$\lceil \frac{2n+1}{4} \rceil$	$2n$	$2n$
$n \equiv 2 \pmod 4$	$\lfloor \frac{2n+1}{4} \rfloor$	$\lceil \frac{2n+1}{4} \rceil$	$\lfloor \frac{2n+1}{4} \rfloor$	$\lfloor \frac{2n+1}{4} \rfloor$	$2n$	$2n$
$n \equiv 3 \pmod 4$	$\lfloor \frac{2n+1}{4} \rfloor$	$\lceil \frac{2n+1}{4} \rceil$	$\lceil \frac{2n+1}{4} \rceil$	$\lceil \frac{2n+1}{4} \rceil$	$2n$	$2n$

Table 1

Theorem 3.2 *The sunflower graph SF_n is 4-prime cordial for all n.*

Proof First we observe that the order and size of SF_n are $2n + 1$ and $4n$ respectively. We consider the following cases.

Case 1. $n \equiv 0 \pmod 4$.

Assign the label 2 to the vertices $u_1, u_2, \cdots u_{\frac{n}{4}}$. Then put the number 4 to the next consecutive vertices $u_{\frac{n}{4}+1}, \cdots, u_{\frac{n}{2}+1}$. The next vertex $u_{\frac{n}{2}+2}$ is labeled by 3. Then the remaining vertices of the cycle, namely, $u_{\frac{n}{2}+3}, \cdots, u_n$ are labeled by 1. For the central vertex u, we use the label 2. We now move to the vertices v_i, $1 \leq i \leq n$. Assign the label 2 to the vertices $v_1, v_2, \cdots v_{\frac{n}{4}-1}$. Then assign the label 4 to the vertices $v_{\frac{n}{4}}, v_{\frac{n}{4}+1}, \cdots, v_{\frac{n}{2}-1}$. The next three vertices $v_{\frac{n}{2}}, v_{\frac{n}{2}+1}, v_{\frac{n}{4}+2}$ are labeled by $1, 3, 1$ respectively. Finally the remaining unlabeled vertices received the integer 3.

Case 2. $n \equiv 1 \pmod 4$.

First we consider the vertices of the cycle C_n. For the vertices $u_1, u_2, \cdots u_{\frac{n-5}{4}}$, we assign the label 2. The successive vertices $u_{\frac{n-1}{4}}, \cdots, u_{\frac{n-1}{2}}$ are labeled by 4. Put the label 3 to the vertex $u_{\frac{n+1}{2}}$. The vertices $u_{\frac{n+3}{2}}, \cdots, u_{n-1}$ are labeled by 1. Put the label 2 to the vertex u_n. Then assign the label 2 to the central vertex u. We now move to the vertices v_i, $1 \leq i \leq n$. Assign the label 2 to the vertices $v_1, v_2, \cdots v_{\frac{n-5}{4}}$. Then put the number 4 to the vertices $v_{\frac{n-1}{4}}, \cdots, v_{\frac{n-3}{2}}$. Put the labels $33, 1$ respectively to the vertices $v_{\frac{n-1}{2}}, v_{\frac{n+1}{2}}$. Assign the label 3 to the vertices $v_{\frac{n+3}{2}}, \cdots, v_{n-1}$. Finally we assign the number 2 to v_n.

Case 3. $n \equiv 2 \pmod 4$.

We first consider the vertex u. Label it by 2. Then we consider the vertices of the cycle C_n. Assign the label 2 to the vertices $u_1, u_2, \cdots u_{\frac{n+2}{4}}$. Then put the integer 4 to the vertices $u_{\frac{n+6}{4}}, \cdots, u_{\frac{n+2}{2}}$. The next vertex $u_{\frac{n+4}{4}}$ is labeled by 3. The remaining vertices of the cycle are labeled by 1. Then consider the vertices v_i, $1 \leq i \leq n$. Assign the label 2 to the vertices v_i where $1 \leq i \leq \frac{n-2}{4}$ then the vertices $v_{\frac{n-2}{4}+i}$ where $1 \leq i \leq \frac{n-2}{4}$ are labeled with 4. Put the labels $1, 3, 1$ to the next consecutive vertices $v_{\frac{n}{2}}, v_{\frac{n+2}{2}}, v_{\frac{n+4}{2}}$. Finally put the number 3 for the unlabeled vertices.

Case 4. $n \equiv 3 \pmod{4}$.

Assign the label 2 to the vertices u_i, where $1 \leq i \leq \frac{n+1}{4}$, then assign 4 to the vertices $u_{\frac{n+1}{4}+i}$ where $1 \leq i \leq \frac{n+1}{4}$. The next vertex $u_{\frac{n+3}{2}}$ received the label 3. Then assign the label 1 to the remaining vertices of the cycle. Put the integer 2 to the vertex u. Then we consider the vertices v_i, $1 \leq i \leq n$. Assign the number 2 to the vertices $v_1, v_2, \cdots v_{\frac{n-3}{4}}$. Then assign 4 to the vertices $v_{\frac{n+1}{4}}, \cdots, v_{\frac{n-1}{2}}$. The next two consecutive vertices $v_{\frac{n+1}{2}}, v_{\frac{n+3}{2}}$ are labeled by 3, 1 respectively. The rest of the unlabeled vertices are labeled by 3.

The Table 2 shows that the above labeling f is a required 4-prime cordial labeling. $\square$

Values of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(0)$	$e_f(1)$
$n \equiv 0 \pmod{4}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}+1$	$2n$	$2n$
$n \equiv 1 \pmod{4}$	$\frac{n-1}{2}$	$\frac{n+1}{2}$	$\frac{n+1}{2}$	$\frac{n+1}{2}$	$2n$	$2n$
$n \equiv 2 \pmod{4}$	$\frac{n}{2}$	$\frac{n}{2}+1$	$\frac{n}{2}$	$\frac{n}{2}$	$2n$	$2n$
$n \equiv 3 \pmod{4}$	$\frac{n-1}{2}$	$\frac{n+1}{2}$	$\frac{n+1}{2}$	$\frac{n+1}{2}$	$2n$	$2n$

Table 2

A 4-prime cordial labeling of SF_8 is given in Figure 1.
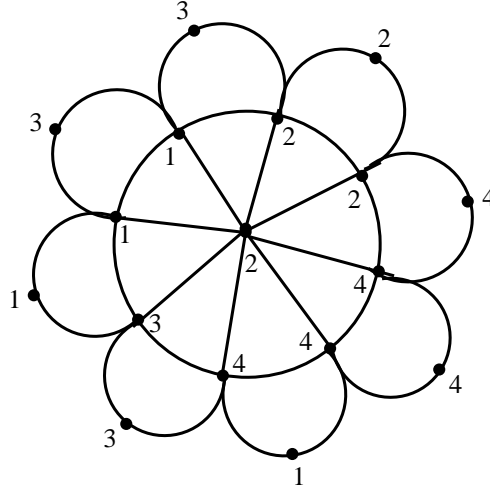
Figure 1

Theorem 3.3 *DH_n is 4-prime cordial.*

Proof Clearly DH_n consists of $4n$ vertices and $6n$ edges. We now give the label to the vertices of DH_n as follows: Assign the label 2 to the vertices u_i, $1 \leq i \leq n$ and assign the label 4 to the vertices v_i, $1 \leq i \leq n$. Now we move to the vertices y_i. Assign the label 1 to the vertices $y_1, y_2, \cdots, y_n$. Finally assign the label 3 to the vertices $x_1, x_2, \cdots, x_n$. This vertex labeling f is obviously a 4-prime cordial labeling of DH_n. Since, $v_f(1) = v_f(2) = v_f(3) = v_f(4) = n$ and $e_f(0) = e_f(1) = 3n$. $\square$

Theorem 3.4 *Let C_n be the cycle $u_1 u_2 \cdots u_n u_1$. Let G_n be the graph with $V(G_n) = V(C_n \bigcup \{v_i :$*

$1 \leq i \leq n\}$ and $E(G_n) = E(C_n) \bigcup \{u_i v_i : 1 \leq i \leq n\} \cup \{u_{i+1} v_i : 1 \leq i \leq n-1\}$. Then G_n is 4-prime cordial for all $n \neq 4$.

Proof Clearly for any vertex labeling of G_4, the maximum possible edges with label 0 is 4. Hence G_4 is not 4-prime cordial.

Case 1. $n \equiv 0 \pmod 4$, $n \neq 4$.

Let $n = 4t$. Assign the label 2 to the vertices $u_1, u_2, \cdots, u_{2t}$ and 4 to the vertices $v_1, v_2, \cdots, v_{2t}$. Next assign the label 3 to the vertices u_{2t+1}, u_{2t+2} and u_{2t+3}. Next assign the label 1 to the cycle vertices $u_{2t+4}, u_{2t+6}, \cdots, u_{4t}$ and 3 to the vertices $v_{2t+4}, v_{2t+5}, \cdots, v_{4t-1}, v_{4t}$. Finally assign the remaining non labeled vertices by 1.

Case 2. $n \equiv 1 \pmod 4$.

Let $n = 4t+1$. In this case, assign the label 2 to the vertices $u_1, u_2, \cdots, u_{2t+1}$ and 4 to the vertices $v_1, v_2, \cdots, v_{2t+1}$. Next assign the label 1 to the cycle vertices $u_{2t+2}, u_{2t+3}, \cdots$ and u_{4t+1}. Finally assign the label 3 to the non labeled vertices $v_{2t+2}, v_{2t+3}, \cdots, v_{4t+1}$.

Case 3. $n \equiv 2 \pmod 4$.

Let $n = 4t+2$. In this case we assign the label 2 to the vertices u_i $(1 \leq i \leq 2t+1)$ and 4 to the vertices v_i $(1 \leq i \leq 2t+1)$. Next assign the label 3 to the cycle vertices u_{2t+2}, u_{2t+3} and u_{2t+4}. Now we assign the label 1 to the vertices $u_{2t+5}, u_{2t+6}, \cdots, u_{4t+2}$. Next assign the label 3 to the vertices $v_{4t+2}, v_{4t+1}, \cdots, v_{2t+4}$. Finally assign the label to the non labeled vertices by 1.

Case 4. $n \equiv 3 \pmod 4$.

Let $n = 4t+3$. Assign the label 2 to the vertices $u_1, u_2, \cdots, u_{2t+2}$ and 4 to the vertices $v_1, v_2, \cdots, v_{2t+2}$. Next assign the label 1 to the vertices $u_{2t+3}, u_{2t+4}, \cdots, u_{4t+3}$. Finally assign the label to the vertices $v_{2t+3}, v_{2t+4}, \cdots, v_{4t+3}$.

Hence the vertex labeling given above is obviously a 4-prime cordial labeling of G_n. □

3.2 Subdivided Graphs

Theorem 3.5 $S(K_2 + mK_1)$ is 4-prime cordial.

Proof Let $V(S(K_2 + mK_1)) = \{u, v, w, u_i, v_i, w_i : 1 \leq i \leq m\}$ and $E(S(K_2 + mK_1)) = \{uv, vw, uu_i, u_i w_i, w_i v_i, v_i u : 1 \leq i \leq m\}$. Note that $S(K_2 + mK_1)$ has $3m+3$ vertices and $4m+2$ edges. The proof is divided into four cases depending upon the nature of n.

Case 1. $n = 4t$.

Assign the label 2 to the vertex u. Next assign the label 2 to the vertices $u_1, u_2, \cdots u_{3t}$. We now assign the label 4 to the vertices $u_{3t+1}, u_{3t+2}, \cdots, u_{4t}$. Next we move to the vertices w_i. Assign the label 4 to the vertices $w_1, w_2, \cdots, w_{2t}$. Next assign 3 to the vertices $w_{3t+1}, w_{3t+2}, \cdots, w_{4t}$. Next assign the label 3 to the vertices $v_{4t}, v_{4-1}, \cdots, v_{3t+1}$. We now assign the labels $4, 3$ respectively to the vertices w, v. Finally, assign 1 to all the remaining non labeled vertices.

Case 2. $n = 4t + 1$.

Assign the labels to the vertices u_i, v_i, w_i, $1 \leq i \leq 4t+1$, u, v, w as in case 1. Finally assign $2, 4$ and 1 respectively to the vertices u_{4t+1}, w_{4t+1} and v_{4t+1}.

Case 3. $n = 4t + 2$.

As in Case 2, assign the labels to the vertices u_i, v_i, w_i, $1 \leq i \leq 4t+1$, u, v, w. Next assign the labels $2, 1, 3$ to the vertices u_{4t+2}, w_{4t+2} and v_{4t+2} respectively.

Case 4. $n = 4t + 3$.

Assign the labels to the vertices u_i, v_i, w_i, $1 \leq i \leq 4t + 2$, u, v, w as in case 3. Finally assign $4, 1, 3$ respectively to the vertices u_{4t+3}, w_{4t+3} and v_{4t+3}.

Clearly, the above labeling is a 4-prime cordial labeling of $S(K_2 + mK_1)$. $\qquad \square$

Theorem 3.6 $S(P_n \odot K_1)$ *is 4-prime cordial.*

Proof Let P_n be the path $u_1 u_2 \ldots_n$ and $V(P_n \odot K_1) = V(P_n) \bigcup \{v_i : 1 \leq i \leq n\}$, $E(P_n \odot K_1) = E(P_n) \bigcup \{u_i v_i : 1 \leq i \leq n\}$. The graph $S(P_n \odot K_1)$ is obtained by subdividing the edge $u_i u_{i+1}$ with w_i and the edge $u_i v_i$ with x_i. Note that $S(P_n \odot K_1)$ has $4n - 1$ vertices and $4n - 2$ edges. The proof is divided into four cases.

Case 1. $n \equiv 0 \pmod 4$.

Let $n = 4t$. Assign the label 2 to the vertices $u_1, u_2, \cdots, u_{4t}$ and 4 to the vertices $w_1, w_2, \cdots, w_{4t-1}$ and x_1. We now assign the label 1 to the vertices $x_2, x_3, \cdots, x_{4t}$. Finally assign the label 3 to the pendent vertices $v_1, v_2, \cdots, v_{4t}$.

Case 2. $n \equiv 1 \pmod 4$.

Assign the label to the vertices u_i, v_i, x_i $(1 \leq i \leq n - 1)$ and w_i $(1 \leq i \leq n - 2)$ as in Case 1. Finally assign the labels $2, 4, 3$ and 1 respectively to the vertices w_{n-1}, u_n, x_n, v_n.

Case 3. $n \equiv 2 \pmod 4$.

As in Case 2, assign the label to the vertices u_i, v_i, x_i $(1 \leq i \leq n-1)$ and w_i $(1 \leq i \leq n-2)$. Next assign the labels $2, 4, 3$ and 1 to the vertices w_{n-1}, u_n, x_n, v_n respectively.

Case 4. $n \equiv 3 \pmod 4$.

In this case also assign labels to the vertices except w_{n-1}, u_n, x_n, v_n as in case 3. Then assign the labels $2, 4, 3$ and 1 to the vertices w_{n-1}, u_n, x_n, v_n respectively.

Clearly, the vertex labeling given in all cases is a 4-prime cordial labeling of $S(P_n \odot K_1)$. $\square$

Theorem 3.7 $S(C_3^{(t)})$ *is 4-prime cordial.*

Proof Note that $S(C_3^{(t)}) \equiv C_6^{(t)}$. Let the ith copy of the C_6 be $u_1^i u_2^i u_3^i u_4^i u_5^i u_6^i$, where $1 \leq i \leq t$ and $u_1^1 = u_1^2 = u_1^3 = u_1^4 = u_1^5 = u_1^6$.

Case 1. t is even, $t \geq 4$.

Assign the label 2 to the central vertex.

Subcase 1.1 $t \equiv 0 \pmod 4$.

Assign the label 2 to all the vertices of first $\frac{t}{4}$ copies of the cycle C_5. Next we move to the $(\frac{t}{4}+1)^{th}$ copy. Assign the label 4 to all the vertices of the $(\frac{t}{4}+1)^{th}$, $(\frac{t}{4}+2)^{th}$, ..., $(\frac{t}{2})^{th}$ copies of the cycle C_5. We now consider the $(\frac{t}{2}+1)^{th}$ copy. Assign the label 1 and 3 alternatively to the vertices of the $(\frac{t}{2}+1)^{th}$ copy of the cycle. In a similar fashion assign the label 1 and 3 alternatively to the vertices of the $(\frac{t}{2}+2)^{th}$, $\cdots$, t^{th} copy of the cycle C_5.

Subcase 1.2 $t \equiv 2 \pmod 4$.

In this case assign the label 2 to all the vertices of first $\frac{t-2}{4}$ copies of the cycle C_5. Now consider the $(\frac{t+2}{4})^{th}$ copy. Assign the label 4 to all the vertices of the cycle $(\frac{t+2}{4})^{th}$ copy. Similarly assign the label 4 to the $(\frac{t+6}{4})^{th}$, ..., $(\frac{t-2}{2})^{th}$ copies of the cycle C_5. We now move to the $(\frac{t}{2})^{th}$ copy. In this copy, assign the label 2 to the vertices u_2^t, u_3^t and 4 to the vertices u_4^t, u_5^t, u_6^t. Next assign 1, 3 alternatively to the vertices of the $(\frac{t}{2}+1)^{th}$, $\cdots$, t^{th} copies of the cycle.

Case 2. t is odd.

Subcase 2.1 $t \equiv 1 \pmod 4$.

As in subcase 1a, assign the label to the vertices of all the ith, $1 \le i \le t-1$ copies of C_5. In the last copy, assign the labels $2, 4, 4, 1$ and 3 respectively to the vertices $u_2^t, u_3^t, u_4^t, u_5^t$ and u_6^t.

Subcase 2.2 $t \equiv 3 \pmod 4$.

Assign the label to the vertices of $(t-1)$ copies of the cycle as in subcase 1b. Finally, assign the labels $2, 4, 3, 3$ and 1 to the vertices $u_2^t, u_3^t, u_4^t, u_5^t$ and u_6^t respectively.

The Table 3 establish that the above vertex labeling f is a 4-prime cordial labeling of $S(C_3^{(t)})$, $t \ge 4$.

Nature of t	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	$e_f(0)$	$e_f(1)$
$t \equiv 0 \pmod 4$	$\frac{5t}{4}$	$\frac{5t}{4}+1$	$\frac{5t}{4}$	$\frac{5t}{4}$	$3t$	$3t$
$t \equiv 1 \pmod 4$	$\frac{5t-1}{4}$	$\frac{5t+3}{4}$	$\frac{5t-1}{4}$	$\frac{5t+3}{4}$	$3t$	$3t$
$t \equiv 2 \pmod 4$	$\frac{5t+2}{4}$	$\frac{5t+2}{4}$	$\frac{5t-2}{4}$	$\frac{5t+2}{4}$	$3t$	$3t$
$t \equiv 3 \pmod 4$	$\frac{5t+1}{4}$	$\frac{5t+1}{4}$	$\frac{5t+1}{4}$	$\frac{5t+1}{4}$	$3t$	$3t$

Table 3

$S(C_3^{(1)}) \cong C_6$ is 4-prime cordial follows from Theorem 2.14. The 4-prime cordial labelings of $S(C_3^{(2)})$ and $S(C_3^{(3)})$ are given in Figure 2.   □
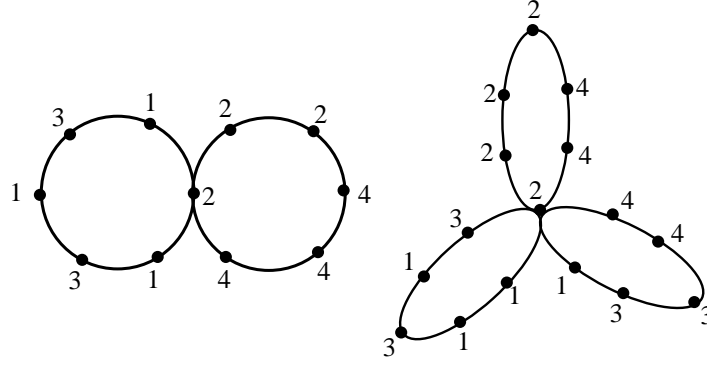
Figure 2

3.3 Miscellaneous Graphs

Theorem 3.8 P_n^2 *is 4-prime cordial if and only if* $n \neq 4$.

Proof Let P_n be the path $u_1, u_2, \cdots, u_n$. Clearly, the order and size of P_n^2 are n and $2n - 3$ respectively. It is easy to verify that P_4^2 does not admits a 4-rime cordial labeling. Let us assume that $n \neq 4$.

Case 1. $n \equiv 0 \pmod 4$.

Let $n = 4t$. Assign the label 1 to the vertices $u_1, u_3, \cdots, u_{2t-1}$. Next assign the label 3 to the vertices $u_2, u_4, \cdots, u_{2t}$. Assign the label 2 to the next t vertices $u_{2t+1}, \cdots, u_{3t}$. Finally, assign the label 4 to the next t non labeled vertices $u_{3t+1}, \cdots, u_{4t}$.

Case 2. $n \equiv 1 \pmod 4$.

Subcase 2.1 $n \equiv 1 \pmod 8$.

Assign the labels to the vertices of u_i, $1 \leq i \leq n-1$ as in Case 1. Finally, assign the label 2 to the vertex u_n.

Subcase 2.1 $n \equiv 5 \pmod 8$.

As in Case 1, assign the labels to the vertices u_i, $1 \leq i \leq n-1$. Then assign the label 1 to the vertex u_n.

Case 3. $n \equiv 2 \pmod 4$.

Subcase 3.1 $n \equiv 2 \pmod 8$.

Assign the labels to the vertices of u_i, $1 \leq i \leq n-1$ as in Subcase 2.1. Then assign the label 1 to the vertex u_n.

Subcase 3.2 $n \equiv 6 \pmod 8$.

As in Case 1, assign the labels to the vertices u_i, $1 \leq i \leq n-2$. Then assign the labels 2 and 1 respectively to the vertices u_{n-1} and u_n.

**Case 4.** $n \equiv 3 \pmod 4$.

Subcase 4.1 $n \equiv 3 \pmod 8$.

As in Subcase 3.1, assign the labels to the vertices of u_i, $1 \le i \le n-1$. Finally, assign the label 4 to the last vertex u_n.

Subcase 4b. $n \equiv 7 \pmod 8$.

Assign the labels to the vertices u_i, $1 \le i \le n-1$ as in Subcase 3.2. Finally, assign the label 4 to the vertex u_n.

It is easy to verify that the above vertex labeling pattern is 4-prime cordial labeling.   □

The following Figure 3 is an example of a 4-prime cordial labeling of P_{10}^2.

1   3   1   3   2   2   4   4   2   1
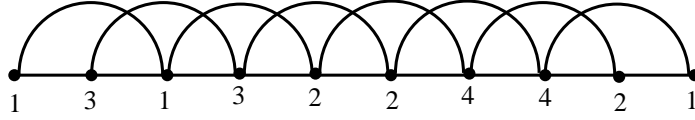
Figure 3

Theorem 3.9   *The graph GL_n is 4-prime cordial for $n > 2$.*

Proof Here we consider the following cases.

Case 1. $n \equiv 0 \pmod 4$.

Clearly GL_n has $2n$ vertices and $4n - 3$ edges. Let $n = 4t$. Assign the label 2 to the vertices $u_1, u_2, \cdots, u_{2t}$ and assign the label 4 to the vertices $v_2, v_3, \cdots, v_{2t+1}$. Next assign the label the labels $3, 1$ alternatively to the vertices to the vertices $u_{2t+1}, u_{2t+2}, \cdots, u_{4t}$. Assign the label 1 to the vertices $v_{4t}, v_{4t-2}, v_{4t-4}, \cdots, v_{2t+5}$ and assign the label 3 to the vertices $v_{4t-1}, v_{4t-3}, v_{4t-5}, \cdots, v_{2t+4}$. Finally assign the labels $1, 1, 3$ and 3 respectively to the vertices $v_{2t+3}, v_{2t+2}, v_{2t+1}$ and v_1.

Case 2. $n \equiv 1 \pmod 4$.

Assign the label to the vertices u_i, v_i, $1 \le i \le n-1$ as in case 1. Next assign the labels $2, 4$ to the vertices u_n, v_n respectively.

Case 3. $n \equiv 2 \pmod 4$.

As in Case 2, assign the label to the vertices u_i, v_i, $1 \le i \le n-1$. Next assign the labels $1, 3$ respectively to the vertices u_n, v_n. Finally interchange the labels of u_{2t+2} and u_{2t+3}, that is the label of u_{2t+2} is 3 and the label of u_{2t+3} is 1.

Case 4. $n \equiv 3 \pmod 4$.

As in Case 3, assign the label to the vertices u_i, v_i, $1 \le i \le n-1$. Then assign the labels $2, 4$ to the vertices u_n, v_n respectively. Clearly the above vertex labeling is a 4 prime cordial labeling of GL_n for all $n \ge 4$ and $n \ne 2$. A 4-prime cordial labeling of GL_3 is shown in the following Figure 4.   □
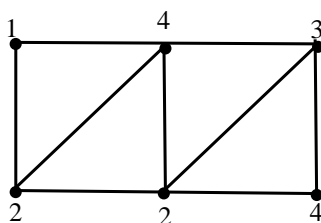
Figure 4

References

[1] I.Cahit, Cordial Graphs: A weaker version of graceful and harmonious graphs, *Ars combin.*, 23 (1987) 201-207.

[2] J.A.Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, 17 (2015) #Ds6.

[3] F.Harary, *Graph Theory*, Addision Wesley, New Delhi (1969).

[4] M.A.Seoud and M.A.Salim, Two upper bounds of prime cordial graphs, *JCMCC*, 75(2010) 95-103.

[5] M.Sundaram, R.Ponraj and S.Somasundaram, Prime cordial labeling of graphs, *J. Indian Acad. Math.*, 27(2005) 373-390.

[6] M. Sundaram, R. Ponraj and S. Somasundaram, Product cordial labeling of graphs, *Bull. Pure and Appl. Sci. (Math. & Stat.)*, 23E (2004) 155-163.

[7] R.Ponraj, Rajpal singh, R.Kala and S. Sathish Narayanan, *k*-prime cordial graphs, *J. Appl. Math. & Informatics*, 34 (3-4) (2016) 227 - 237.

[8] R.Ponraj, Rajpal Singh, and S.Sathish Narayanan, A note on 3-Prime cordial graphs, *Journal of Algorithms and Computation*, **48(1)**(2016), 45- 55.

[9] R.Ponraj, Rajpal singh and S. Sathish Narayanan, 4-prime cordiality of some classes of graphs, *Journal of algorithms and computation*, 48(1)(2016), 69- 79.

[10] R.Ponraj, Rajpal singh and S. Sathish Narayanan, 4-prime cordiality of some cycle related graphs, *Applications and Applied Mathematics: an international journal*, 12(1)(2017), 230-240.

Some Results on α-graceful Graphs

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Abstract: In this paper we have proved that the graph obtained by merging t consecutive vertices of two cycles C_{4r} and C_{4s} is α-graceful graph. We also proved that G_1 be an α-graceful graph and G_2 be a graceful graph joining by path P_n is graceful in addition we proved G_1 and G_2 be α-graceful graphs joining by path P_n is an α-graceful graph too.

Key Words: Cycle, path, graceful labeling, α-labeling.

AMS(2010): 05C78.

§1. Introduction

A.Rosa [1] defined α-labeling or (α-valuation) as a graceful labeling with the additional property that there is an integer k $(0 \leq k < E(G))$ such that for every $e = (x, y) \in E(G)$, either $f(x) \leq k < f(y)$ or $f(y) \leq k < f(x)$. If follows that such a k must be the smaller of the two vertex labels that yield the edge labeled 1.

In [4] Kaneria et al. proved that the α-labeling of double path union of some α-graceful graph like cycle, complete bipartite graph and path. We will consider a simple finite and undirected graph $G = (V, E)$ on $|V| = p$ vertices and $|E| = q$ edges. For a comprehensive bibliography of papers on graph labeling we have refereed Gallian [3]. Here we recall some definitions which are used in this paper.

Definition 1.1 *A function f is called graceful labeling of a graph $G = (V, E)$ if $f : V \longrightarrow \{0, 1, \cdots, q\}$ is injective and the induced function $f^\star : E \longrightarrow \{1, 2, \cdots, q\}$ defined as $f^\star(e) = |f(u) - f(v)|$ is bijective for every edge $e = (u, v) \in E$. A graph G is called graceful graph if it admits a graceful labeling.*

Definition 1.2 *A function f is called α-labeling of a graph $G = (V, E)$ if f is a graceful labeling for G and there exist an integer k $(0 \leq k < q - 1)$ such that for every $e = (x, y) \in E(G)$, either $f(x) \leq k < f(y)$ or $f(y) \leq k < f(x)$. A graph G with an α-labeling is necessarily bipartite graph. A graph which admits α-labeling, we call here α-graceful graph.*

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§2. Main Results

Theorem 2.1 *The graph obtained by merging t consecutive vertices of two cycle C_{4r} and C_{4s} is α-graceful, where $t \leq 2[min\{r, s\}]$ and $r, s, t \in N$.*

Proof Let $u_i (i \leq i \leq 4r)$ be consecutive vertices of the cycle C_{4r} and $v_1 = u_{4r-t+1}$, $v_2 = u_{4r-t+2}$, $\cdots$, $v_t = u_{4r}$, v_{t+1}, v_{t+2}, $\cdots$, v_{4s} be consecutive vertices of the second cycle C_{4s}, among the consecutive vertices v_1, v_2, $\cdots$, v_t of the second cycle are merged with the consecutive vertices u_{4r-t+1}, u_{4r-t+2}, $\cdots$, u_{4r} of the first cycle C_{4r} respectively.

Let G be the graph obtained by merging t consecutive vertices stated above of the cycle C_{4r} and C_{4s}, where $t \leq 2[min\{r, s\}]$ and $r, s, t \in N$.

The vertex labeling function $f : V(C_{4r}) \longrightarrow \{0, 1, \cdots, 4r\}$ is defined by

$$f(u_i) = \begin{cases} 4r - \frac{i-1}{2}, \forall i = 1, 3, \cdots, 4r - 1, \\ \frac{i-2}{2}, \forall i = 2, 4, \cdots, 2r, \\ \frac{i}{2}, \forall i = 2r + 2, 2r + 4, \cdots, 4r \end{cases}$$

is an α-graceful labeling for the cycle C_{4r}, where $k_1 = 2r$.

Also the vertex labeling function $g_i : V(C_{4s}) \longrightarrow \{0, 1, \cdots, 4s\}$ $(i = 1, 2)$ is defined by

$$g_1(v_j) = \begin{cases} 4s - \frac{j-1}{2}, \forall j = 1, 3, \cdots, 4s - 1, \\ \frac{j-2}{2}, \forall j = 2, 4, \cdots, 2s, \\ \frac{j}{2}, \forall j = 2s + 2, 2s + 4, \cdots, 4s. \end{cases}$$

$$g_2(v_j) = \begin{cases} \frac{j-1}{2}, \forall j = 1, 3, \cdots, 2s - 1, \\ \frac{j+1}{2}, \forall j = 2s + 1, 2s + 3, \cdots, 4s - 1, \\ 4s - \frac{j-2}{2}, \forall j = 2, 4, \cdots, 4s. \end{cases}$$

are α-graceful labeling for the cycle C_{4s}, where $k_2 = 2s$.

To define a vertex labeling function $h : V(G) \longrightarrow \{0, 1, \cdots, q\}$, where $q = 4(r + s) - t + 1$, we take following two cases.

Case 1. t is odd.

Define $h(u) = f(u)$ if $f(u) \leq 2r$ and $f(u) + 4s - t + 1$if $f(u) > 2r$ for $\forall u \in V(C_{4r})$, $h(v) = g_2(v) + 2r - \frac{t-1}{2}$ for $\forall v \in V(C_{4s}) - V(C_{4r})$.

Case 2. t is even.

Define $h(u) = f(u)$ if $f(u) \leq 2r$ and $f(u) + 4s - t + 1$ if $f(u) > 2r$ for $\forall u \in V(C_{4r})$, $h(v) = g_1(v) + 2r - \frac{t-2}{2}$ for $\forall v \in V(C_{4s}) - V(C_{4r})$.

Above defined labeling pattern give the vertex labels $0, 1, \cdots, r - 1, r + 1, r + 2, \cdots, 2r, q$, $q - 1, \cdots, q + 1 - 2r$ to the vertices of C_{4r} and the vertex labels $2r + 1, 2r + 2, \cdots, 2r + \lfloor \frac{2s-t}{2} \rfloor$, $2r + \lfloor \frac{2s+4-t}{2} \rfloor, 2r + \lfloor \frac{2s+6-t}{2} \rfloor, \cdots, q - 2r$ to the vertices of C_{4s} which are not common vertices of C_{4r} and C_{4s}.

So f is an injective map. Moreover it produces the edge labels $q, q-1, \cdots, q-4r+1$ to the edges of C_{4r} and the edge labels $1, 2, \cdots, q-4r$ to the edges of C_{4s}, which are not common edges of C_{4r} and C_{4s}. Thus $f^\star : E(G) \longrightarrow \{0, 1, \cdots, q\}$ defined by absolute difference of end vertices labels is a bijective map.

It is observed that $h(v_{4s}) = 2(r+s) + 1 - \frac{t-1}{2}$, $h(v_{4s-1}) = 2(r+s) - \frac{t-1}{2}$ in Case 1 and $h(v_{4s}) = 2(r+s) - \frac{t-2}{2}$, $h(v_{4s-1}) = 2(r+s) + 1 - \frac{t-2}{2}$ in the Case 2. These produce $h^\star(e = (v_{4s-1}, v_{4s})) = 1$. Take $k = \lceil \frac{q}{2} \rceil$. Now,

$$
\begin{aligned}
\left\lceil \frac{q}{2} \right\rceil &= \left\lceil \frac{4(r+s) - t + 1}{2} \right\rceil \\
&= 2(r+s) - \left\lfloor \frac{t-1}{2} \right\rfloor \\
&= \begin{cases} 2(r+s) - \frac{t-1}{2} & \text{when } t \text{ is odd} \\ 2(r+s) - \frac{t-2}{2} & \text{when } t \text{ is even} \end{cases} \\
&= \min\{h(v_{4s}), h(v_{4s-1})\}.
\end{aligned}
$$

Therefore, $k = \lceil \frac{q}{2} \rceil$ is non-negative integer $(0 \leq k < q)$, which satisfied for every $e = (x, y) \in E(G)$, $\min\{h(x), h(y)\} \leq k < \max\{h(x), h(y)\}$. Hence, G admits an α-graceful labeling h and so, it is an α-graceful graph. $\qquad\square$

Illustration 2.2   Graph obtained by merging 7 consecutive vertices of C_{16}, C_{20} and its α-graceful labeling are shown in Figure 1.
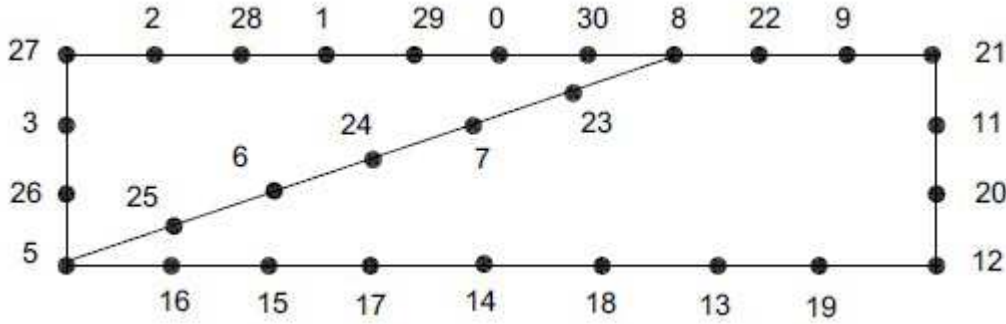
Figure 1 $C_{16} \bigcup C_{20}$ with seven consecutive common vertices and its α-labeling, here $k = 15$.

Theorem 2.3   *Let G_1 be an α-graceful graph, G_2 be a graceful graph and $q_1 = |E(G_1)|$, $q_2 = |E(G_2)|$. Let f_1 be an α-graceful labeling for G_1 and f_2 be a graceful labeling for G_2. Let k $(0 \leq k < q_1)$ be a non-negative integer such that for every $e_1 = (x, y) \in E(G_1)$, $\min\{f_1(x), f_1(y)\} \leq k < \max\{f_1(x), f_1(y)\}$. Let $v \in V(G_1)$ with $f_1(v) = k$ and $w = V(G_2)$ with $f_2(w) = 0$. Then the graph obtained by joining v with w by a path P_n is graceful.*

Proof   Take $V_1 = \{u \in V(G_1)/f_1(u) \leq k\}$ and $V_2 = V(G_1) - V_1$. Let $u_1 = v$, $u_2, u_3, \cdots, u_n = w$ be the vertices of path P_n. Let G be the graph obtained by joining v with w by a path P_n.

The vertex labeling function $g : V(P_n) \longrightarrow \{0, 1, \cdots, n-1\}$ defined by

$$g(u_i) = \begin{cases} \frac{i-1}{2} & \text{if } i \text{ is odd}, \\ q - \frac{i-2}{2} & \text{if } i \text{ is even}, \forall i = 1, 2, \cdots, n \end{cases}$$

is an α-graceful for the path P_n, where $k_1 = \lfloor \frac{n-1}{2} \rfloor$. Let $V_3 = \{u \in V(P_n)/f(u) \leq k_1\}$ and $V_4 = V(P_n) - V_3$. To define a vertex labeling function $h : V(G) \longrightarrow \{0, 1, \ldots, q\}$, where $q = q_1 + q_2 + n - 1$, we take following two cases.

Case 1. n is odd.

Let $h|_{V_1} = f_1|_{V_1}$, $h|_{V_2} = f_1|_{V_2} + q_2 + n - 1$, $h|_{V_3} = g|_{V_3} + k$, $h|_{V_4} = g|_{V_4} + k + q_2$ and $h|_{V(G_2)} = f_2|_{V(G_2)} + k + \frac{n-1}{2}$.

Case 2. n is even.

Let $h|_{V_1} = f_1|_{V_1}$, $h|_{V_2} = f_1|_{V_2} + q_2 + n - 1$, $h|_{V_3} = g|_{V_3} + k$, $h|_{V_4} = g|_{V_4} + k + q_2$ and $h|_{V(G_2)} = q_2 + k + \frac{n}{2} - f_2|_{V(G_2)}$.

It is observe that $h(v) = f_1(v) = g(u_1) + k = k$, which is common vertex of G_1 and P_n. Also

$$
\begin{aligned}
h(w) &= \begin{cases} g(w) + k = k + \lfloor \frac{n}{2} \rfloor, & \text{where } n \text{ is odd as } w \in V_3 \\ g(w) + k + q_2 = k + q_2 + \lfloor \frac{n}{2} \rfloor, & \text{where } n \text{ is even as } w \in V_4 \end{cases} \\
&= \begin{cases} k + \lfloor \frac{n}{2} \rfloor + f_2(w), & \text{where } n \text{ is odd as } f_2(w) = 0 \\ k + q_2 + \lfloor \frac{n}{2} \rfloor - f_2(w), & \text{where } n \text{ is even as } f_2(w) = 0 \end{cases},
\end{aligned}
$$

which is common vertex of P_n and G_2. Above defined labeling pattern give the vertex labels from $\{0, 1, \cdots, k, q_2 + k + n, \cdots, q\}$ to the vertices of G_1, the vertex labels $k + 1, k + 2, \cdots, k + \lfloor \frac{n-2}{2} \rfloor, k + q_2 + \lfloor \frac{n+2}{2} \rfloor, \ldots, q_2 + k + n - 1$ to the vertices of P_n except terminal vertices v, w and the vertex labels from $\{k + \lfloor \frac{n}{2} \rfloor, k + \lfloor \frac{n}{2} \rfloor + 1, \cdots, q_2 + k + \lfloor \frac{n}{2} \rfloor\}$ to the vertices of G_2.

So, h is an injective map. Moreover it produced the edge labels $q, q - 1, \cdots, q_2 + n$ to the graph G_1, the edge labels $q_2 + 1, q_2 + 2, \cdots, q_2 + n - 1$ to the graph P_n and the edge labels $1, 2, \cdots, q_2$ to the graph G_2. Thus $f^\star : E(G) \longrightarrow \{0, 1, \cdots, q\}$ defined by absolute difference of end vertices labels is a bijective map. Therefore G admits a graceful labeling h and so, it is a graceful graph. $\square$

Illustration 2.4 The graphs obtained by joining a vertex of an α-graceful graph $K_{4,3}$ and a vertex of a graceful graph C_7 by a path P_4 with their require labeling are shown in Figures 2 and 3.
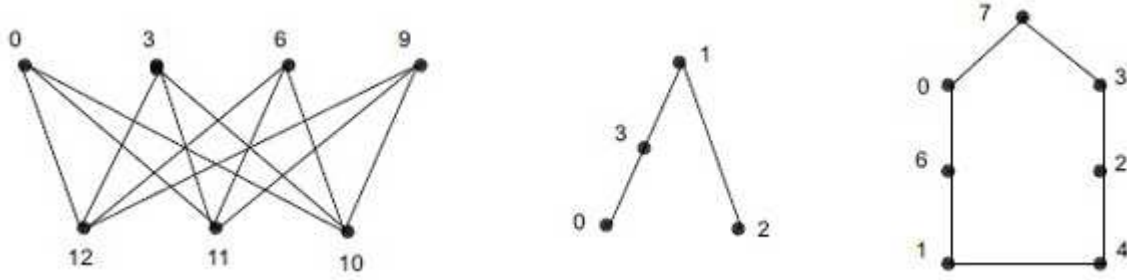
Figure 2 α-graceful labeling for $K_{4,3}$ and P_4,where $k = 9, 1$ respectively and graceful labeling for C_7.
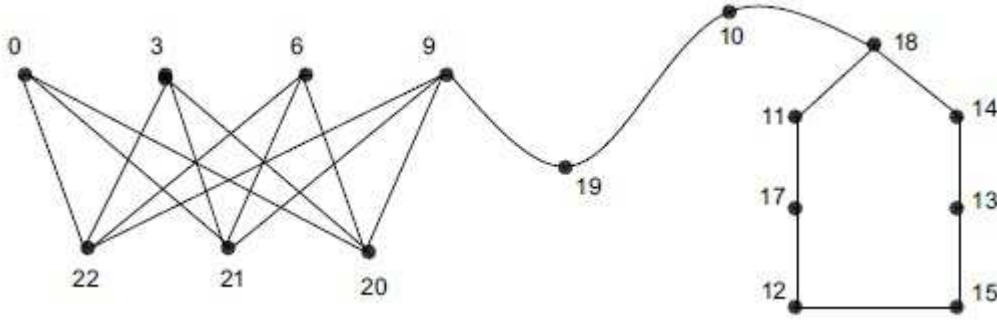
Figure 3 A graceful labeling for the graph obtained by joining $K_{4,3}$, C_7 by a path P_4.

Theorem 2.5 *Let G_1, G_2 be α-graceful graphs and $q_1 = |E(G_1)|$, $q_2 = |E(G_2)|$. Let f_1, f_2 be α-graceful labeling for G_1 and G_2 respectively. Let k_1 $(0 \le k_1 < q_1)$, k_2 $(0 \le k_2 < q_2)$ be two non-negative integers such that for every $e_i = (x_i, y_i) \in E(G_i)$, $\min\{f_i(x_i), f_i(y_i)\} \le k_i < \max\{f_i(x_i), f_i(y_i)\}$, $\forall\ i = 1, 2$. Let $v \in V(G_1)$ with $f_1(v) = k_1$ and $w \in V(G_2)$ with $f_2(w) = 0$. Then the graph obtained by joining v with w by a path P_n is α-graceful graph.*

Proof Take $V_1 = \{u \in V(G_1)/f_1(u) \le k_1\}$ and $V_2 = V(G_1) - V_1$. Let $u_1 = v, u_2, u_3, \cdots, u_n = w$ be the vertices of path P_n. Let G be the graph obtained by joining v with w by a path P_n. The vertex labeling function g on P_n defined in Theorem 2.3 is an α-graceful labeling for the path P_n, where $k_3 = \lfloor \frac{n-1}{2} \rfloor$. Let $V_3 = \{u \in V(P_n)/f(u) \le k_3\}$ and $V_4 = V(P_n) - V_3$.

We define a vertex labeling function $h : V(G) \longrightarrow \{0, 1, \cdots, q\}$, where $q = q_1 + q_2 + n - 1$ as it defined in Theorem 2.3.

It is observed that $h(v) = f_1(v) = g(u_1) + k = k$, which is common vertex of G_1 and P_n.

Also

$$
\begin{aligned}
h(w) &= \begin{cases} g(w) + k, & \text{where } n \text{ is odd as } w \in V_3 \\ g(w) + k + q_2, & \text{where } n \text{ is even as } w \in V_4 \end{cases} \\
&= \begin{cases} k + \lfloor \frac{n}{2} \rfloor + f_2(w), & \text{where } n \text{ is odd} \\ k + q_2 + \lfloor \frac{n}{2} \rfloor - f_2(w), & \text{where } n \text{ is even} \end{cases},
\end{aligned}
$$

which is common vertex of P_n and G_2. Above defined labeling pattern give rise a graceful labeling h to the graph G, as discussed in Theorem 2.3.

Since f_2 is an α-graceful labeling for G_2, Therefor $e = (u, v) \in E(G)$ such that $f_2^\star(e) = |f_2(u) - f_2(v)| = 1$. Science $\min\{f_2(u), f_2(v)\} \le k_2 < \max\{f_2(u), f_2(v)\}$ and $|f_2(u) - f_2(v)| = 1$, we must say $\min\{f_2(u), f_2(v)\} = k_2$ and $\max\{f_2(u), f_2(v)\} = k_2 + 1$.

Now

$$
\begin{aligned}
h^\star(e) &= |h(u) - h(v)| = |f_2(u) - f_2(v)| \\
&= f^\star(e) = 1.
\end{aligned}
$$

Take

$$
\begin{aligned}
k &= \min\{h(u), h(v)\} \\
&= \begin{cases} \min\{f_2(u) + k_1 + \lfloor \frac{n}{2} \rfloor, f_2(v) + k_1 + \lfloor \frac{n}{2} \rfloor\}, & \text{where } n \text{ is odd} \\ \min\{q_2 + k_1 + \lfloor \frac{n}{2} \rfloor - f_2(u), q_2 + k_1 + \lfloor \frac{n}{2} \rfloor - f_2(v)\}, & \text{where } n \text{ is even} \end{cases} \\
&= \begin{cases} k_1 + \lfloor \frac{n}{2} \rfloor + \min\{f_2(u), f_2(v)\}, & \text{where } n \text{ is odd} \\ q_2 + k_1 + \lfloor \frac{n}{2} \rfloor - \max\{f_2(u), f_2(v)\}, & \text{where } n \text{ is even} \end{cases} \\
&= \begin{cases} k_1 + k_2 + \lfloor \frac{n}{2} \rfloor, & \text{where } n \text{ is odd} \\ q_2 + k_1 - (k_2 + 1) + \lfloor \frac{n}{2} \rfloor, & \text{where } n \text{ is even} \end{cases} \\
&= \begin{cases} k_1 + k_2 + \frac{n-1}{2}, & \text{where } n \text{ is odd} \\ q_2 - k_2 + k_1 + \frac{n-2}{2}, & \text{where } n \text{ is even} \end{cases}.
\end{aligned}
$$

Then it is observed that for any $e = (x, y) \in E(G)$, $\min\{f(x), f(y)\} \le k < max\{f(x), f(y)\}$ and so, h is an α-graceful labeling for G. Hence G is an α-graceful graph. $\qquad \square$

Illustration 2.6 The graphs obtained by joining a vertex of an α-graceful graph $P_3 \times P_4$ and a vertex of another α-graceful graph C_8 by a path P_5 and their related labelings are shown in Figures 4 and 5.
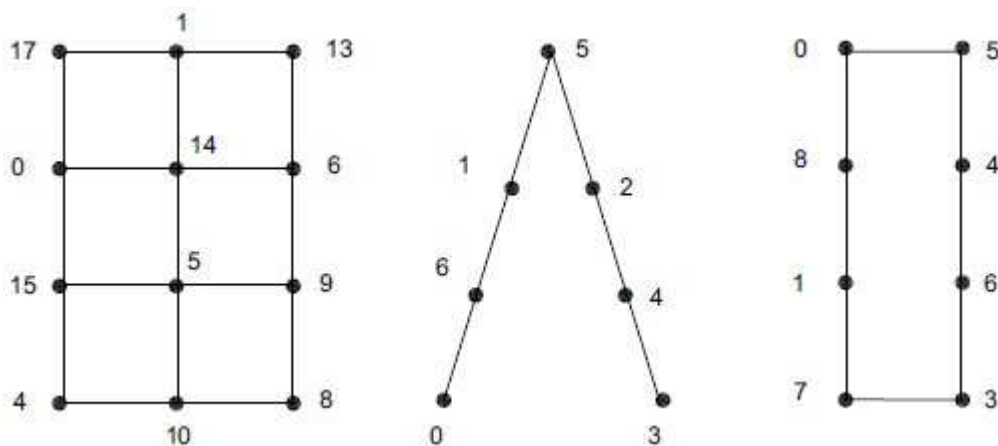
Figure 4 α-graceful labeling for $P_3 \times P_4$,P_7 and C_8, where $k = 8, 3, 4$ respectively.
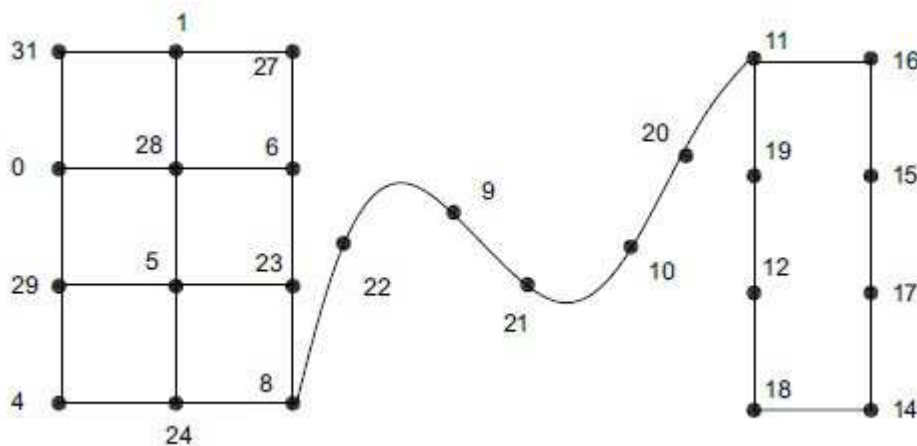

Figure 5 α-graceful labeling for a graph obtained by joining $P_3 \times P_4$ and C_8 by a path P_7.

Acknowledgements

References

[1] A. Rosa, On certain valuation of graph, *Theory of Graphs (Rome, July 1966)*, Goden and Breach, N. Y. and Paris, (1967) pp. 349-355.

[2] F. Harary, *Graph Theory*, Addition Wesley, Massachusetts, 1972.

[3] J. A. Gallian, A dynamic survey of graph labeling, *The Electronics Journal of Combinatorics*, 19, ♯DS6 (2013).

[4] V. J. Kaneria, Om Teraiya and Meera Meghpara, Double path union of α-graceful graph with its α-labeling, *J. of Graph Label.*, Vol.2, 2(2016), pp. 107-114.

Supereulerian Locally Semicomplete Multipartite Digraphs

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Abstract: A digraph D is *eulerian* if D is strongly connected and for every vertex $v \in V(D)$, $d^+(v) = d^-(v)$. Bang-Jensen and Thomassé gave the conjecture for a digraph D that if $\lambda(D) \geq \alpha(D)$, then D is supereulerian. Bang-Jensen and Maddaloni [Journal of Graph Theory, 79(2015)8-20] proved that this conjecture holds for every semicomplete multipartite digraph. In this paper, we generalized the above known results and show that this conjecture holds for every strong locally semicomplete multipartite digraph.

Key Words: Digraph, supereulerian digraph, semicomplete digraph, locally semicomplete multipartite digraph.

AMS(2010): 05C10, 05C25, 20B25.

§1. Introduction

A graph G is *eulerian* if G is connected without vertices of odd degree, and G is supereulerian if G has a spanning eulerian subgraph. The purpose of this paper is to investigate the digraph version of the supereulerian problem. Given a digraph D, its underling graph, denoted by $\overline{D}$, is gotten from D by overlooking the directions. Let $A(D)$ and $V(D)$ be the set of arcs and vertices in D, respectively. A *walk* in D is an alternating sequence $W = x_1 e_1 x_2 e_2 \ldots x_{k-1} e_{k-1} x_k$ of vertices x_i and arcs e_j from D such that $e_i = (x_i, x_{i+1}) \in A(D)$ for $i = 1, \ldots, k-1$. A walk is *closed* if $x_1 = x_k$. If all the arcs of a closed walk are distinct, we call it a *closed trail* (also called a *cycle*). If all the vertices of a walk W are distinct, we call it an (x_1, x_k)-*dipath*. A digraph $D = (V(D), A(D))$ is *strongly connected* if there is a (u, v)-dipath for any two vertices u and v. A digraph D is *weakly connected* if $\overline{D}$ is connected. Furthermore, D is said to be *eulerian* if D is strongly connected and for every vertex $v \in V(D)$, $d^+(v) = d^-(v)$. Equivalently, D is eulerian if and only if D itself is a spanning closed trail. A digraph D is *supereulerian* if D has a spanning eulerian subdigraph. If X and Y are disjoint subsets of $V(D)$, let $A(X, Y) := \{(x, y) \in A(D) : x \in X, y \in Y\}$ (sometime, $A(X, Y)$ is simply written as (X, Y)) and $d(X, Y) := |(X, Y)|$. A *circuit* C of a digraph D is a connected subdigraph of D such that

$d_C{}^+(v) = d_C{}^-(v) = 1$ for each vertex $v \in C$. Let $[n] = \{1, 2, \ldots, n\}$.

A *semicomplete digraph* is a digraph without nonadjacent vertices. A *locally semicomplete digraph* [3] is a digraph D that satisfies the following conditions: for every vertex x of D, the set $N^+(x)$ of vertices dominated by x (respectively, the set $N^-(x)$ of vertices that dominate x) induces a semicomplete digraph. Hence a semicomplete digraph is also locally semicomplete. If each vertex in a semicomplete digraph is blown up, then we get a semicomplete multipartite digraph. A digraph $D = (V, A)$ is *semicomplete multipartite* if there is a partition $V_1, V_2, \ldots, V_c$ of V into independent sets so that every vertex in V_i shares an arc with every vertex in V_j for $1 \leq i < j \leq c$.

Similarly, a *locally semicomplete multipartite digraph* D is obtained from a locally semicomplete digraph F with $V(F) = \{v_1, v_2, \ldots, v_c\}$ by blowing up each vertex $v_i \in V(F)$ into one independent set V_i in D, such that $N_D^\lambda(x) = V_{i_1} \cup \ldots \cup V_{i_p}$ for any $x \in V_i$ if and only if $N_F^\lambda(v_i) = \{v_{i_1}, \ldots, v_{i_p}\}$, where $\lambda \in \{+, -\}$ and $\{v_{i_1}, \ldots, v_{i_p}\} \subseteq V(F)$. If a digraph $D = (V, A)$ is locally semicomplete multipartite, then there is a partition $V_1, V_2, \ldots, V_c$ of V satisfying the following two conditions:

(1) each V_i, called a partite set, for $i \in [c]$ is an independent set;

(2) the set of vertices dominated by x (respectively, the set of vertices that dominate x) induces a semicomplete multipartite digraph for any vertex $x \in V$.

It can be seen that a semicomplete multipartite digraph is locally semicomplete multipartite but the inverse is not right.

Let $\kappa(D)$ and $\lambda(D)$ be the vertex-strong connectivity and the arc-strong connectivity of a digraph D, respectively. While $\alpha(D)$ and $\alpha'(D)$ denote the independence number and the matching number of D respectively, which equal to the independence number and the matching number of a graph $\overline{D}$, respectively.

An eulerian factor of D is a collection of arc-disjoint cycles spanning $V(D)$. There are much results about the graph to be supereulerian, for examples to see the surveys [7],[8],[12]. Contrary to the supereulerian properties of graphs, not much work has been done yet for supereulerian digraphs. Gutin [8] discussed supereulerian digraphs by using the connected (g, f)-factors; Hong et al. [10] and J. Bang-Jensen et al. [4] gave the degree sum conditions for a graph being supereulerian; Bang-Jensen and Thomassé gave the following conjecture in 2011, Bang-Jensen and Maddaloni [4] said the conjecture is unpublished.

Conjecture 1.1 *Let D be a digraph. If $\lambda(D) \geq \alpha(D)$, then D is supereulerian.*

Then by using the Hoffman's circulation theorem, the first author and Maddaloni [4] proved that for a digraph D if $\lambda(D) \geq \alpha(D)$, then D has an eulerian factor and the Conjecture 1.1 hold for all semicomplete multipartite digraphs. Algefari and Lai [2] proved that if $\lambda(D) \geq \alpha'(D) > 0$, then D has a spanning eulerian subdigraph. These two authors, Alsatami and Liu [1] gave the sufficient condition of symmetric digraphs to be supereulerian, and got the result that partially symmetric digraphs are supereulerian.

In this paper, we prove the Conjecture 1.1 for strong locally semicomplete multipartite digraphs.

§2. **Locally Semicomplete Multipartite Digraphs**

Lemma 2.1([4]) *Let D be a digraph. If $\lambda(D) \geq \alpha(D)$, then D has an Eulerian factor.*

From the definition of a locally semicomplete multipartite digraph, one has the following Lemma 2.2.

Lemma 2.2 *Let D be a locally semicomplete multipartite digraph. Let $x \in V(D)$ and v_1, v_2 be two distinct vertices in $N_D^+(x)$ (or $N_D^-(x)$), then each of the following results holds.*

(1) *either v_1 and v_2 belong to the same partite set in which there exists no arc between v_1 and v_2 or there exists at least an arc between v_1 and v_2;*

(2) *If v_1 and v_2 belong to the same partite set, then $N_D^-(v_1) = N_D^-(v_2)$ and $N_D^+(v_1) = N_D^+(v_2)$.*

Applying the similar method used in [4] we can prove the following Lemma 2.3.

Lemma 2.3 *Let D be a locally semicomplete multipartite digraph. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two vertex disjoint closed trails. If $A(\mathcal{E}_1, \mathcal{E}_2) \neq \emptyset$ and $A(\mathcal{E}_2, \mathcal{E}_1) \neq \emptyset$, then there exists a closed trail $\mathcal{E}$ of D such that $V(\mathcal{E}) = V(\mathcal{E}_1) \cup V(\mathcal{E}_2)$.*

Proof Consider the bipartite digraph B with partitions $V(\mathcal{E}_1)$, $V(\mathcal{E}_2)$ and arcs $A(\mathcal{E}_1, \mathcal{E}_2) \cup A(\mathcal{E}_2, \mathcal{E}_1)$. We have two distinct cases.

Case 1. Every vertex of B has positive in- and out- degree. Then, clearly, B contains a closed trail $\mathcal{E}$ that connects $\mathcal{E}_1$ and $\mathcal{E}_2$, so $\mathcal{E} \cup \mathcal{E}_1 \cup \mathcal{E}_2$ is the desired trail.

Case 2. There exists one vertex of B which has no positive in-degree or out-degree.

Let $v_1 v_2 \ldots v_h v_1$ be a spanning trail of $\mathcal{E}_1$ and let $w_1 w_2 \ldots w_k w_1$ be a spanning trail of $\mathcal{E}_2$. Without loss of generality, assume that there is a vertex, say x, of $\mathcal{E}_1$ such that x has no out-neighbors in $\mathcal{E}_2$. As $A(\mathcal{E}_1, \mathcal{E}_2) \neq \emptyset$, there exists another vertex, say v_s of $\mathcal{E}_1$ with at least one out-neighbor in $\mathcal{E}_2$. Assume $(v_s, w_t) \in A(\mathcal{E}_1, \mathcal{E}_2)$.

Now consider the out neighbors v_{s+1} and w_t of v_s. If v_{s+1} and w_t are in the same partite set V_i. By Lemma 2.2(2), $(w_{t-1}, v_{s+1}) \in A(\mathcal{E}_2, \mathcal{E}_1)$, thus, $\mathcal{E}_1 \cup \mathcal{E}_2 \cup \{(w_{t-1}, v_{s+1}), (v_s, w_t)\} \setminus \{(w_{t-1}, w_t), (v_s, v_{s+1})\}$ is the desired trail. Otherwise, by Lemma 2.2(1), there exists an arc between w_t and v_{s+1}. If $(w_t, v_{s+1}) \in A(\mathcal{E}_2, \mathcal{E}_1)$, then $\mathcal{E}_1 \cup \mathcal{E}_2 \cup \{(w_t, v_{s+1}), (v_s, w_t)\} \setminus \{(v_s, v_{s+1})\}$ is the desired trail. If $(w_t, v_{s+1}) \notin A(\mathcal{E}_2, \mathcal{E}_1)$, then $(v_{s+1}, w_t) \in A(\mathcal{E}_2, \mathcal{E}_1)$.

Next we consider two out neighbors v_{s+2} and w_t of v_{s+1}, by the similar discussion as considering v_{s+1} and w_t, one has that either D contains a closed trail $\mathcal{E}$ such that $V(\mathcal{E}) = V(\mathcal{E}_1) \bigcup V(\mathcal{E}_2)$ or there exists an arc $(v_{s+2}, w_t) \in A(\mathcal{E}_2, \mathcal{E}_1)$. If the former holds, the theorem is proved. Otherwise, $(v_{s+2}, w_t) \in A(\mathcal{E}_2, \mathcal{E}_1)$.

Repeatedly using this procedures, one of the following two results holds: there exists a closed trail $\mathcal{E}$ of D such that $V(\mathcal{E}) = V(\mathcal{E}_1) \bigcup V(\mathcal{E}_2)$ or every vertex in $V(\mathcal{E}_1)$ has the out neighbor w_t. Since x has no out-neighbors in $\mathcal{E}_2$, the result that every vertex in $V(\mathcal{E}_1)$ has the out neighbor w_t can not happen, thus, there exists a closed trail $\mathcal{E}$ of D such that $V(\mathcal{E}) = V(\mathcal{E}_1) \cup V(\mathcal{E}_2)$. $\square$

By the proof of Lemma 2.3, one has the following Lemma 2.4.

Lemma 2.4   *Let D be a locally semicomplete multipartite digraph and $\mathcal{E}_1, \mathcal{E}_2$ be two vertex disjoint closed trails. Let $(x, y) \in A(\mathcal{E}_1, \mathcal{E}_2)$ and $A(\mathcal{E}_2, \mathcal{E}_1) = \emptyset$, then $(v, y) \in A(\mathcal{E}_1, \mathcal{E}_2)$ for each $v \in V(\mathcal{E}_1)$.*

Theorem 2.5   *Let D be a locally semicomplete multipartite digraph. Then D is supereulerian if and only if it is strong and has an eulerian factor.*

Proof   If D is supereulerian, equivalently, there is a spanning eulerian subdigraph in D, then clearly it is strong and has an eulerian factor which consists of the union of arc disjoint cycles spanning $V(D)$.

Now assume that D is strong and has an eulerian factor $\mathcal{C}$, we will show that D has a spanning eulerian subdigraph. The following is a procedure to produce a spanning eulerian subdigraph of D for a given eulerian factor.

Form a minimal collection of vertex disjoint closed trails by merging those trails of $\mathcal{C}$ having common vertices. For any two closed trails $\mathcal{E}, \mathcal{F}$ in the collection with $A(\mathcal{E}, \mathcal{F}) \neq \emptyset$ and $A(\mathcal{F}, \mathcal{E}) \neq \emptyset$, join $\mathcal{E}$ and $\mathcal{F}$ into a closed trail as in Lemma 2.3.

Let $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_t$ be the collection of closed trails of D obtained after the first step is no more applicable. Note that all the trails have at least two vertices, and for any two distinct trails $\mathcal{E}_i$ and $\mathcal{E}_j$, at least one of $A(\mathcal{E}_i, \mathcal{E}_j)$ and $A(\mathcal{E}_j, \mathcal{E}_i)$ is empty. Let D' be the digraph with the set of vertices $\{a_1, \ldots, a_t\}$ and the set of arcs $\{(a_i, a_j) | A(\mathcal{E}_i, \mathcal{E}_j) \neq \emptyset$ for $1 \leq i, j \leq t\}$. Note that D' has no 2-circuit. By the fact that D is strong, D' is strong. Let $f : \{a_i, i \in [t]\} \to \{\mathcal{E}_i, i \in [t]\}$ be a bijective map such that $f(a_i) = \mathcal{E}_i$ for each $i \in [t]$. We has the following claim.

Claim 1. D' has a hamiltonian (directed) circuit.

Proof of Claim 1.   Let C be a circuit of D' with the maximum number of vertices. If $V(D') = V(C)$, then C is our desired circuit. So suppose by contradiction that $|V(C)| < t$. Since D' is strong, there exists a dipath Q with only two ends in C and Q has at least three vertices. Let Q be chosen such that the length of the shortest dipath P in C between the endpoints of Q is minimum. Let x, v, y be the first, second, and last vertex of Q (to see Fig.1). Let $P = \langle x, x_1, \cdots, y \rangle$ and $Q = \langle x, v, \cdots, y \rangle$.

Since C is a circuit with the maximum length, y can not be x_1 nor a vertex x. In deed, if $y = x_1$, then circuit, gotten from C by replacing P with Q, has larger length than that of C which is a contradiction.

If $y = x$, the out-neighbor of x in C is denoted by x_1. We will show that $(x_1, v) \in A(D')$. In fact, we consider v and x_1. Since $(x, v), (x, x_1) \in D'$ and D' has no 2-circuits, by Lemma **??**, there exist $\varepsilon_1 \in V(f(x)), \varepsilon_2 \in V(f(x_1))$ and $\varepsilon_3 \in V(f(v))$ such that $(\varepsilon_1, \varepsilon_2), (\varepsilon_1, \varepsilon_3) \in A(D)$. By the definition of the locally semicomplete multipartite digraph, either ε_2 and ε_3 belong to the same partite set or there exists an arc between ε_2 and ε_3 in D. If ε_2 and ε_3 belong to the same partite set in D. Since $f(v)$, as a closed trail of D, has at least two vertices and there exists a vertex, say $\varepsilon_4 \in V(f(v))$ such that $(\varepsilon_4, \varepsilon_3) \in A(D)$. By Lemma 2.2(2), $(\varepsilon_4, \varepsilon_2) \in A(D)$ which implies that $(v, x_1) \in A(D')$, then $C_1 := C \cup \{(x, v), (v, x_1)\} - \{(x, x_1)\}$ is the circuit of D' and $|C_1| \geq |C|$ which contradicts the choice of C. Now assume that there exists an arc between

ε_2 and ε_3 in D. If $(\varepsilon_3, \varepsilon_2) \in A(D)$, it implies that $(v, x_1) \in A(D')$ which is a contradiction. So $(\varepsilon_2, \varepsilon_3) \in A(D)$, it implies that $(x_1, v) \in A(D')$.
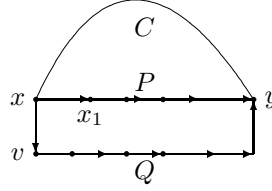
Fig.1 Circuit C with the dipath Q

By the same reason as considering v and x_1, then consider v and the out-neighbor, say x_2, of x_1 in $C \cap D'$. We get that $(x_2, v) \in A(D')$. Repeatedly using this procedures, we can derive that $(t, v) \in A(D')$ for every $t \in V(C)$. Let z be the in-neighbor of x in C. Note that $y = x$. By $(z, v) \in A(D')$, then $C_1 := C \cup Q \cup \{(z, v)\} - \{(z, x), (x, v)\}$ is the circuit of D' with $|C_1| \ge |C|$ which contradicts the choice of C. As a result, y can not equal x_1 nor a vertex x.

Since (x, v) and (x, x_1) are distinct arcs of D', there exists $t_1, t_2 \in V(f(x))$, $z_1 \in V(f(v))$ and $z_2 \in V(f(x_1))$ such that $(t_1, z_1) \in A(f(x), f(v))$ and $(t_2, z_2) \in A(f(x), f(x_1))$ in D. By Lemma 2.4 and no 2-circuit in D', $(t_1, z_2) \in A(f(x), f(x_1))$ in D. Since z_1 and z_2 are the out-neighbors of t_1 in D. By Lemma 2.2, either z_1 and z_2 are in the the same partite set or there exists an arc between z_1 and z_2 in D. The later case can not happen. In fact, the arc between z_1 and z_2 in D implies that there exists an arc between v and x_1 in D'. If $(v, x_1) \in D'$, then $C \bigcup \{(x, v), (v, x_1)\} - \{(x, x_1)\}$ is the circuit with the length longer than that of C, it contradicts with the choice of C. If $(x_1, v) \in D'$, then it contradicts with the shortest dipath P in C. As a result, there exists no arc between v and x_1 in D'. Now assume that z_1 and z_2 are in the same partite set in D, since $f(v)$ contains at least two vertices and $f(v)$ is a closed trail of D, there exists $z_3 \in f(v)$ such that $(z_1, z_3) \in A(f(v)) \subseteq D$, by Lemma 2.2(2), $(z_2, z_3) \in A(D)$, it implies that $(x_1, v) \in A(D')$ which is a contradiction. The proof of Claim is finished.

By Claim 1, D' has a hamiltonian circuit C. Without loss of generality, let $C = a_1 a_2 \cdots a_t a_1$. By Lemma 2.4 and no 2-circuits in D', there exists $w_i \in \mathcal{E}_i$ for each $i \in [t]$ such that $(v_{i-1}, w_i) \in A(\mathcal{E}_{i-1}, \mathcal{E}_i)$ for each $v_{i-1} \in V(\mathcal{E}_{i-1})$ and $2 \le i \le t$, and $(v_t, w_1) \in A(\mathcal{E}_t, \mathcal{E}_1)$ for each $v_t \in V(\mathcal{E}_t)$. As a result, $w_1 w_2 \cdots w_t w_1$ is a circuit of D. Then D is supereulerian with a eulerian digraph $(\bigcup_{i=1}^{t} \mathcal{E}_i) \cup \{w_1 w_2 \cdots w_t w_1\}$. $\square$

From Theorem 2.5 and Lemma 2.1, we can derive the following Theorem 2.6.

Theorem 2.6 *Let D be a strong locally semicomplete multipartite digraph. If $\lambda(D) \ge \alpha(D)$, then D is supereulerian.*

Since a semicomplete multipartite digraph is the strong locally semicomplete multipartite digraph, thus the following known result can be gotten directly.

Corollary 2.7([4]) *Let D be a semicomplete multipartite digraph. If $\lambda(D) \ge \alpha(D)$, then D is supereulerian.*

References

[1] Mansour J. Algefari, Khalid A. Alsatami, Hong-Jian Lai, Juan Liu, Supereulerian digraphs with given local structures, *Information Processing Letters*, 116(2016) 321-326.

[2] Mansour J. Algefari and Hong-Jian Lai, Supereulerian digraphs with large arc-strong connectivity, *Journal of Graph Theory*, 81(4)(2016) 393-402.

[3] J. Bang-Jensen, Locally semicomplete digraphs: A generalization of tournaments, *Journal of Graph Theory*, 14(3)(1990) 371-390.

[4] J. Bang-Jensen and A. Maddaloni, Sufficient conditions for a digrapht to be supereulerian, *Journal of Graph Theory*, 79(1)(2015) 8-20.

[5] J. Bang-Jensen and G.Gutin, *Digraphs: Theory, Algorithms and Applications*, 2nd., Springer-Verlag, London, 2009.

[6] J. Bang-Jensen and S.Bessy, (Arc-)disjoint flows innet works, *Theor. Comput. Sci.*, 526(2014) 28-40.

[7] P. A. Catlin, Supereulerian graphs: A survey, *J. Graph Theory*, 16(2)(1992) 177-196.

[8] Z. H. Chen H.-J. Lai, Reduction techniques for super-Eulerian graphs and related topics-a survey, *Combinatorics and Graph Theory'95*, World Sci. Publishing, RiverEdge, NJ, 1(1995) 53-69.

[9] G. Gutin, Connected (g, f)-factors and supereulerian digraphs, *Ars Combin.*, 54(2000) 311-317.

[10] Y.M. Hong, H.-J. Lai, Q.H. Liu, Supereulerian digraphs, *Discrete Math.*, 330(2014) 87-95.

[11] L. Han, H.-J. Lai, L. Xiong, and H. Yan, The Chvátal-Erdös, condition for supereulerian graphs and the Hamiltonian index, *Discrete Math.*, 310(2010) 2082-2090.

[12] H.-J. Lai, Y. Shao and H. Yan, An update on supereulerian graphs, *WSEAS Trans Math.*, 12(2013) 926-940.

Non-Existence of Skolem Mean Labeling for Five Star

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Abstract: In this paper, we prove if $\ell \leq m < n$, the five star $G = K_{1,\ell} \bigcup K_{1,\ell} \bigcup K_{1,\ell} \bigcup K_{1,m} \bigcup K_{1,n}$ is not a skolem mean graph if $|m - n| > 4 + 3\ell$ for $\ell = 2, 3, \cdots$ and $m = 2, 3, \cdots$.

Key Words: Labeling, Smarandachely edge m-labeling f_S^*, skolem mean labeling, skolem mean graph, star.

AMS(2010): 05C78.

§1. Introduction

Let G be a simple graph. A vertex labeling of G is an assignment $f : V(G) \rightarrow \{1, 2, 3, \cdots, p+q\}$ be an injection. For a vertex labeling f, the induced Smarandachely edge m-labeling f_S^* for an edge $e = uv$, an integer $m \geq 2$ is defined by $f_S^*(e) = \left\lceil \dfrac{f(u) + f(v)}{m} \right\rceil$. Then f is called a Smarandachely super m-mean labeling if $f(V(G)) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, 3, \ldots, p + q\}$. Particularly, in the case of $m = 2$, we know that

$$
f^*(e) = \begin{cases} \dfrac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even;} \\ \dfrac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd.} \end{cases}
$$

Such a labeling is usually called a mean labeling. A graph that admits a Smarandachely super mean m-labeling is called a Smarandachely super m-mean graph, particularly, a skolem mean graph if $m = 2$

In [2], we proved the following theorems to study the existence of skolem mean graphs. We proved the three star $K_{1,\ell} \bigcup K_{1,m} \bigcup K_{1,n}$ is a skolem mean graph if $|m - n| = 4 + \ell$ for $\ell = 1, 2, 3, \cdots$; $m = 1, 2, 3, \cdots$ and $\ell \leq m < n$. The three star $K_{1,\ell} \bigcup K_{1,m} \bigcup K_{1,n}$ is not a skolem mean graph if $|m - n| > 4 + \ell$ for $\ell = 1, 2, 3, \cdots$; $m = 1, 2, 3, \cdots$ and $\ell \leq m < n$. The four star $K_{1,\ell} \bigcup K_{1,\ell} \bigcup K_{1,m} \bigcup K_{1,n}$ is a skolem mean graph if $|m - n| = 4 + 2\ell$ for $\ell = 2, 3, \cdots$; $m = 2, 3, \cdots$ and $\ell \leq m < n$. The four star $K_{1,\ell} \bigcup K_{1,\ell} \bigcup K_{1,m} \bigcup K_{1,n}$ is not a skolem mean graph if $|m - n| > 4 + 2\ell$ for $\ell = 2, 3, \cdots$; $m = 2, 3, \cdots$ and $\ell \leq m < n$. In [3]. The five star $K_{1,\ell} \bigcup K_{1,\ell} \bigcup K_{1,\ell} \bigcup K_{1,m} \bigcup K_{1,n}$ is a skolem mean graph if $|m - n| = 4 + 3\ell$ for $\ell = 2, 3, \cdots$;

$m = 2, 3, \cdots$ and $\ell \leq m < n$. Further, we prove the four star $K_{1,1} \bigcup K_{1,1} \bigcup K_{1,m} \bigcup K_{1,n}$ is a skolem mean graph if $|m - n| = 7$ for $m = 1, 2, 3, \cdots$ and $1 \leq m < n$; The four star $K_{1,1} \bigcup K_{1,1} \bigcup K_{1,m} \bigcup K_{1,n}$ is not a skolem mean graph if $|m - n| > 7$ for $m = 1, 2, 3, \cdots$ and $1 \leq m < n$; The five star $K_{1,1} \bigcup K_{1,1} \bigcup K_{1,1} \bigcup K_{1,m} \bigcup K_{1,n}$ is a skolem mean graph if $|m - n| = 8$ for $m = 1, 2, 3, \cdots$ and $1 \leq m < n$.

Definition 1.1   *The five star is the disjoint union of $K_{1,a}, K_{1,b}, K_{1,c}, K_{1,d}$ and $K_{1,e}$ and is denoted by $K_{1,a} \bigcup K_{1,b} \bigcup K_{1,c} \bigcup K_{1,d} \bigcup K_{1,e}$.*

§2. Main Result

Theorem 2.1   *The five star $G = K_{1,\ell} \bigcup K_{1,\ell} \bigcup K_{1,\ell} \bigcup K_{1,m} \bigcup K_{1,n}$ is not a skolem mean graph if $|m - n| > 4 + 3\ell$ for $\ell = 2, 3, \cdots$ and $m = 2, 3, \cdots$.*

Proof Let $G = 4K_{1,2} \bigcup K_{1,13}$, where $V(G) = \{v_{i,j} : 1 \leq i \leq 4; 0 \leq j \leq 2\} \bigcup \{v_{5,j} : 0 \leq j \leq 13\}$ and $E(G) = \{v_{i,0} : v_{i,j} : 1 \leq i \leq 4; 1 \leq j \leq 2\} \bigcup \{v_{5,0}v_{5,j} : 1 \leq j \leq 13\}$. Then, $p = 26$ and $q = 21$. Suppose G is a skolem mean graph. Then there exists a function f from the vertex set of G to $\{1, 2, 3, \cdots p\}$ such that the induced map f^* from the edge set of G to $\{2, 3, 4, \cdots p\}$ defined by

$$f^*(e = uv) = \begin{cases} \dfrac{f(u) + f(v)}{2} & \text{if} f(u) + f(v) \text{is even} \\ \dfrac{f(u) + f(v) + 1}{2} & \text{if} f(u) + f(v) \text{is odd} \end{cases}$$

then the resulting edges get distinct labels from the set $\{2, 3, \cdots p\}$.

Let $t_{i,j}$ be the label given to the vertex $v_{i,j}$ for $1 \leq i \leq 4; 0 \leq j \leq 2$ and $v_{5,j}$ for $0 \leq j \leq 13$ and $X_{i,j}$ be the corresponding edge label of the edge $v_{i,0}v_{i,j}$ for $1 \leq i \leq 5; 0 \leq j \leq 2$ and $v_{5,0}v_{5,j}$ for $1 \leq j \leq 13$.

Let us first consider the case that $t_{5,0} = 26$. If $v_{5,j} = 2n$ and $t_{5,k} = 2n + 1$ for some n and for some j and k then $f^*(v_{5,0}v_{5,j}) = \dfrac{26 + 2n}{2} = 13 + n = f^*(v_{5,0}v_{5,k})$. This is not possible as f^* is a bijection.

Therefore the thirteen vertices $t_{5,j}$ for $1 \leq j \leq 13$ are among the 13 numbers (1 or 2), (3 or 4), (5 or 6), (7 or 8), (9 or 10), (11 or 12), (13 or 14), (15 or 16), (17 or 18), (19 or 20), (21 or 22), (23 or 24) and 25.

Primarily, $t_{5,2}$ is either of 23 or 24. We first consider the case that $t_{5,2} = 23$.

Case 1. $t_{5,2} = 23$.

We have $t_{5,0} = 26$; $t_{5,1} = 25$; $t_{5,2} = 23$; $t_{1,0} = 24$. Now 24 is a label of either $t_{i,0}$ for $1 \leq i \leq 4$ or $t_{i,j}$ for $1 \leq i \leq 4; 1 \leq j \leq 2$. That is 24 is a label of pendent or non pendent vertex in a $k_{1,2}$ component of G. Let us assume that $t_{1,0} = 24$.

Subcase 1.1 $t_{1,0} = 24$.

We have $t_{5,0} = 26$; $t_{5,1} = 25$; $t_{5,2} = 23$; $t_{1,0} = 24$. If $t_{1,0} = 24$ then $t_{1,1}$ take the values 1 or 2. As $t_{1,1} \geq 3$ would imply that $X_{1,1} \geq 14$ this is not possible . The corresponding edge labels are $X_{1,1} = 13$.

Next $t_{5,3}$ is either 21 or 22. If $t_{5,3} = 21$ then $t_{2,0} = 22$. If $t_{2,1} = 3$ or 4 then $X_{2,1} = \dfrac{22 + 3 \text{ or } 4}{2} = 13$ this is not possible.

Similarly, if $t_{5,3} = 22$ then $t_{2,0} = 21$. Then $t_{2,1}$ take the value 3 or 4. The corresponding edge labels are $X_{2,1} = 12$, $X_{1,1} = 13$.

If $t_{2,2} \geq 5$ then $X_{2,2} \geq 14$ this is not possible. Hence it is not possible that $t_{1,0} = 24$. That is 24 is not a label of a non-pendent vertex in $k_{1,2}$ component of G. Next we consider the case that 24 is a label of a pendent vertex in a $k_{1,2}$ component of G. Let us assume that $t_{1,1} = 24$.

Subcase 1.2 $t_{1,1} = 24$.

We have $t_{5,0} = 26$; $t_{5,1} = 25$; $t_{5,2} = 23$; $t_{1,1} = 24$. If $t_{1,0} \geq 3$ then $X_{1,1} \geq 14$. This is not possible. Hence the value of $t_{1,0}$ is 1 or 2.

First, $t_{1,0} = 1$ or 2. We have $t_{5,0} = 26$; $t_{5,1} = 25$; $t_{5,2} = 23$; $t_{1,1} = 24$. Then $X_{1,1} = 13$. Now $t_{5,3}$ is either of 21 or 22.

Next case let, $t_{5,3} = 21$ and hence $t_{2,1} = 22$. If $t_{2,0} \geq 5$ then $X_{2,1} \geq 14$. This is not possible. If $t_{2,0} = 3 \text{ or } 4$ then $X_{2,1} = \left(\dfrac{26 + 3 \text{ or } 4}{2}\right) = 15$ this is not possible.

Suppose $t_{5,3} = 22$ and hence $t_{2,1} = 21$. We have $t_{5,0} = 26$; $t_{5,1} = 25$; $t_{5,2} = 23$; $t_{1,1} = 24$; $t_{1,0} = 1$ or 2; $t_{2,1} = 21$; $t_{2,0} = 3$. Then $X_{1,1} = 13$, $X_{2,1} = 12$. Now $t_{5,4}$ is either of 19 or 20.

Consider the case that $t_{5,4} = 19$ hence $t_{3,1} = 20$. We have $t_{5,0} = 26$; $t_{5,1} = 25$; $t_{5,2} = 23$; $t_{1,1} = 24$; $t_{5,3} = 22$; $t_{2,1} = 21$; $t_{2,0} = 3$. Here the value $t_{3,0} \geq 4$ then $X_{3,1} \geq 13$ this is not possible. If $t_{5,4} = 20$, then $t_{3,1} = 19$. Notice that $t_{5,0} = 26$; $t_{5,1} = 25$; $t_{5,2} = 23$; $t_{1,1} = 24$; $t_{5,3} = 22$; $t_{2,1} = 21$; $t_{2,0} = 3$. Here the value $t_{3,0} \geq 4$ then $X_{3,1} \geq 12$. This is not possible. Hence $t_{5,4} \neq 19$.

Similarly $t_{5,4} \neq 20$; $t_{5,3} \neq 22$; $t_{5,3} \neq 21$. Hence $t_{1,0} \neq 1 \text{ or } 2$ therefore $t_{1,1} \neq 24$; $t_{5,2} \neq 23$.

Case 2. $t_{5,2} = 24$.

Now 23 is a label of either $t_{i,0}$ for $1 \leq i \leq 4$ or $t_{i,j}$ for $1 \leq i \leq 4$; $1 \leq j \leq 2$; that is 23 is a label of pendent or non-pendent vertex in a $K_{1,2}$ component of G.

Subcase 2.1 $t_{1,0} = 23$.

We have $t_{5,0} = 26$; $t_{5,1} = 25$; $t_{5,2} = 24$; $t_{1,0} = 23$). If $t_{1,0} = 23$ then $t_{1,1}$ and $t_{1,2}$ take the values of 1 and 2 or 3 as $t_{1,1} \geq 4$ would imply that $X_{1,1} \geq 14$ is not possible. The corresponding edge labels are $X_{1,1} = 12$; $X_{1,2} = 13$.

Now $t_{5,3}$ is either of 21 or 22. If $t_{5,3} = 21$ then $t_{2,0} = 22$ then $X_{5,3} = \dfrac{26 + 21}{2} = 24$ and $t_{2,j} \geq 4$ and this is not possible. As $t_{2,j} \geq 4$ would imply that $X_{2,j} \geq 13$ and this not possible.

Similarly $t_{5,3} = 22$ then $X_{5,3} = \dfrac{26 + 22}{2} = 24$; $t_{2,0} = 21$ and also $t_{2,j} \geq 4$ this is not possible. As $t_{2,j} \geq 4$ would imply that $X_{2,j} \geq 13$ and this not possible.

Hence, it is not possible that $t_{1,0} = 23$ that is 23 is not a label of non-pendent vertex in $K_{1,2}$ component of G.

Next we consider the case that 23 is a label of a pendent vertex in a $K_{1,2}$ component of G. Let us assume that $t_{1,1} = 23$.

Subcase 2.2 $t_{1,1} = 23$.

We have $t_{5,0} = 26$; $t_{5,1} = 25$; $t_{5,2} = 24$; $t_{1,1} = 23$. If $t_{1,0} \geq 4$ then $X_{1,1} \geq 14$ and this is not possible. Hence the value of $t_{1,0}$ can either be 1 or 2 or 3. There exist two cases, i.e., $t_{1,0} = 1$ and $t_{1,0} = 2$ or 3.

Subcase 2.2.1 $t_{1,0} = 1$.

We have $t_{5,0} = 26$; $t_{5,1} = 25$; $t_{5,2} = 24$; $t_{1,0} = 1$; $t_{1,1} = 23$. Then $X_{1,1} = 12$. Now $t_{5,3}$ is either of 21 or 22.

Let $t_{5,3} = 21$ hence $t_{2,1} = 22$. If $t_{2,0} \geq 5$ then $X_{2,1} \geq 14$ and is not possible. If $t_{2,0} = 2$ then $X_{2,1} = \dfrac{26 + 2}{2} = 12$ and this is not possible. Hence $t_{2,0}$ is either of 3 or 4. We have $t_{5,0} = 26; t_{5,1} = 25; t_{5,2} = 24$; $t_{1,0} = 1$; $t_{5,3} = 21$; $t_{2,1} = 22$; $t_{2,0} = 3$ or 4 then $X_{1,1} = 12$; $X_{2,1} = 13$.

Now $t_{5,4}$ is either 19 or 20. Assume $t_{5,4} = 19$ Hence $t_{3,1} = 20$. If $t_{3,0} \geq 5$ then $X_{3,1} \geq 13$ and is not possible. Hence $t_{3,0}$ is 2. Notice that $t_{5,0} = 26$; $t_{5,1} = 25$; $t_{5,2} = 24$; $t_{5,3} = 21$; $t_{1,0} = 1$; $t_{2,1} = 22$; $t_{2,0} = 3$ or 4; $t_{3,1} = 20$; $t_{3,0} = 2$ then $X_{1,1} = 12$; $X_{2,1} = 13$; $X_{3,1} = 11$.

Now $t_{5,5}$ is either 17 or 18. Consider $t_{5,5} = 17$. Hence $t_{4,1} = 18$. We have $t_{5,0} = 26$; $t_{5,1} = 25$; $t_{5,2} = 24$; $t_{5,3} = 21$; $t_{5,4} = 19$; $t_{1,0} = 1$; $t_{1,1} = 23$; $t_{2,1} = 22$; $t_{2,0} = 3$ or 4; $t_{3,0} = 2$; $t_{3,1} = 20$; $t_{4,1} = 18$. Here the value $t_{4,0} \geq 5$ then $X_{4,1} \geq 12$, which is not possible.

Let $t_{5,5} = 18$ and hence $t_{4,1} = 17$. We have $t_{5,0} = 26$; $t_{5,1} = 25$; $t_{5,2} = 24$; $t_{5,3} = 21$; $t_{5,4} = 19$; $t_{5,5} = 18$; $t_{1,0} = 1$; $t_{1,1} = 23$; $t_{2,1} = 22$; $t_{2,0} = 3$ or 4; $t_{3,0} = 2$; $t_{3,1} = 20$; $t_{4,1} = 17$.

If the value $t_{4,0} \geq 5$ then $X_{4,1} \geq 11$, which is not possible. Hence $t_{5,4} \neq 19$. Similarly we can prove $t_{5,4} \neq 20$ and therefore $t_{5,3} \neq 21$.

Consider the case that $t_{5,3} = 22$ and hence $t_{2,1} = 21$. If $t_{2,0} \geq 6$ then $X_{2,1} \geq 14$ and is not possible. Hence the value of $t_{2,0}$ can either of 4 or 5.

First we consider $t_{2,0} = 4$ or 5. We have $t_{5,0} = 26$; $t_{5,1} = 25$; $t_{5,2} = 24$; $t_{5,3} = 22$; $t_{1,1} = 23$; $t_{1,0} = 1$; $t_{2,1} = 21$; $t_{2,0} = 4$ or 5, then $X_{1,1} = 12$ and $X_{2,1} = 13$.

Now $t_{5,4}$ is either 19 or 20. Considering, $t_{5,4} = 19$ and $t_{3,1} = 20$. If $t_{3,0} \geq 7$ then $X_{3,1} \geq 14$ and is not possible. Hence the value of $t_{3,0}$ can either be 2 or 6.

If $t_{3,0} = 6$ then $X_{3,1} = \dfrac{20 + 6}{2} = 13$, which is not possible. Hence $t_{3,0}$ is 2. Notice that $t_{5,0} = 26$; $t_{5,1} = 25$; $t_{5,2} = 24$; $t_{5,3} = 22$; $t_{1,1} = 23$; $t_{1,0} = 1$; $t_{2,1} = 21$; $t_{2,0} = 4$ or 5; $t_{3,1} = 20$; $t_{3,0} = 2$.

Now $t_{5,5}$ is either 17 or 18. Let us consider $t_{5,5} = 17$ and $t_{4,1} = 18$. Notice that $t_{5,0} = 26$; $t_{5,1} = 25$; $t_{5,2} = 24$; $t_{5,3} = 22$; $t_{1,1} = 23$; $t_{1,0} = 1$; $t_{2,1} = 21$; $t_{2,0} = 4$ or 5; $t_{3,1} = 20$; $t_{3,0} = 2$; $t_{4,1} = 18$.

Here the value $t_{4,0} = 3$ then $X_{4,1} = \dfrac{18 + 3}{2} = 11$, which is not possible.

Now $t_{5,5}$ is either 18 or 17. Let $t_{5,5} = 18$ and $t_{4,1} = 17$. Notice that $t_{5,0} = 26$; $t_{5,1} = 25$; $t_{5,2} = 24$; $t_{5,3} = 22$; $t_{1,1} = 23$; $t_{1,0} = 1$; $t_{2,1} = 21$; $t_{2,0} = 4$ or 5; $t_{3,1} = 20$; $t_{3,0} = 2$; $t_{4,1} = 17$; $t_{4,0} = 3$.

Now $t_{5,6}$ is either 15 or 16. If $t_{5,6} = 15$ and $t_{5,1} = 16$, we have $t_{5,0} = 26; t_{5,1} = 25$; $t_{5,2} = 24$; $t_{5,3} = 22$; $t_{1,1} = 23$; $t_{1,0} = 1$; $t_{2,1} = 21$; $t_{2,0} = 4$ or 5; $t_{3,1} = 20$; $t_{3,0} = 2$; $t_{4,1} = 17$; $t_{4,0} = 3$; $t_{5,1} = 16$. Here the value of $t_{5,0} \geq 6$. This is not possible.

If $t_{5,6} = 16$ and $t_{5,1} = 15$, we have $t_{5,0} = 26$; $t_{5,1} = 25$; $t_{5,2} = 24$; $t_{5,3} = 22$; $t_{1,1} = 23$; $t_{1,0} = 1$; $t_{2,1} = 21$; $t_{2,0} = 4$or 5; $t_{3,1} = 20$; $t_{3,0} = 2$; $t_{4,1} = 17$; $t_{4,0} = 3$; $t_{5,1} = 15$. Here the value of $t_{5,0} \geq 6$. This is not possible. Hence $t_{5,4} \neq 19$.

Similarly $t_{5,4} \neq 20$ and $t_{2,0} \neq 4 or 5$. Therefore $t_{5,3} \neq 18$. Hence $t_{1,0} \neq 1$.

Subcase 2.2.2 $t_{1,0} = 2$ or 3.

In this case, we have $t_{5,0} = 26$; $t_{5,1} = 25$; $t_{5,2} = 24$; $t_{1,1} = 23$. Then $X_{1,1} = 13$.

Now $t_{5,3}$ is either 21 or 22. If $t_{5,3} = 21$ and $t_{2,1} = 22$. If $t_{2,0} \geq 4$ then $X_{2,1} \geq 13$. This is not possible. Hence $t_{2,0}$ is 1. Notice that $t_{5,0} = 26$; $t_{5,1} = 25$; $t_{5,2} = 24$; $t_{5,3} = 21$; $t_{1,1} = 23$; $t_{1,0} = 2$ or 3; $t_{2,0} = 1$.

Now $t_{5,4}$ is either 19 or 20. Suppose $t_{5,4} = 19$ and $t_{3,1} = 20$. Notice that $t_{5,0} = 26$; $t_{5,1} = 25$; $t_{5,2} = 24$; $t_{5,3} = 21$; $t_{1,1} = 23$; $t_{1,0} = 2$ or 3; $t_{2,0} = 1$; $t_{5,4} = 19$; $t_{3,1} = 20$. Here the value of $t_{3,0} \geq 4$, which is not possible.

Let $t_{5,4} = 20$ and $t_{3,1} = 19$. Notice that $t_{5,0} = 26$; $t_{5,1} = 25$; $t_{5,2} = 24$; $t_{5,3} = 21$; $t_{1,1} = 23$; $t_{1,0} = 2$ or 3; $t_{2,0} = 1$; $t_{5,4} = 19$; $t_{3,1} = 20$. Here the value of $t_{3,0} \geq 4$, which is not possible. Hence $t_{5,3} \neq 21$.

Similarly $t_{5,3} \neq 22$ and $t_{5,4} \neq 19$; $t_{5,4} \neq 20$ therefore $t_{1,0} \neq 2$ or 3. Hence $t_{5,2} \neq 24$. Therefore $t_{5,0} \neq 26$ and hence $t_{5,1} \neq 25$. We have proved that if $t_{5,0} = 26$ the five star $G = 4K_{1,2} \bigcup K_{1,13}$ does not admit a skolem mean labelling .

Similarly, we can prove the result for other values of $t_{5,0}$. Hence the five star

$$
\begin{aligned}
G &= K_{1,\ell} \bigcup K_{1,\ell} \bigcup K_{1,\ell} \bigcup K_{1,m} \bigcup K_{1,n} \\
&= K_{1,2} \bigcup K_{1,2} \bigcup K_{1,2} \bigcup K_{1,2} \bigcup K_{1,13} \\
&= 4K_{1,2} \bigcup K_{1,13}
\end{aligned}
$$

is not a skolem mean graph. That is G is not a skolem mean graph if $|m - n| = 5 + 3\ell$.

In a similar way, we can prove that $G = 4K_{1,2} \bigcup K_{1,14}$ is not a skolem mean graph if $|m - n| = 6 + 3\ell$. Hence on generalizing, $G = K_{1,\ell} \bigcup K_{1,\ell} \bigcup K_{1,\ell} \bigcup K_{1,m} \bigcup K_{1,n}$ is not a skolem mean graph if $|m - n| > 4 + 3\ell$. □

References

[1] Abraham K.Samuel, D.S.T.Ramesh, M.Elakkiya, V.Balaji, On Relaxed skolem mean labeling for four star, *International Journal of Mathematics And its Application*, Vol 4,(2016), 17-23.

[2] J.C. Bermond, Graceful graphs, radio antennae and French windmills, *Graph Theory and Combinatorics*, pitman,London,(1979), 13-37.

[3] V. Balaji, D.S.T. Ramesh and A. Subramanian, Skolem Mean Labeling, *Bulletian of Pure and Applied Sciences*, Vol. 26E, No. 2,2007 ,245-248.

[4] V. Balaji, D.S.T. Ramesh and A. Subramanian, Some results on skolem mean graphs, *Bulletian of Pure and Applied Sciences*, vol. 27E, No. 1,2008 ,67-74.

[5] V. Balaji, D.S.T. Ramesh and A. Subramanian, Some results on relaxed skolem mean graphs, *Bulletian of Kerala Mathematics Association*, Vol. 5(2), December 2009 ,33-44.

[6] V. Balaji, D.S.T. Ramesh and A. Subramanian, Relaxed skolem mean labeling, *Advances and Application in Discrete Mathematics*, Vol.5(1), January 2010 ,11-22.

[7] V.Balaji, Solution of a conjecture on skolem mean graphs of stars $K_{1,\ell} \bigcup K_{1,m} \bigcup K_{1,n}$, *International Journal of Mathematical Combinatorics*, Vol.4,(2011), 115-117.

[8] V. Balaji, D.S.T. Ramesh and V. Maheswari, Solution of a conjecture on skolem mean graphs of stars $K_{1,\ell} \bigcup K_{1,\ell} \bigcup K_{1,m} \bigcup K_{1,n}$, *International Journal of Scientific and Engineering Research*, 3(11)2012, 125-128.

[9] V. Balaji, D.S.T. Ramesh and S. Ramarao, Skolem mean labeling for four star, *International Research Journal of Pure Algebra*, Vol.6 No.1, Jan. - 2016, 221 - 226.

[10] V. Balaji, D.S.T. Ramesh and K. Valarmathy, On relaxed skolem mean labeling for three star, *International Journal of Mathematical Archieve*, 7(2),2016,1-7.

[11] V. Balaji, D.S.T. Ramesh and V. Maheswari, Solution of a conjecture on skolem mean graphs of stars $K_{1,\ell} \bigcup K_{1,1} \bigcup K_{1,m} \bigcup K_{1,n}$, *Sacred Heart Journal of Science and Humanities*, Volume 3, July 2013.

[12] J.A. Gallian, A dynamic survay of graph labeling, *The Electronic Journal of Combinatorics* , 6(2010),DS6.

[13] F.Harary, *Graph Theory*,Addison - Wesley, Reading, 1969.

[14] V. Maheswari, D.S.T. Ramesh and V. Balaji, On skolem mean labeling, *Bulletin of Kerala Mathematics Association*, Vol.10, No.1, 2013, 89-94.

[15] D.S.T. Ramesh, P.Alayamani, V. Balaji and M.Elakkiya, On relaxed skolem mean labeling for four star, *IOSR Journal of Mathematics*, Vol.12(2016), 19-26.

[16] S.Somasundaram and R.Ponraj, Mean labeling of graphs, *National Academy Science letters*, 26(2003), 210 - 213.

[17] S.Somasundaram and R.Ponraj, Non - existence of mean labeling for a wheel,*Bulletin of Pure and Applied Sciences (section E:Mathematics and Statistics)*, 22E (2003), 103 - 111.

[18] S.Somasundaram and R.Ponraj, Some results on mean graphs, *Pure and Applied Mathematics Sciences*, 58 (2003), 29 - 35.

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[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

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[6]Linfan Mao, Mathematics on non-mathematics - A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.

[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

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