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A New Approach on the Striction Curves Belonging to Bertrandian Frenet Ruled Surfaces

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Abstract: In this study, we think six special ruled surfaces associated to the Bertrand curves pair $\{\alpha, \alpha^*\}$. We describe Bertrandian Frenet ruled surfaces and striction curves of these surfaces are expressed, as depending on the angle between the tangent vectors of the Bertrand curves pair $\{\alpha, \alpha^*\}$. Also, we examined the situation of the tangent vectors belonging to Striction curves of Frenet and Bertrandian Frenet ruled surfaces.

Key Words: Striction curves, Bertrand curves pair, ruled surfaces, Bertrandian Frenet ruled surface, Frenet ruled surface.

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§1. Introduction and Preliminaries

A surface is told to be ruled if it is generated by moving a direct line continuously in Euclidean space [8]. Ruled surfaces are one of the simplest objects in geometric modeling. The basis notions on ruled surfaces in are given in [4]. A ruled surface is a surface swept out by a straight line L moving along a curve $\alpha(u) \in \mathbb{R}^3$. The various positions of the generating line L are called the rulings of the surface. Such a surface always has a ruled parametrization

$$\varphi(u, v) = \alpha(u) + vX(u), \quad u \in I, \quad v \in \mathbb{R} \tag{1.1}$$

We call α base curve and X the director curve, although X is usually pictured as a vector field on α pointing along the line L. The striction point on a ruled surface is the foot of the common normal between two consecutive generators. The set of striction points defines the striction curve given by ([2])

$$c(s) = \alpha(s) - \frac{\langle \alpha', X' \rangle}{\langle X', X' \rangle} X. \tag{1.2}$$

Bertrand curves discovered by J. Bertrand in 1850 are one of the important and interesting

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topics of classical special curve theory. These well known properties of Bertrand curves in Euclidean 3-space was extended by L. R. Pears in [9]. The set, whose elements are frame vectors and curvatures of a curve α, is called Frenet-Serret apparatus of the curves. Let Frenet vector fields be $V_1(s), V_2(s), V_3(s)$ of α and let the first and second curvatures of the curve $\alpha(s)$ be $k_1(s)$ and $k_2(s)$, respectively. The quantities $\{V_1, V_2, V_3, k_1, k_2\}$ are collectively Frenet-Serret apparatus of the curves, ([3], [6]). The Frenet formulae are known as

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

Let $\alpha : I \to \mathbb{E}^3$ and $\alpha^* : I \to \mathbb{E}^3$ be the C^2- class differentiable two curves and let $V_1(s), V_2(s), V_3(s)$ and $V_1^*(s), V_2^*(s), V_3^*(s)$ be the Frenet frames of the curves α and α^*, respectively. If the principal normal vector V_2 of the curve α is linearly dependent on the principal normal vector V_2^* of the curve α^*, then the pair $\{\alpha, \alpha^*\}$ are called Bertrand curves pair, [4], [7]. Also α^* is called Bertrand mate. If the curve α^* is Bertrand mate of α, then we may write that

$$\alpha^*(s) = \alpha(s) + \lambda V_2(s). \tag{1.3}$$

If the curve α^* is Bertrand mate $\alpha(s)$, then we have that $\langle V_1^*(s), V_1(s)\rangle = \cos\theta =$ *constant*. The distance between corresponding points of the Bertrand curves pair in $\mathbb{E}^3$ is constant.$\alpha(s)$ is a Bertrand curve if and only if there exist nonzero real numbers λ and β such that constant $\lambda k_1 + \beta k_2 = 1$ for any $s \in I$, [4], [7]. The relationship between Frenet apparatus belonging to α and the Bertrand mate α^* is as follows:

$$\begin{cases} V_1^* = \cos\theta V_1 + \sin\theta V_3 \\ V_2^* = V_2 \\ V_3^* = -\sin\theta V_1 + \cos\theta V_3. \end{cases} \tag{1.4}$$

$$k_1^* = \frac{\lambda k_1 - \sin^2\theta}{\lambda(1-\lambda k_1)}, \; k_2^* = \frac{\sin^2\theta}{\lambda^2 k_2}, \tag{1.5}$$

where $\angle(V_1, V_1^*) = \theta$, [10]. Due to this equations, we can write

$$\begin{aligned} \frac{k_1^*}{\eta^*} &= \frac{\lambda k_1 - \sin^2\theta}{\lambda\big(1-\lambda k_1\big)\left(\frac{\left(\lambda k_1 - \sin^2\theta\right)^2}{\lambda^2\left(1-\lambda k_1\right)^2} + \frac{\sin^4\theta}{\lambda^4 k_2^2}\right)}, \\ \left(\frac{k_1^*}{\eta^*}\right)' &= \left[\frac{(\lambda k_1 - \sin^2\theta)\lambda^3 k_2^2\big(1-\lambda k_1\big)}{\left(\left(\lambda k_1 - \sin^2\theta\right)^2\lambda^2 k_2^2 + \sin^4\theta\left(1-\lambda k_1\right)^2\right)}\right]' \frac{1}{k_2\sqrt{\lambda^2+\beta^2}}, \\ \frac{ds}{ds^*} &= \frac{1}{k_2\sqrt{\lambda^2+\beta^2}}, \end{aligned}$$

where $\eta^* = {k_1^*}^2 + {k_2^*}^2$. Frenet ruled surface can be generated by the motion of a Frenet vector of curve in $\mathbb{E}^3$. Tangent, Normal and Binormal ruled surfaces are collectively named Frenet ruled surfaces, [5].

Theorem 1.1([5]) *Striction curves belonging to Frenet ruled surfaces are given in equations*

$$\begin{aligned} c_1(s) &= c_3(s) = \alpha(s) \\ c_2(s) &= \alpha(s) + \frac{k_1(s)}{k_2^2(s) + k_2^2(s)} V_2(s) \end{aligned} \tag{0.1}$$

Theorem 1.2([5]) *Tangent vector fields T_1, T_2 and T_3 of striction curves belonging to Frenet ruled surface are given by*

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{k_2^2}{\eta \left\| c_2'(s) \right\|} & \frac{\left(\frac{k_1}{\eta}\right)'}{\left\| c_2'(s) \right\|} & \frac{k_1 k_2}{\eta \left\| c_2'(s) \right\|} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \tag{1.7}$$

where $\eta = k_1^2 + k_2^2$.

§2. A New Approach on the Striction Curves Belonging to Bertrandian Frenet Ruled Surfaces

In this section first, we give Bertrandian tangent, Bertrandian normal and Bertrandian binormal Frenet ruled surfaces depending on the angle between the tangent vectors of the Bertrand curves pair. The later, we examined the situation of the tangent vectors belonging to Stricion curves of Frenet and Bertrandian Frenet ruled surfaces.

Definition 2.1 *Let Bertrand curves pair $\{\alpha, \alpha^*\}$. Bertrandian Frenet ruled surfaces which are depended on the angle between the tangent vectors of Bertrand curves pair are defined as follow, respectively*

$$\begin{aligned} \varphi_1^*(s, w_1) &= \alpha^*(s) + w_1 V_1^*(s) = \alpha + \lambda V_2 + w_1(\cos\theta V_1 + \sin\theta V_3), \\ \varphi_2^*(s, w_2) &= \alpha^*(s) + w_2 V_2^*(s) = \alpha + (\lambda + w_2) V_2, \qquad (2.1) \\ \varphi_3^*(s, w_3) &= \alpha^*(s) + w_3 V_3^*(s) = \alpha + \lambda V_2 + w_3(-\sin\theta V_1 + \cos\theta V_3). \end{aligned}$$

Theorem 2.2([11]) *Striction curves belonging to Bertrandian Frenet ruled surfaces depending on the angle between the tangent vectors of Bertrand curves pair can be expressed as follows,*

$$\begin{aligned} c_1^* &= \alpha + \lambda V_2, \\ c_2^* &= \alpha + \lambda V_2 + \frac{(\lambda k_1 - \sin^2\theta)V_2}{\lambda(1-\lambda k_2)\Big(\frac{(\lambda k_1 - \sin^2\theta)^2}{\lambda^2(1-\lambda k_2)^2} + \frac{\sin^4\theta}{\lambda^4 {k_2}^2}\Big)}, \\ c_3^* &= \alpha + \lambda V_2. \end{aligned}$$

Theorem 2.3([11]) *Tangent vector fields* T_1^*, T_2^*, T_3^*, *and* T_4^* *of striction curves belonging to Bertarndian Frenet ruled surfaces are given by*

$$\begin{bmatrix} T_1^* \\ T_2^* \\ T_3^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a^* & b^* & c^* \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \end{bmatrix} \tag{2.2}$$

where

$$\begin{aligned} a^* &= \frac{{k_2^*}^2}{\eta^* \left\| {c_2^*}'(s) \right\|} = \frac{\sin^4\theta}{\lambda^4 {k_2}^2\Big(\frac{(\lambda k_1 - \sin^2\theta)^2}{\lambda^2(1-\lambda k_1)^2} + \frac{\sin^4\theta}{\lambda^4 {k_2}^2}\Big)\sqrt{{\psi_1}^2 + {\psi_2}^2 + {\psi_3}^2}}, \\ b^* &= \frac{\left(\frac{k_1^*}{\eta^*}\right)'}{\left\| {c_2^*}'(s) \right\|} = \frac{\left[(\lambda k_1 - \sin^2\theta)\lambda^{-1}\big(1-\lambda k_1\big)^{-1}\left(\frac{\left(\lambda k_1 - \sin^2\theta\right)^2}{\lambda^2\left(1-\lambda k_1\right)^2} + \frac{\sin^4\theta}{\lambda^4 {k_2}^2}\right)^{-1}\right]'}{k_2\sqrt{(\lambda^2+\beta^2)({\psi_1}^2 + {\psi_2}^2 + {\psi_3}^2)}}, \\ c^* &= \frac{k_1^* k_2^*}{\eta^* \left\| {c_2^*}'(s) \right\|} = \frac{(\lambda k_1 - \sin^2\theta)\sin^2\theta}{\lambda^3 k_2 (1-\lambda k_1)\Big(\frac{(\lambda k_1 - \sin^2\theta)^2}{\lambda^2(1-\lambda k_1)^2} + \frac{\sin^4\theta}{\lambda^4 {k_2}^2}\Big)\sqrt{{\psi_1}^2 + {\psi_2}^2 + {\psi_3}^2}}. \end{aligned}$$

$$\begin{aligned} \psi_1 &= \left[\frac{\sin^4\theta\cos\theta}{\lambda^4 {k_2}^2\Big(\frac{(\lambda k_1 - \sin^2\theta)^2}{\lambda^2(1-\lambda k_1)^2} + \frac{\sin^4\theta}{\lambda^4 {k_2}^2}\Big)} - \frac{(\lambda k_1 - \sin^2\theta)\sin^3\theta}{\lambda^3 k_2(1-\lambda k_1)\Big(\frac{(\lambda k_1 - \sin^2\theta)^2}{\lambda^2(1-\lambda k_1)^2} + \frac{\sin^4\theta}{\lambda^4 {k_2}^2}\Big)}\right], \\ \psi_2 &= \left[\frac{\lambda k_1 - \sin^2\theta}{\lambda\big(1-\lambda k_1\big)\Big(\frac{\left(\lambda k_1 - \sin^2\theta\right)^2}{\lambda^2\left(1-\lambda k_1\right)^2} + \frac{\sin^4\theta}{\lambda^4 {k_2}^2}\Big)}\right]' \frac{1}{k_2\sqrt{\lambda+\beta}}, \\ \psi_3 &= \left[\frac{\sin^5\theta}{\lambda^4 {k_2}^2\Big(\frac{(\lambda k_1 - \sin^2\theta)^2}{\lambda^2(1-\lambda k_1)^2} + \frac{\sin^4\theta}{\lambda^4 {k_2}^2}\Big)} + \frac{(\lambda k_1 - \sin^2\theta)\sin^2\theta\cos\theta}{\lambda^3 k_2(1-\lambda k_1)\Big(\frac{(\lambda k_1 - \sin^2\theta)^2}{\lambda^2(1-\lambda k_1)^2} + \frac{\sin^4\theta}{\lambda^4 {k_2}^2}\Big)}\right]. \end{aligned}$$

Theorem 2.4 *The matrix of tangent vector fields on striction curves belonging to Frenet and*

Bertrandian Frenet ruled surfaces are given by

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \begin{bmatrix} T_1^* \\ T_2^* \\ T_3^* \end{bmatrix}^{\mathbf{T}} = \begin{bmatrix} \cos\theta & a^*\cos\theta - c^*\sin\theta & \cos\theta \\ X & a^*X + \frac{\left(\frac{k_1}{\eta}\right)'}{\|c_2'(s)\|}b^* + c^*Y & X \\ \cos\theta & a^*\cos\theta - c^*\sin\theta & \cos\theta \end{bmatrix}$$

where $X = \frac{k_2^2}{\eta\|c_2'(s)\|}\cos\theta + \frac{k_1k_2}{\eta\|c_2'(s)\|}\sin\theta,\ \ Y = -\frac{k_2^2}{\eta\|c_2'(s)\|}\sin\theta + \frac{k_1k_2}{\eta\|c_2'(s)\|}\cos\theta.$

Proof Let be $[T] = [A]\,[V]$ and $[T^*] = [A^*]\,[V^*]$. By using the properties of the matrix, the following result is obtained.

$$\begin{aligned} [T]\,[T^*]^{\mathbf{T}} &= [A]\,[V]\,([A^*]\,[V^*])^{\mathbf{T}} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ \frac{k_2^2}{\eta\|c_2'(s)\|} & \frac{\left(\frac{k_1}{\eta}\right)'}{\|c_2'(s)\|} & \frac{k_1k_2}{\eta\|c_2'(s)\|} \\ 1 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \begin{bmatrix} V_1{}^* \\ V_2{}^* \\ V_3{}^* \end{bmatrix}^{\mathbf{T}} \right) \begin{bmatrix} 1 & 0 & 0 \\ a^* & b^* & c^* \\ 1 & 0 & 0 \end{bmatrix}^{\mathbf{T}} \\ &= \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ X & \frac{\left(\frac{k_1}{\eta}\right)'}{\|c_2'(s)\|} & Y \\ \cos\theta & 0 & -\sin\theta \end{bmatrix} \begin{bmatrix} 1 & a^* & 1 \\ 0 & b^* & 0 \\ 0 & c^* & 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & a^*\cos\theta - c^*\sin\theta & \cos\theta \\ X & a^*X + \frac{\left(\frac{k_1}{\eta}\right)'}{\|c_2'(s)\|}b^* + c^*Y & X \\ \cos\theta & a^*\cos\theta - c^*\sin\theta & \cos\theta \end{bmatrix} \end{aligned}$$ □

Corollary 2.1 (i) *If the Bertrand mate is helix curve, then tangent vector fields of striction curves belonging to Tangent and Bertrandian normal ruled surfaces can be orthogonal;*

(ii) *If the Bertrand mate is helix curve, then tangent vector fields of striction curves belonging to Binormal and Bertrandian normal ruled surfaces can be orthogonal.*

Proof i) If inner product of T_1 and T_2^* is zero, then T_1 and T_2^* are orthogonal vectors.

$$\begin{aligned} \langle T_1, T_2^*\rangle = 0 \Rightarrow \langle T_1, T_2^*\rangle &= a^*\cos\theta - c^*\sin\theta = 0 \Longrightarrow \frac{a^*}{c^*} = \frac{\sin\theta}{\cos\theta} = \tan\theta \\ \frac{k_2^*}{k_1^*} &= \tan\theta = \mathit{constant} \end{aligned}$$

this completes the proof.

For (ii), since T_1 and T_3 are equivalent vectors, then the proof is clear. □

Corollary 2.2 (i) *If Bertrand curve is helix curve, then tangent vector fields of striction curves belonging to Normal and Bertrandian tangent ruled surfaces can be orthogonal;*

(ii) *If the Bertrand curve is helix curve, then tangent vector fields of striction curves belonging to Normal and Bertrandian binormal ruled surfaces can be orthogonal.*

Proof (i) If inner product of T_2 and T_1^* is zero, then T_2 and T_1^* are orthogonal vectors.

$$\begin{aligned} \langle T_2, T_1^* \rangle = 0 \Rightarrow \langle T_2, T_1^* \rangle = X &= \frac{k_2^2}{\eta \left\| c_2'(s) \right\|} \cos\theta + \frac{k_1 k_2}{\eta \left\| c_2'(s) \right\|} \sin\theta = 0 \\ &\Rightarrow \frac{k_2}{k_1} = -\tan\theta = constant. \end{aligned}$$

This completes the proof.

For (ii), since T_1^* and T_3^* are equivalent vectors, the proof is clear. □

Corollary 2.3 *Tangent vector fields of striction curves belonging to Normal and Bertrandian normal ruled surfaces have orthogonal under the condition*

$$\left[\frac{(\lambda k_1 - \sin^2\theta)\lambda^3 k_2^2 (1 - \lambda k_1)}{\left((\lambda k_1 - \sin^2\theta)^2 \lambda^2 k_2^2 + \sin^4\theta (1 - \lambda k_1)^2 \right)} \right]' = \frac{\frac{\sin^2\theta}{\lambda^2 k_2} \left(\frac{\sin^2\theta}{\lambda^2 k_2} X + \frac{\lambda k_1 - \sin^2\theta}{\lambda(1 - \lambda k_1)} Y \right)}{-\frac{\left(\frac{k_1}{\eta}\right)'}{\left\| c_2'(s) \right\|} \left(\frac{(\lambda k_1 - \sin^2\theta)^2}{\lambda^2 (1 - \lambda k_1)^2} + \frac{\sin^4\theta}{\lambda^4 {k_2}^2} \right)}.$$

Proof $\langle T_2, T_2^* \rangle = 0 \Rightarrow \langle T_2, T_2^* \rangle = a^* X + \frac{\left(\frac{k_1}{\eta}\right)'}{\left\| c_2'(s) \right\|} b^* + c^* Y = 0$

$$\begin{aligned} a^* X + \frac{\left(\frac{k_1}{\eta}\right)'}{\left\| c_2'(s) \right\|} b^* + c^* Y &= 0 \\ {k_2^*}^2 X + \frac{\left(\frac{k_1}{\eta}\right)'}{\left\| c_2'(s) \right\|} \left(\frac{k_1^*}{\eta^*} \right)' \eta^* + k_1^* k_2^* Y &= 0 \\ k_2^* (k_2^* X + k_1^* Y) &= -\frac{\left(\frac{k_1}{\eta}\right)'}{\left\| c_2'(s) \right\|} \left(\frac{k_1^*}{\eta^*} \right)' \eta^* \\ \left(\frac{k_1^*}{\eta^*} \right)' &= \frac{k_2^* (k_2^* X + k_1^* Y)}{-\frac{\left(\frac{k_1}{\eta}\right)'}{\left\| c_2'(s) \right\|} \eta^*} \\ \left[\frac{(\lambda k_1 - \sin^2\theta)\lambda^3 k_2^2 (1 - \lambda k_1)}{\left((\lambda k_1 - \sin^2\theta)^2 \lambda^2 k_2^2 + \sin^4\theta (1 - \lambda k_1)^2 \right)} \right]' &= \frac{\frac{\sin^2\theta}{\lambda^2 k_2} \left(\frac{\sin^2\theta}{\lambda^2 k_2} X + \frac{\lambda k_1 - \sin^2\theta}{\lambda(1 - \lambda k_1)} Y \right)}{-\frac{\left(\frac{k_1}{\eta}\right)'}{\left\| c_2'(s) \right\|} \left(\frac{(\lambda k_1 - \sin^2\theta)^2}{\lambda^2 (1 - \lambda k_1)^2} + \frac{\sin^4\theta}{\lambda^4 {k_2}^2} \right)}. \end{aligned}$$

This completes the proof. □

Corollary 2.4 *Four pairs of tangent vectors fields of Frenet ruled surface belonging to Bertrand pair* $\{\alpha, \alpha^*\}$ *are equal.*

Proof From the equations (1.7) and (2.2)

$$\langle T_1, T_1^* \rangle = \langle T_1, T_3^* \rangle = \langle T_3, T_1^* \rangle = \langle T_3, T_3^* \rangle = \langle V_1, \cos\theta V_1 + \sin\theta V_3 \rangle = \cos\theta. \quad \square$$

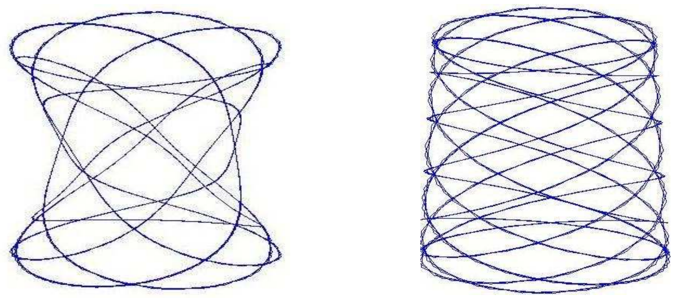
Example 2.1 Let us consider the following Bertrand curve α and Bertrand mate α^*, respectively.

$$\begin{aligned}
\alpha(s) &= \Big(\frac{9}{208}\sin 16s - \frac{1}{117}\sin 36s, -\frac{9}{208}\cos 16s + \frac{1}{117}\cos 36s, \frac{6}{65}\sin 10s\Big),\\
\alpha^*(s) &= \Big(\frac{9}{208}\sin 16s - \frac{1}{117}\sin 36s + \frac{12}{13}\lambda\cos(26s), -\frac{9}{208}\cos 16s + \frac{1}{117}\cos 36s\\
&+ \frac{12}{13}\lambda\sin 26s, \frac{6}{65}\sin 10s - \frac{5}{13}\lambda\Big), [?].
\end{aligned}$$

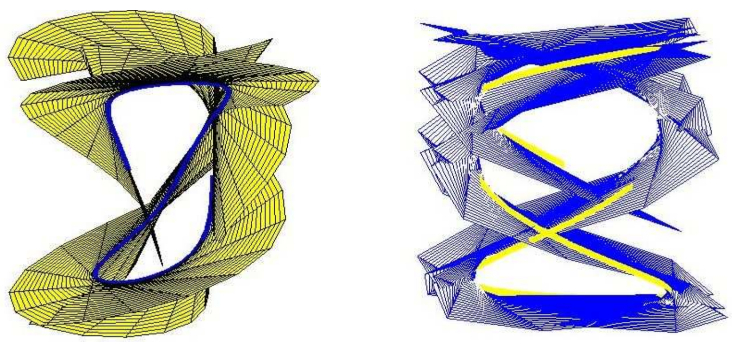
Striction curves of the ruled surface formed by Frenet vector of this curves is as follows:

$$\begin{cases}
c_1(s) = \Big(\frac{9}{208}\sin 16s - \frac{1}{117}\sin 36s, -\frac{9}{208}\cos 16s + \frac{1}{117}\cos 36s, \frac{6}{65}\sin 10s\Big),\\
c_2(s) = \Big(\frac{9}{208}\sin 16s - \frac{1}{117}\sin 36s + \frac{1}{24}\sin 10s(\frac{9}{13}\cos 16s - \frac{4}{13}\cos 36s),\\
\quad -\frac{9}{208}\cos 16s + \frac{1}{117}\cos 36s + \frac{1}{24}\sin 10s(\frac{9}{13}\sin 16s - \frac{4}{13}\sin 36s),\\
\quad \frac{6}{65}\sin 10s + \frac{1}{26}\sin 10s\cos 10s\Big),\\
c_3(s) = \Big(\frac{9}{208}\sin 16s - \frac{1}{117}\sin 36s, -\frac{9}{208}\cos 16s + \frac{1}{117}\cos 36s, \frac{6}{65}\sin 10s\Big)
\end{cases}$$

$$\begin{cases}
{c_1}^*(s) = \Big(\frac{9}{208}\sin 16s - \frac{1}{117}\sin 36s + \frac{12}{13}\lambda\cos(26s), -\frac{9}{208}\cos 16s + \frac{1}{117}\cos 36s\\
\quad +\frac{12}{13}\lambda\sin 26s, \frac{6}{65}\sin 10s - \frac{5}{13}\lambda\Big),\\
{c_2}^*(s) = \Big[\big(\frac{9}{208}\sin 16s - \frac{1}{117}\sin 36s + \frac{12}{13}\lambda\cos 26s\big)\cos^2 10s + \frac{1}{14976}\sin 10s\cos 26s\\
\quad -\frac{1}{624}\lambda\cos 26s, \cos^2 10s\big(-\frac{9}{208}\cos 16s + \frac{1}{117}\cos 36s + \frac{12}{13}\lambda\sin 26s\big)\\
\quad +\frac{1}{14976}\sin 10s\sin 26s - \frac{1}{624}\lambda\sin 26s, \cos^2 10s\big(\frac{6}{65}\sin 10s - \frac{5}{13}\lambda\big)\\
\quad -\frac{5}{179712}\sin 10s + \frac{5}{7488}\lambda\Big]\cos^{-2} 10s,\\
{c_3}^*(s) = \Big(\frac{9}{208}\sin 16s - \frac{1}{117}\sin 36s + \frac{12}{13}\lambda\cos(26s), -\frac{9}{208}\cos 16s + \frac{1}{117}\cos 36s\\
\quad +\frac{12}{13}\lambda\sin 26s, \frac{6}{65}\sin 10s - \frac{5}{13}\lambda\Big),
\end{cases}$$

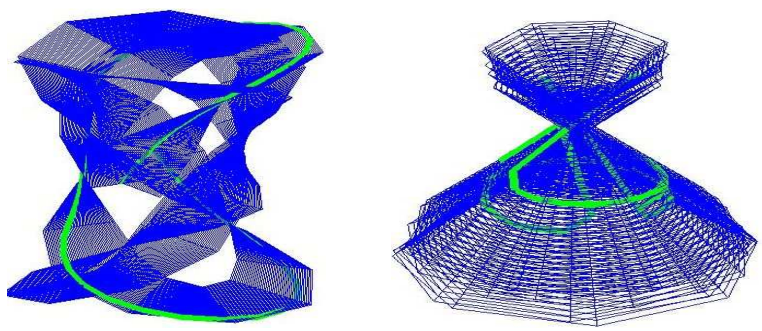


Bertrand Curve Bertrand mate curve



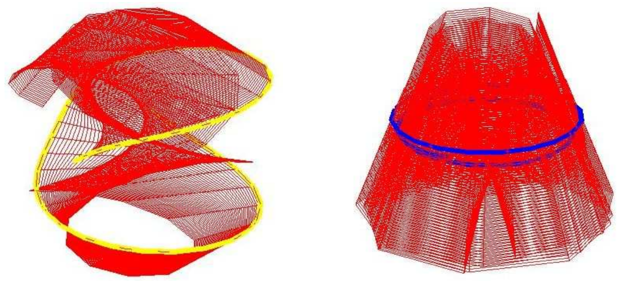
a) b)

Figure 1. The figures a) and b) show, respectively, Tangent ruled surface and Bertrandian Tangent ruled surface. Blue and yellow colors are striction curves on these surfaces, respectively.



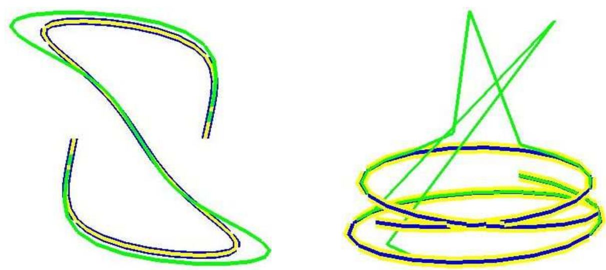
c) d)

Figure 2. The figures c) and d) show, respectively, Normal ruled surface and Bertrandian Normal ruled surface. Green colors are striction curves on these surfaces, respectively.



e) f)

Figure 3. The figures e) and f) show, respectively, Binormal ruled surface and Bertrandian Binormal ruled surface. Yellow and blue colors are striction curves on these surfaces, respectively.



k) m)

Figure 4. The figures k) and m) show Striction curves of Frenet ruled surfaces and Bertrandian Frenet ruled surfaces.

These figures are drawn with Mapple program for $\lambda = 1$.

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