International J.Math. Combin. Vol.2(2016), 29-42

Tutte Polynomial of Generalized Flower Graphs

Nithya Sai Narayana

(N.E.S. Ratnam College of Arts, Science and Commerce, Bhandup, Mumbai-400078, India)

E-mail: narayana_nithya@yahoo.com

Abstract: The book graph denoted by $B_{n,2}$ is the Cartesian Product $S_{n+1} \times P_2$ where S_{n+1} is a star graph with n vertices of degree 1 and one vertex of degree n and P_2 is the path graph of 2 vertices. Let $X_{n,p}$ denote the generalized form of Book graph where a family of p cycles which are n in number, is merged at a common edge. The generalized flower graph is obtained by merging t copies of $X_{n,p}$ with a base cycle C_t of length t at the common edges. The resultant structure looks like flower with petals. In this paper we discuss some properties satisfied by Tutte polynomial of this special graph and the related graphs.

Key Words: Tutte polynomial, recurrence relation, flower graph.

AMS(2010): 05C30, 05C99, 68R05.

§1. Introduction and Preliminaries

Tutte polynomial is a polynomial in two variables x, y with remarkable properties and it can be defined for a graph, matrix and more generally for matroids. Tutte polynomial is closely associated with many graphical invariants and in fact the following are the special cases of Tutte polynomial along particular curves of (x, y) plane.

(1) The chromatic and flow polynomial of a graph;

(2) The partition function of a $\mathbb{Q}$-state Pott's model;

(3) The Jone's polynomial of an alternating knot;

(4) The weight enumerator of a linear code over $GF(q)$;

(5) The all terminal reliability probability of a network;

(6) The number of spanning trees,number of forests, number of connected spanning subgraphs, the dimension of bicycle space and so on.

Tutte polynomial is widely studied for the reason that it provides structural information about the graph.

Definition 1.1 *(i) Let $G = (V, E)$ be an undirected connected multi-graph. The Tutte polyno-*

[1]Received November 3, 2015, Accepted May 12, 2016.

mial of the graph G is given by

$$\begin{aligned} T(G,x,y) &= 1 \; if \; E(G)=\phi; \\ &= xT(G.e,x,y) \; if \; e\in E \; and \; e \; is \; a \; cut \; edge; \\ &= yT(G-e,x,y) \; if \; e\in E \; and \; e \; is \; a \; loop; \\ &= T(G-e,x,y)+T(G.e,x,y) \; if \; e\in E \; and \; e \; is \; neither \; a \; loop \; nor \; a \; cut \; edge. \end{aligned}$$

(ii) If G is a disconnected graph with connected components $G_1, G_2, \cdots, G_t$ with $t \geq 2$, then the Tutte Polynomial of G denoted by $T(G,x,y)$ is defined as $T(G,x,y) = \prod_{i=1}^{t} T(G_i,x,y)$.

Tutte polynomial of some of standard graphs are given below.

Theorem 1.2 *Let T_n be a tree on n vertices and let C_n be a cycle on n vertices then*

(1) $T(T_n,x,y) = x^{n-1}$;

(2) $T(C_n,x,y) = y + \sum_{i=1}^{n-1} x^i$.

Theorem 1.3 *Let G be a bi connected graph. Let u,v be two vertices in G such that u,v are joined by a path P^s of length s where degree of each vertex in P^s is two except possibly for u,v then*

$$T(G) = (1+x+x^2+\cdots x^{s-1})T(G-P^s)+T(G.P^s).$$

Proof Let $e_1, e_2, \cdots e_s$ be the s edges in the path P^s, then

$$\begin{aligned} T(G) &= T(G-e_1)+T(G.e_1) \\ &= x^{s-1}T(G-P^s)+x^{s-2}T(G-P^s)+\cdots x^{s-s}T(G-P^s)+T(G.P^s) \\ &\quad (G \text{ is bi}-\text{connected}, e^t \text{ is not a bridge in } G-G_1-G_2\cdots-G_{t-1}) \\ &= (1+x+x^2+\cdots x^{s-1})T(G-P^s)+T(G.P^s). \end{aligned}$$

□

We study the Tutte polynomial of generalized Book graph. Cartesian product of two graphs G_1, G_2 denoted by $G_1 \times G_2$ is a graph with $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and two vertices $(u_1,v_1),(u_2,v_2)$ of $G_1 \times G_2$ are adjacent if and only if either $u_1 = u_2$ and (v_1,v_2) is an edge in G_2 or $v_1 = v_2$ and (u_1,u_2) is an edge of G_1. The book graph denoted by $B_{n,2}$ or simplyB_n is the Cartesian Product $S_{n+1} \times P_2$ where S_{n+1} is a star graph with n vertices of degree 1 and one vertex of degree n and P_2 is the path graph of 2 vertices. It can be observed that book graphs are planar. Some book graphs and their planar representation are given below.

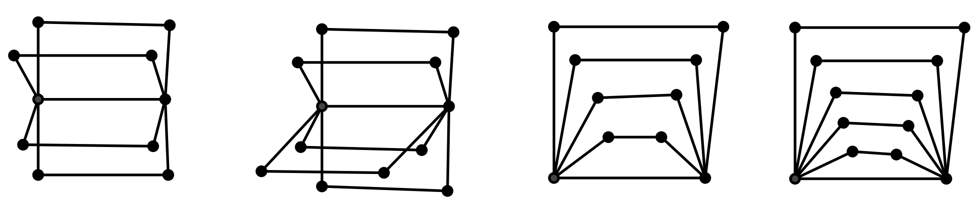

Figure 1 Book graph B_4 , B_5 and their Planar representation

We make a generalization of this graph. Through out this section $T(G,x,y) = T(G)$ denotes the Tutte polynomial for the graph G. We make use of the following notation. $X_{n,p}$ denote a graph with n number of p-cycles with a common edge $e = xy$ and let $Y_{n,p} = X_{n,p} - e$ and $Z_{n,p} = X_{n,p}.e$. Note that $Z_{n,p}$ is actually a graph with n number of $p-1$ cycles with a common vertex.

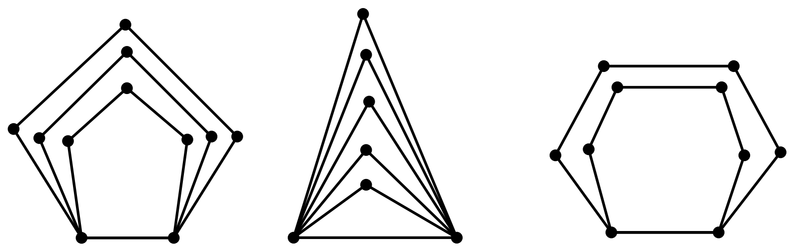

Figure 2 $X_{3,5}$, $X_{5,3}$,$X_{2,6}$

Thus $B_n = X_{n,4}$ is a particular case of the graph we have defined which we call as generalized book graph. We first arrive at some recurrence relation satisfied by these graphs. Before we prove the relations satisfied by these graphs we will prove some preliminary results.

Notations and Conventions 1.4

(1) Let G_1 and G_2 be two disjoint graphs each of them having a unique identified vertex. The graph obtained by merging an identified vertex of G_1 to an identified vertex of G_2 is denoted by $G_1 \times G_2$.

(2) Let G_1, G_2 be two disjoint graphs each of them having two designated vertices namely x, y and x', y'. The graph obtained by merging the identified vertex x with x' and the vertex y with y' is denoted by $G_1 * G_2$.

(3) Let G be a graph with an identified vertex v. The graph obtained by taking n copies of G and joining all the copies at the identified vertex v is denoted by $G^{(n)}$.

(4) Let G_1, G_2 be two disjoint graphs each with two identified vertices x, y and x', y' respectively. Let $xy \in E(G)$ and $x'y' \in E(G')$. The graph obtained by merging the two vertices x, x' and y, y' and the edges xy and $x'y'$ to a single edge is denoted by $G_1 \odot G_2$.

Proposition 1.5 *G be a graph which can be expressed as $G = H \times P_l$ where P_l is a tree of order $l+1$, then $T(G) = x^l T(H)$.*

Proposition 1.6 $T(C_p^{(n)}) = T(C_p)^n = \left[y + \sum\limits_{k=1}^{p-1} x^k\right]^n$.

Proof Induction on n. For $n = 1, T(C_p) = y + \sum\limits_{k=1}^{p-1} x^k$ by Theorem 1.2. Let v be the identified vertex. Assume that the result is true for $n-1$. Let $G = C_p^{(n)}$. Note that $C_p^{(n)} = C_p^{(n-1)} \times C_p$.

Let e be any edge adjacent to v. By recurrence relation

$$\begin{aligned}
T(G) &= T(G-e)+T(G.e)\\
&= T(C_p^{(n-1)}\times P_{p-1})+T(C_p^{(n-1)}\times C_{p-1})\\
&= x^{p-1}T(C_p^{(n-1)})+T(C_p^{(n-1)}\times C_{p-1})\\
&= x^{p-1}T(C_p^{(n-1)})+T(C_p^{(n-1)}\times P_{p-2})+T(C_p^{(n-1)}\times C_{p-2})\\
&= x^{p-1}T(C_p^{(n-1)})+x^{p-2}T(C_p^{(n-1)})+T(C_p^{(n-1)}\times C_{p-2})\\
&\qquad \text{by using Proposition 1.5}\\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&= (x^{p-1}+x^{p-2}+\cdots+x)T(C_p^{(n-1)})+T(C_p^{(n-1)}\times C_1)\\
&= (x^{p-1}+x^{p-2}+\cdots+x)T(C_p^{(n-1)})+yT(C_p^{(n-1)}) \text{ as } C_1 \text{ is a loop}\\
&= (x^{p-1}+x^{p-2}+\cdots+x+y)T(C_p^{(n-1)})\\
&= \{y+\sum_{k=1}^{p-1}x^k\}\left[y+\sum_{k=1}^{p-1}x^k\right]^{n-1} = \left[y+\sum_{k=1}^{p-1}x^k\right]^{n}.
\end{aligned}$$

□

§2. Tutte Polynomial of Generalized Book Graph

Theorem 2.1 *Let $X_{n,p}$ denote a graph with n number of p-cycles with a common edge $e=xy$ and let $Y_{n,p}=X_{n,p}-e$ then,*
$X_{n,p}$ and $Y_{n,p}$ satisfy the following recurrence relations

(i) $T(X_{n,p})=T(Y_{n,p})+T(C_{p-1})^n$;

(ii) $T(Y_{n,p})=\left[\sum_{k=1}^{p-2}x^k\right]T(Y_{n-1,p})+T(X_{n-1,p})$ *for* $n\geq 2$ *with* $T(Y_{1,p})=x^{p-1}$.

Proof *(i)* e is neither a loop nor a cut edge and hence using recurrence relation of Tutte polynomial

$$\begin{aligned}
T(X_{n,p}) &= T(X_{n,p}-e)+T(X_{n,p}.e)\\
&= T(Y_{n,p})+T(C_{p-1}^{(n)})=T(Y_{n,p})+T(C_{p-1})^n
\end{aligned}$$

using Proposition 1.6. This proves (1).

(ii) Clearly, $Y_{1,p}=$ a path of length $p-1$ and hence $T(Y_{1,p})=x^{p-1}$. We prove this result by induction on n.

For $n=2$, $T(Y_{2,p})=T(Y_{2,p}-e')+T(Y_{2,p}.e')$ where e' is any edge of $Y_{2,p}$ adjacent to x

other than e.

$$\begin{aligned} T(Y_{2,p}) &= x^{p-2}T(Y_{1,p}) + \mathbf{T(Y_{1,p} * P_{p-2})} \\ &= x^{p-2}T(Y_{1,p}) + x^{p-3}T(Y_{1,p}) + \mathbf{T(Y_{1,p} * P_{p-3})} \\ &= x^{p-2}T(Y_{1,p}) + x^{p-3}T(Y_{1,p}) + \cdots + xT(Y_{1,p}) + \mathbf{T(Y_{1,p} * P_1)} \\ &= \left[\sum_{k=1}^{p-2} x^k\right] T(Y_{1,p}) + T(X_{1,p}), \end{aligned}$$

which proves the result for $n = 2$.

Assume that the result is true for a graph $Y_{n-1,p}$. Consider $Y_{n,p}$ and let e' is any edge of $Y_{n-1,p}$ adjacent to x other than e. Then,

$$\begin{aligned} T(Y_{n,p}) &= T(Y_{n-1,p} - e') + T(Y_{n-1,p}.e') \\ &= x^{p-2}T(Y_{n-1,p}) + \mathbf{T(Y_{n-1,p} * P_{p-2})} \\ &= x^{p-2}T(Y_{n-1,p}) + x^{p-3}T(Y_{n-1,p}) + \mathbf{T(Y_{n-1,p} * P_{p-3})} \\ &= x^{p-2}T(Y_{n-1,p}) + x^{p-3}T(Y_{n-1,p}) + \cdots + xT(Y_{n-1,p}) + \mathbf{T(Y_{n-1,p} * P_1)} \\ &= \left[\sum_{k=1}^{p-2} x^k\right] T(Y_{n-1,p}) + T(X_{n-1,p}), \end{aligned}$$

which proves the result for $n \geq 2$. □

Theorem 2.2 *(i)* $T(Y_{n,p}) = b^{n-1}x^{p-1} + \left[\sum_{k=0}^{n-2} b^k(b+y-1)^{n-1-k}\right]$ *for* $n \geq 2$;

(ii) $T(X_{n,p}) = b^{n-1}x^{p-1} + \left[\sum_{k=0}^{n-2} b^k(b+y-1)^{n-1-k}\right] + (b+y-1)^n$, *where* $b = \sum_{k=0}^{p-2} x^k$ *for* $n \geq 2$.

Proof By Theorem 2.1,

$$\begin{aligned} T(Y_{n,p}) &= \left[\sum_{k=1}^{p-2} x^k\right] T(Y_{n-1,p}) + T(X_{n-1,p}) \\ &= \left[\sum_{k=1}^{p-2} x^k\right] T(Y_{n-1,p}) + T(Y_{n-1,p}) + T(C_{p-1})^{n-1} \\ &= \left[\sum_{k=0}^{p-2} x^k\right] T(Y_{n-1,p}) + T(C_{p-1})^{n-1} \\ &= bT(Y_{n-1,p}) + T(C_{p-1})^{n-1}. \end{aligned}$$

Note that $T(C_{p-1}) = x^{p-2} + x^{p-3} + \cdots + x + y = b + y - 1$. We solve the recurrence relation,

$$\begin{aligned}
T(Y_{1,p}) &= x^{p-1}, \\
T(Y_{2,p}) &= bT(Y_{1,p}) + T(C_{p-1}) \\
&= bx^{p-1} + (y+b-1) = b^1 x^{p-1} + \left[\sum_{k=0}^{2-2} b^k (b+y-1)^{2-1-k}\right], \\
T(Y_{3,p}) &= bT(Y_{2,p}) + T(C_{p-1})^2 \\
&= b^2 x^{p-1} + bT(C_{p-1}) + (y+b-1)^2 \\
&= b^2 x^{p-1} + b(y+b-1) + (y+b-1)^2 \\
&= b^{3-1} x^{p-1} + \left[\sum_{k=0}^{3-2} b^k (b+y-1)^{3-1-k}\right], \\
T(Y_{4,p}) &= bT(Y_{3,p}) + T(C_{p-1})^3 \\
&= b^3 x^{p-1} + b^2 T(C_{p-1}) + bT(C_{p-1})^2 + T(C_{p-1})^3 \\
&= b^3 x^{p-1} + b^2(b+y-1) + b(b+y-1)^2 + (b+y-1)^3 \\
&= b^{4-1} x^{p-1} + \left[\sum_{k=0}^{4-2} b^k (b+y-1)^{4-1-k}\right].
\end{aligned}$$

Assume that by induction

$$\begin{aligned}
T(Y_{n-1,p}) &= b^{n-2} x^{p-1} + \left[\sum_{k=0}^{n-3} b^k (b+y-1)^{n-2-k}\right], \\
T(Y_{n,p}) &= bT(Y_{n-1,p}) + T(C_{p-1})^{n-1} \\
&= b\left\{b^{n-2} x^{p-1} + \left[\sum_{k=0}^{n-3} b^k (b+y-1)^{n-2-k}\right]\right\} + (b+y-1)^{n-1} \\
&= b^{n-1} x^{p-1} + b[(b+y-1)^{n-2} + b(b+y-1)^{n-3} + \cdots + b^{n-3}] + (b+y-1)^{n-1} \\
&= b^{n-1} x^{p-1} + (b+y-1)^{n-1} + b(b+y-1)^{n-2} + \cdots + b^{n-2}(b+y-1) \\
&= b^{n-1} x^{p-1} + \left[\sum_{k=0}^{n-2} b^k (b+y-1)^{n-1-k}\right].
\end{aligned}$$

This completes the proof of (i).

$$\begin{aligned}
T(X_{n,p}) &= T(Y_{n,p}) + (b+y-1)^n \\
&= b^{n-1} x^{p-1} + \left[\sum_{k=0}^{n-2} b^k (b+y-1)^{n-1-k}\right] + (b+y-1)^n,
\end{aligned}$$

which completes proof of (ii). □

Remark 2.3 For $n = 1$,

$$\begin{aligned} T(X_{n,p}) &= T(C_p) \\ &= x^{p-1} + x^{p-2} + \cdots + x + yx^{p-1} + (b + y - 1), \end{aligned}$$

which matches with the Theorem 2.2

An equivalent representation of Tutte polynomial for generalized book graph is the following.

Theorem 2.4 $T(X_{n,p}) = xb^n + \left[\sum_{k=0}^{n-1} b^k (y + b - 1)^{n-1-k}\right] y$

Proof By Theorem 2.2,

$$T(X_{n,p}) = b^{n-1}x^{p-1} + \left[\sum_{k=0}^{n-2} b^k (b + y - 1)^{n-1-k}\right] + (b + y - 1)^n <$$

$$\begin{aligned} b &= x^{p-2} + x^{p-3} + \cdots + x + 1 = \frac{x^{p-1} - 1}{x - 1} \\ &\Rightarrow x^{p-1} = b(x - 1) + 1 = bx - b + 1 \\ &\Rightarrow x^{p-1}b^{n-1} = (bx - b + 1)b^{n-1} = xb^n - b^n + b^{n-1}. \end{aligned}$$

Hence,

$$\begin{aligned} T(X_{n,p}) &= xb^n - b^n + b^{n-1} + \left[\sum_{k=0}^{n-2} b^k (b + y - 1)^{n-1-k}\right] + (b + y - 1)^n \\ &= xb^n - b^n + b^{n-1} + (b + y - 1)^{n-1} + b(b + y - 1)^{n-2} + \cdots \\ &\quad + b^{n-2}(b + y - 1) + (b + y - 1)^n. \qquad (*) \end{aligned}$$

Now consider

$$\begin{aligned} & xb^n + \left[\sum_{k=0}^{n-1} b^k (y + b - 1)^{n-1-k}\right] y \\ &= xb^n + y\left[(y + b - 1)^{n-1} + b(y + b - 1)^{n-2} + \cdots + b^{n-1}\right] \\ &= xb^n + (y + b - 1 - (b - 1))\left[(y + b - 1)^{n-1} + b(y + b - 1)^{n-2} + \cdots + b^{n-1}\right] \\ &= xb^n + (y + b - 1)\left[(y + b - 1)^{n-1} + b(y + b - 1)^{n-2} + \cdots + b^{n-1}\right] \\ &\quad -(b - 1)\left[(y + b - 1)^{n-1} + b(y + b - 1)^{n-2} + \cdots + b^{n-1}\right] \\ &= xb^n + \left[(y - b - 1)^n + b(y - b + 1)^{n-1} + \cdots + b^{n-1}(y + b - 1)\right] \\ &\quad -b(\left[(y + b - 1)^{n-1} + b(y + b - 1)^{n-2} + \cdots + b^{n-1}\right] \\ &\quad + \left[(y + b - 1)^{n-1} + b(y + b - 1)^{n-2} + \cdots + b^{n-1}\right] \end{aligned}$$

$$= xb^n + (y+b-1)^n + b(y+b-1)^{n-1} + \cdots + (y+b-1)b^{n-1}$$
$$-b(y+b-1)^{n-1} - b^2(y+b-1)^{n-2} - \cdots - b^{n-1}(b+y-1) - b^n$$
$$+(y+b-1)^{n-1} + b(y+b-1)^{n-2} + \cdots + b^{n-2}(y+b-1) + b^{n-1}$$
$$= xb^n + (y+b-1)^n - b^n + (y+b-1)^{n-1} + b(y+b-1)^{n-2} + \cdots$$
$$+b^{n-2}(y+b-1) + b^{n-1}. \qquad (**)$$

From (*) and (**) we get

$$T(X_{n,p}) = xb^n + \left[\sum_{k=0}^{n-1} b^k (y+b-1)^{n-1-k}\right] y. \qquad \Box$$

§3. The Generalized Flower Graph

The generalized flower graph is obtained by merging $X_{n,p}$ at each of the edge of a basic cycle of length t. We define the generalized complete flower graph and generalized Flower graph with k petals.

Definition 3.1 (*i*) *A graph in which i copies of $X_{n,p}$ is taken and is merged with any of the i out of t edges of the base cycle C_t of length t where, $1 \leq i \leq t-1$ is called a generalized flower graph with i petals and is denoted by $G^{(i)}_{n,p,t}$.*

(*ii*) *A graph obtained by taking a base cycle C_t of length t and t copies of $X_{n,p}$ and merging the two graphs at the common edge of $X_{n,p}$ with each of the edge of the basic cycle C_t is referred to as Generalized Flower Graph or Generalized Complete Flower graph and is denoted by $G_{n,p,t}$. In fact $G_{n,p,t} = G^{(t)}_{n,p,t}$.*

(*iii*) *The graph obtained by taking i copies of $X_{n,p}$ with each of the cycle containing a designated edge and joining the i copies at the end vertices of the designated edges is denoted by $H^{(i)}_{n,p}$.*

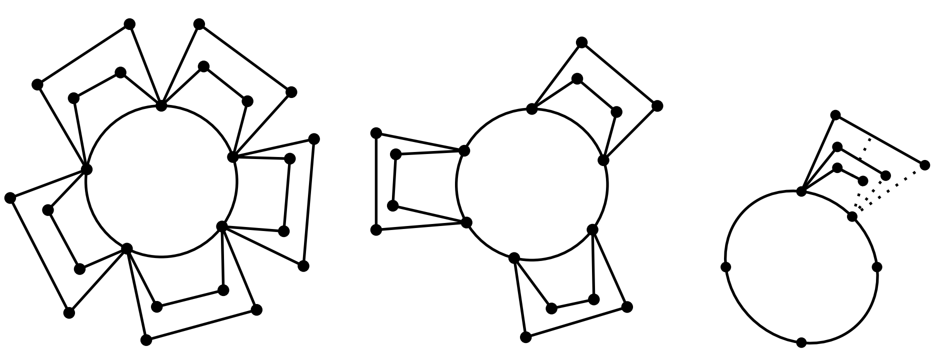

Figure 3 Generalized complete flower graph $G_{2,4,5}$ and flower graph $G^{(3)}_{2,4,6}$ with 3 petals and generalized flower graph with one petal

Theorem 3.2 *Let $X_{n,p}$ has the common edge e. Let $G^{(1)}_{n,p,t}$ be the generalized flower graph with* *one petal. Then,*

$$T(G^{(1)}_{n,p,t}) = (1 + x + x^2 + \cdots + x^{t-2})T(X_{n,p}) + y(y + b - 1)^n.$$

Proof Let e_1 be any edge on C_t other than e. By deletion contraction formula we get

$$\begin{aligned} T(G_{n,p,t}) &= x^{t-2}T(X_{n,p}) + T(G_{n,p,t-1}) \\ &= x^{t-2}T(X_{n,p}) + x^{t-3}T(X_{n,p}) + \cdots + xT(X_{n,p}) + T(G_{n,p,2}) \\ &= (x + x^2 + \cdots + x^{t-2})T(X_{n,p}) + T(X_{n,p}) + yT(C^n_{p-1}) \\ &= (1 + x + x^2 + \cdots + x^{t-2})T(X_{n,p}) + y(y + b - 1)^n, \end{aligned}$$

which completes the proof. □

Corollary 3.3 $T(G_{n,p,p}) = b^n x^{n-1} + \sum\limits_{k=1}^{n} b^k(b + y - 1)^{n-k} + (y + b - 1)^{n+1}$

Proof By Theorem 2.5 taking $p = t$ we get

$$\begin{aligned} T(G_{n,p,p}) &= (1 + x + x^2 + \cdots + x^{p-2})T(X_{n,p}) + y(y + b - 1)^n \\ &= b\left\{b^{n-1}x^{p-1} + \left[\sum_{k=0}^{n-2} b^k(b + y - 1)^{n-1-k}\right] + (b + y - 1)^n\right\} + y(y + b - 1)^n \\ &= b^n x^{p-1} + \left[\sum_{k=0}^{n-2} b^{k+1}(b + y - 1)^{n-1-k}\right] + b(b + y - 1)^n + y(y + b - 1)^n \\ &= b^n x^{p-1} + \left[\sum_{k=0}^{n-2} b^{k+1}(b + y - 1)^{n-1-k}\right] + (b + y)(b + y - 1)^n \\ &= b^n x^{p-1} + \left[\sum_{k=0}^{n-2} b^{k+1}(b + y - 1)^{n-1-k}\right] + (b + y - 1 + 1)(y + b - 1)^n \\ &= b^n x^{p-1} + \left[\sum_{k=0}^{n-2} b^{k+1}(b + y - 1)^{n-1-k}\right] + (b + y - 1)^n + (y + b - 1)^{n+1} \\ &= b^n x^{p-1} + b(b + y - 1)^{n-1} + b^2(b + y - 1)^{n-2} + \cdots + b^{n-1}(b + y - 1) \\ &\quad +(b + y - 1)^n + (b + y - 1)^{n+1} \\ &= b^n x^{p-1} + \left[\sum_{k=0}^{n} b^{k+1}(b + y - 1)^{n-k}\right] + (y + b - 1)^{n+1}. \end{aligned}$$

Lemma 3.4 *Let u, v be two vertices of a graph G which are joined by n disjoint paths of length $p - 1$, namely $P_1, P_2, \cdots P_n$ such that degree of each of vertices in $P_1, P_2, \cdots P_n$ other than u, v is 2 in G and removal of these n paths does not disconnect u and v, then*

$$T(G) = b^n T(G'') + \left[\sum_{k=0}^{n-1} b^k(y + b - 1)^{n-1-k}\right] T(G'),$$

where G' is obtained from G by removing the n disjoint paths of length $p-1$ between u, v and identifying u, v and G'' is the graph is obtained by removing the n disjoint paths of length $p-1$ from G.

Proof Using Theorem 1.3, $T(G) = bT(G - P_1) + T(G.P_1)$. But

$$T(G.P_1) = T(C_{p-1}^{n-1} \times G') = (y + b - 1)^{n-1} T(G').$$

Hence

$$\begin{aligned} T(G) &= bT(G - P_1) + (y + b - 1)^{n-1} T(G') \\ &= b\left[bT(G - P_1 - P_2) + (y + b - 1)^{n-2} T(G')\right] + (y + b - 1)^{n-1} T(G') \\ &= b^2 T(G - P_1 - P_2) + \left[b(y + b - 1)^{n-2} + (y + b - 1)^{n-1}\right] T(G') \\ &= b^2 T(G - P_1 - P_2 - P_3) + \left[b^2 (y + b - 1)^{n-3} + b(y + b - 1)^{n-2}\right. \\ &\quad \left. + (y + b - 1)^{n-1}\right] T(G') \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &= b^n T(G - P_1 - P_2 \cdots P_n) + \left[b^{n-1}(y + b - 1)^0 + b^{n-2}(y + b - 1)^1 + \cdots\right. \\ &\quad \left. + b(y + b - 1)^{n-2} + (y + b - 1)^{n-1}\right] T(G') \\ &= b^n T(G'') + \left[\sum_{k=0}^{n-1} b^k (y + b - 1)^{n-1-k}\right] T(G'), \end{aligned}$$

which completes the proof. □

From the above theorem we get another method of proving Theorem 2.4.

Corollary 3.5 $T(X_{n,p}) = xb^n + \left[\sum\limits_{k=0}^{n-1} b^k (y + b - 1)^{n-1-k}\right] y.$

Proof Applying Lemma 3.5 to $X_{n,p}$we get $G'' = K_2$ and $G' =$ a single loop so that $T(G'') = x$, $T(G') = y$ and we obtain the result. □

Theorem 3.6 *Let $H_{n,p}^{(i)}$ denote a graph obtained by taking i copies of $X_{n,p}$ and joining it at a common vertex in succession. Then, $H_{n,p}^{(i)} = [T(X_{n,p})]^i$.*

Proof Let $e = uv$ be the common edge of the i^{th} copy of $X_{n,p}$. If G'' is the graph obtained by removing the n distinct paths of length $p-1$, then the resultant graph is a graph obtained by joining the $i-1$ copies of $X_{n,p}$ with edge $e = uv$ at the vertex u and hence $T(G'') = xT(H_{n,p}^{(i-1)})$. If G' is obtained by removing n disjoint paths of length $p-1$ between u, v and identifying u and v then, $T(G') = yT(H_{n,p}^{(i-1)})$. Thus using Lemma 3.4

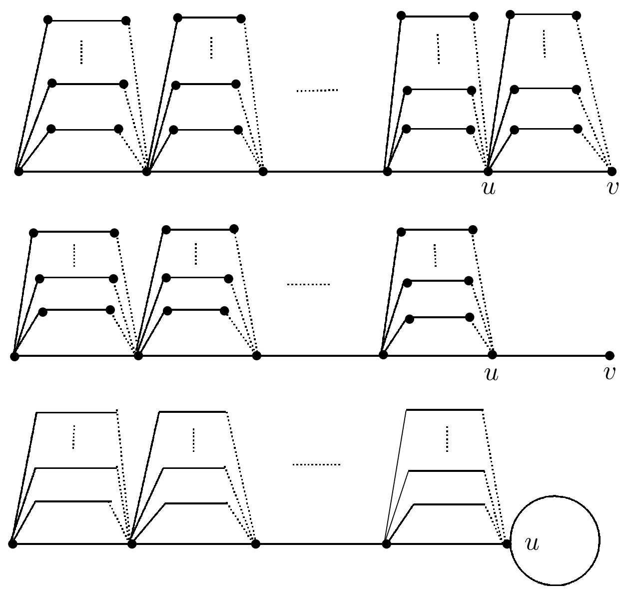

Figure 4 $H_{n,p}^{(i)}$ graph G'' and G' segregation

$$\begin{aligned} T(H_{n,p}^{(i)}) &= xb^nT(H_{n,p}^{(i-1)}) + y\left[\sum_{k=0}^{n-1} b^k(y+b-1)^{n-1-k}\right]T(H_{n,p}^{(i-1)}) \\ &= \left\{xb^n + y\left[\sum_{k=0}^{n-1} b^k(y+b-1)^{n-1-k}\right]\right\}T(H_{n,p}^{(i-1)}) \\ &= [T(X_{n,p})]\,T(H_{n,p}^{(i-1)}) \\ &= [T(X_{n,p})]^2\,T(H_{n,p}^{(i-2)}), \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

$$[T(X_{n,p})]^{i-1}\,T(H_{n,p}^{(1)}) = [T(X_{n,p})]^{i-1}\,T(X_{n,p}) = \left[T(X_{n,p})^i\right]. \qquad \square$$

Corollary 3.7 *Let G denote a graph obtained by taking i copies of $X_{n,p}$ and t copies of K_2 and joining it in succession in any order then,*

$$T(G) = x^tT(H_{n,p}^{(i)}) = x^t\,[T(X_{n,p})]^i.$$

Theorem 3.8 *Let $G_{n,p,t}$ denote a graph obtained by taking t copies of $X_{n,p}$ and taking $\odot$ product with C_t in succession, then*

$$\begin{aligned} T(G_{n,p,t}) &= b^n\sum_{k=0}^{t-2}(b^n+\alpha y)^k(b^nx+\alpha y)^{t-1-k} \\ &\quad +(b^n+\alpha y)^{t-2}\alpha y(b+y-1)^n + (b^n+\alpha y)^{t-2}y(b+y-1)^{2n}, \end{aligned}$$

where $\alpha = \left[\sum_{k=0}^{n-1} b^k(y+b-1)^{n-1-k}\right]$.

Proof Let G'' be the graph obtained by removing n distinct paths of length $p-1$ between the two vertices which are end points of common edge e of any copy of $X_{n,p}$ on the cycle C_t and let G' be the graph obtained by removing n distinct paths of length $p-1$ as described for G'' and identifying the two end vertices of e in C_t. Then by Lemma 3.4.

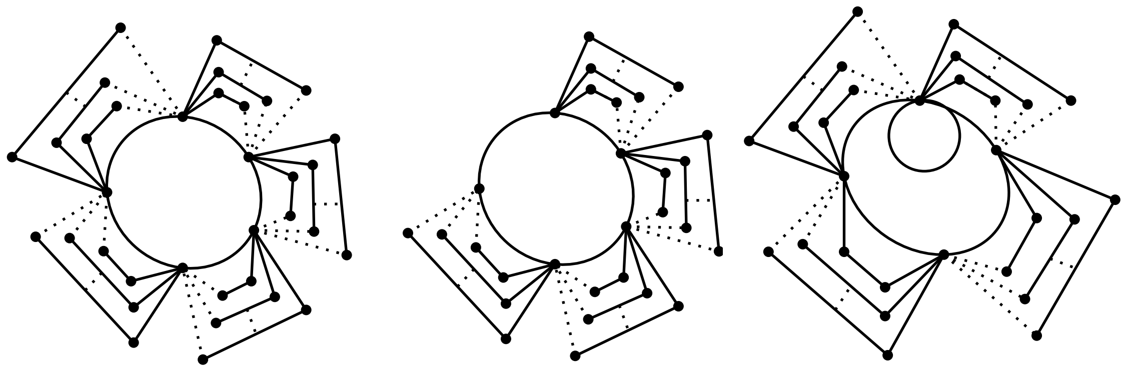

Figure 5 $G_{n,p,t}$ graph G'' and G' segregation

$$T(G_{n,p,t}) = b^n T(G'') + \alpha T(G'),$$

where $\alpha = \left[\sum_{k=0}^{n-1} b^k (y+b-1)^{n-1-k}\right]$. Also using deletion contraction formula of Tutte polynomial $T(G') = yT(G_{n,p,t-1})$ and

$$T(G'') = T(H_{n,p}^{(t-1)}) + T(G_{n,p,t-1}) = T(X_{n,p})^{t-1} + T(G_{n,p,t-1})$$

from Theorem 3.6. But by Corollary 3.5,

$$T(X_{n,p}) = xb^n + \left[\sum_{k=0}^{n-1} b^k (y+b-1)^{n-1-k}\right] y = b^n x + \alpha y$$

Thus, $T(G'') = (b^n x + \alpha y)^{t-1} + T(G_{n,p,t-1})$ and

$$\begin{aligned}
T(G_{n,p,t}) &= b^n (b^n x + \alpha y)^{t-1} + b^n T(G_{n,p,t-1}) + \alpha y T(G_{n,p,t-1}) \\
&= b^n (b^n x + \alpha y)^{t-1} + (b^n + \alpha y) T(G_{n,p,t-1}) \\
&= b^n (b^n x + \alpha y)^{t-1} + (b^n + \alpha y)\left[b^n (b^n x + \alpha y)^{t-2} + (b^n + \alpha y) T(G_{n,p,t-2})\right] \\
&= b^n (b^n x + \alpha y)^{t-1} + b^n (b^n x + \alpha y)^{t-2}(b^n + \alpha y) + (b^n + \alpha y)^2 T(G_{n,p,t-2}) \\
&= b^n (b^n x + \alpha y)^{t-1} + b^n (b^n x + \alpha y)^{t-2}(b^n + \alpha y) \\
&\quad + (b^n + \alpha y)^2 \left[b^n (b^n x + \alpha y)^{t-3} + (b^n + \alpha y) T(G_{n,p,t-3})\right] \\
&= b^n (b^n x + \alpha y)^{t-1} + b^n (b^n x + \alpha y)^{t-2}(b^n + \alpha y) \\
&\quad + b^n (b^n x + \alpha y)^{t-3}(b^n + \alpha y)^2 + (b^n + \alpha y)^3 T(G_{n,p,t-3}) \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

$$\begin{aligned} &= b^n(b^nx+\alpha y)^{t-1}+b^n(b^nx+\alpha y)^{t-2}(b^n+\alpha y) \\ &\quad +b^n(b^nx+\alpha y)^{t-3}(b^n+\alpha y)^2+\cdots+(b^n+\alpha y)^{t-2}T(G_{n,p,t-(t-2)}) \\ &= b^n(b^nx+\alpha y)^{t-1}+b^n(b^nx+\alpha y)^{t-2}(b^n+\alpha y)+b^n(b^nx+\alpha y)^{t-3}(b^n+\alpha y)^2 \\ &\quad +\cdots+(b^n+\alpha y)^{t-2}T(G_{n,p,2})). \end{aligned}$$

But

$$\begin{aligned} T(G_{n,p,2}) &= b^nT(X_{n,p})+\alpha y(b+y-1)^n+y(b+y-1)^{2n} \\ &= b^n(b^nx+\alpha y)+\alpha y(b+y-1)^n+y(b+y-1)^{2n}. \end{aligned}$$

Thus

$$\begin{aligned} T(G_{n,p,t}) &= b^n(b^nx+\alpha y)^{t-1}+b^n(b^nx+\alpha y)^{t-2}(b^n+\alpha y) \\ &\quad +b^n(b^nx+\alpha y)^{t-3}(b^n+\alpha y)^2+\cdots+b^n(b^nx+\alpha y)^2(b^n+\alpha y)^{t-3} \\ &\quad +(b^n+\alpha y)^{t-2}\left[b^n(b^nx+\alpha y)+\alpha y(b+y-1)^n+y(b+y-1)^{2n}\right] \\ &= b^n(b^nx+\alpha y)^{t-1}+b^n(b^nx+\alpha y)^{t-2}(b^n+\alpha y) \\ &\quad +b^n(b^nx+\alpha y)^{t-3}(b^n+\alpha y)^2+\cdots \\ &\quad +b^n(b^nx+\alpha y)^2(b^n+\alpha y)^{t-3}+b^n(b^nx+\alpha y)(b^n+\alpha y)^{t-2} \\ &\quad +(b^n+\alpha y)^{t-2}\alpha y(b+y-1)^n+(b^n+\alpha y)^{t-2}y(b+y-1)^{2n} \\ &= b^n\sum_{k=0}^{t-2}(b^n+\alpha y)^k(b^nx+\alpha y)^{t-1-k} \\ &\quad +(b^n+\alpha y)^{t-2}\alpha y(b+y-1)^n+(b^n+\alpha y)^{t-2}y(b+y-1)^{2n}, \end{aligned}$$

which completes the proof. □

Theorem 3.9 $T(G^{(i)}_{n,p,t})=(1+x+\cdots+x^{t-i-1})(b^nx+\alpha y)^i+b^n\sum_{k=0}^{i-2}(b^n+\alpha y)^k(b^nx+\alpha y)^{i-1-k}+(b^n+\alpha y)^{i-2}\alpha y(b+y-1)^n+(b^n+\alpha y)^{i-2}y(b+y-1)^{2n}$.

Proof In $G^{(i)}_{n,p,t}$ there are $t-i$ sides of C_t without petals. Let e be any side of $G^{(i)}_{n,p,t}$ without petal. Clearly e is neither a loop nor a bridge. Applying deletion and contraction formula

$$\begin{aligned} G^{(i)}_{n,p,t} &= x^{t-i-1}T(H^{(i)}_{n,p})+T(G^{(i)}_{n,p,t-1}) \\ &= x^{t-i-1}T(H^{(i)}_{n,p})+x^{t-i-2}T(H^{(i)}_{n,p})+T(G^{(i)}_{n,p,t-2}) \\ &= (x^{t-i-1}+x^{t-i-2}+\cdots+x^{t-i-(t-i)})T(H^{(i)}_{n,p})+T(G^{(i)}_{n,p,t-(t-i)}) \\ &= (1+x+\cdots+x^{t-i-1})T(H^{(i)}_{n,p}+T(G^{(i)}_{n,p,i}) \\ &= (1+x+\cdots+x^{t-i-1})T(X^i_{n,p})+T(G_{n,p,i}) \\ &= (1+x+\cdots+x^{t-i-1})(b^nx+\alpha y)^i+b^n\sum_{k=0}^{i-2}(b^n+\alpha y)^k(b^nx+\alpha y)^{i-1-k} \\ &\quad +(b^n+\alpha y)^{i-2}\alpha y(b+y-1)^n+(b^n+\alpha y)^{i-2}y(b+y-1)^{2n}. \end{aligned}$$

§4. Conclusion

Tutte Polynomial has been an open topic for research for mathematicians for the last 30 years. It is a two variable polynomial which reduces to many graph polynomials associated with the graph. It gives various information about the graph like the number of spanning trees, number of cyclic orientations not resulting in oriented cycles and colorability of graphs.

In this research paper, Tutte polynomial of many specialized graphs have been studied in detail. The properties and Tutte polynomials of , generalized Book graph, Generalized Book graph with petals, Complete Generalized Book graph have been arrived at which in turn reveal various other related information by substituting appropriate values for the two variables.

References

[1] Gary Haggard, David J. Pearce, and Gordon Royle *Computing Tutte Polynomials*, www.newton.ac.uk/ files/preprints/ ni09024.pdf.

[2] Ira M. Gessel, Bruce E. Sagan, The Tutte polynomial of a graph, depth-first search, and simplicial complex partitions, *Electronic J. Combin.* (Foata Festschrift Volume), 3, (1996) Volume 69, Issue 3, pages 341 - 348, March 2012.

[3] Joanna A. Ellis-Monaghan, Criel Merino, Graph Polynomials and Their Applications I: The Tutte Polynomial, *Structural Analysis of Complex Networks*, 2011, pp 219-255,http://arxiv.org/pdf/0803.3079v2.pdf. Ser. B 84 (2002), pages 260 - 269.

[4] Martin Trinks, From spanning forests to edge subsets, *ARS Mathematica Contemporanea*, 7(2014), pages 141 - 151.

[5] A. R. Nizami, M. Munir, Ammara Usman, Saima Jabeen, M. Athar, Some specializations and Evaluations of Tutte Polynomial of a family of graphs, *Asian Journal of Natural & Applied Sciences*, Vol.2, No.2, June 2013.

[6] Mphako-Banda.E, Tutte Polynomials of Flower Graphs, *Bulletin of the Iranian Mathematical Society*, Vol. 35 No. 2 (2009), pp 179-190.

[7] Tutte W.T., Chromatic polynomials, in *Selected Topics in Graph Theory*(III), Academic Press, 1988, p. 15-42.

[8] Tutte W.T., A contribution to the theory of chromatic polynomials, *Can. J. Math.*, 6, pages 80 - 91 (1954)