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Traversability and Covering Invariants of Token Graphs

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Abstract: Let $F_k(G), k \geq 1$, G be the token graph of a connected graph G. In this paper, we investigate the Eulerian and Hamiltonian property of token graphs and obtain the covering invariants for complete graph of token graph.

Key Words: Token graph, α_0, α_1, β_0, β_1, symmetric difference.

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§1. Introduction

All graphs considered here are simple, connected,undirected graphs. Let G be a graph with vertex set V(G)=$\{v_1, v_2, v_3, \cdots, v_n\}$ and edge set $E(G) = \{e_1, e_2, e_3, \cdots, e_m\}$. We refer the reader to Harary [3].

R.Fabila-Monroy and et.al. introduced a model in which, k indistinguished tokens move from vertex to vertex along the edges of a graph. This idea is formalized as follows, for a graph G and integer $k \geq 1$, we define $F_k(G)$ to be the graph with vertex set $\binom{V(G)}{k}$, where two vertices A and B of $F_k(G)$ are adjacent whenever their symmetric difference $A \triangle B$ is a pair $\{a, b\}$ such that $a \in A$, $b \in B$ and $ab \in E(G)$. Thus the vertices of $F_k(G)$ correspond to configurations of k-indistinguishable tokens placed at distinct vertices of G, where two configurations are adjacent whenever one configuration can be reached from the other by moving one token along an edge from its current position to an unoccupied vertex. The $F_k(G)$ is called the *k-token graph* of G.

Many problems in mathematics and computer science are modeled by moving objects on the vertices of a graph according to certain prescribed rules. In *graph pebbeling*, a pebbling step consists of removing two pebbels from a vertex and placing one pebble on an adjacent vertex; [4] and [5] for surveys. Related pebbling games have been used to study rigidity [6,7], motion planning[1,9], and as models of computation[10]. In the "chip firing game", a vertex v fires by distributing one chip to each of its neighbors(assuming the number of chips at v is at least its degree). This model has connections with matroid, the Tutte polynomial, and mathematical physics [8].

Inspired by this we investigate the some more properties like traversability and covering invariants of token graphs.

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Remark 1.1([2]) Let G be a graph and $F_k(G)$ be the token graph of G with $k \geq n-1$, $|V(F_k(G))|=\binom{n}{k}$, $|E(F_k(G))|=\binom{n-2}{k-1}|E(G)|$.

Remark 1.2([2]) Two vertices A and B are adjacent in $F_k(G)$ if and only if $V(G) \setminus A$ and $V(G) \setminus B$ are adjacent in $F_{n-1}(G)$, $F_k(G) \cong F_{n-k}(G)$, with only one token, the token graph is isomorphic to G. Thus, $F_1(G) \cong G$.

Remark 1.3 Degree of vertices in $F_k(G)$ is

$$
\begin{aligned}
deg(V_{F_k}(G)) &= \sum_{i=1}^{n-1} deg_G(V_i) - 2 \\
&\quad \text{(sum of pairs of vertices } v_i; i \in k \text{ of } G \text{ which are the elements of } V\text{)}
\end{aligned}
$$

if v_i and v_j are two vertices in G, then in $F_k(G)$

$$
\begin{aligned}
v_i v_j &= 1, if\ \ v_i\ is\ adjacent\ \ to\ \ v_j\ \ in\ G. \\
&= 0,\ \ if\ v_i\ is\ not\ adjacent\ to\ v_j\ \ in\ G.
\end{aligned}
$$

Remark 1.4 If degree of all the vertices in a graph G is even or even regular then by the Remark 3 degree of all the vertices in $F_k(G)$ is even, irrespective of tokens being odd or even.

Remark 1.5 If degree of all the vertices in a graph G is odd or odd regular then by Remark 3, degree of all the vertices in $F_k(G)$ is even, only when k is even token.

Remark 1.6 If G contains both even and odd degree vertices then the vertices in $F_k(G)$ are also of odd and even degree, irrespective of tokens being odd or even.

§2. Traversability of Token Graphs

In this section we obtain the traversability properties of token graphs.

Theorem 2.1 *Let G be a connected graph. Then* $F_k(G)$ *is Eulerian if and only if it satisfies either of the following conditions.*

(i) *Every vertex in G is of even degree;*

(ii) *Every vertex in G is of odd degree and k-is even.*

Proof Let $F_k(G)$ be a token graph of graph G. Assume $F_k(G)$ is Eulerian, that is each vertices in $F_k(G)$ is of even degree. By the Remark 1.3, we have, $d(V_{F_k(G)}) = deg(u) + deg(v)$-2(sum of pair of adjacent elements of G in V of $F_k(G)$).

Depending upon the degree, we consider the following cases.

Case 1. Suppose $deg(u)+deg(v)$ is odd, then by Remark 3, $d(V_{F_k(G)})$ is odd, a contradiction. Thus condition (i) is satisfied.

Case 2. Suppose $deg(u)+deg(v)$ is even, where u and v are odd or odd regular with odd tokens then by Remarks 1.3 and 1.5. $F_k(G)$ is non-Eulerian, a contradiction. Thus the condition (ii) is satisfied.

If G is Eulerian, that is it contains even degree of vertices. Then by the Remark 4, $F_k(G)$ is Eulerian. That is it contains even degree vertices.

The converse follows from Remarks 1.4 and 1.5. □

Corollary 2.2 *If G be Eulerian graph, then $F_{n-1}(G)$ is also Eulerian.*

Proof Let G be Eulerian graph. Then by the Remark 1.2, $F_{n-1}(G)$ is Eulerian. □

Lemma 2.3 *If G is hamiltonian, then $F_{n-1}(G)$ is also hamiltonian.*

Proof Suppose G is hamiltonian, by the Remark 1.2, we know that

$$G \cong F_1(G) \text{ and } F_k(G) \cong F_{n-1}(G).$$

If $k = 1$ then,

$$F_1(G) \cong F_{n-1}(G)$$

Therefore,

$$G \cong F_1(G), \quad G \cong F_{n-1}(G).$$

Thus, $F_{n-1}(G)$ is also hamiltonian. □

Theorem 2.4 *$F_k(G)$ is hamiltonian if and only if G is complete graph.*

Proof Let G be complete graph and let $\{v_1, v_2, v_3, \cdots, v_n\}$ be the vertices in a graph G. In complete graph all vertices are mutually adjacent and G is hamiltonian.

By the definition of token graph, $F_k(G)$ contains $\binom{n}{k}$ number of vertices and by the Lemma 2.3, $F_1(G)$ and $F_{n-1}(G)$ are hamiltonian.

Now, we have to prove for $k = 2, 3, 4, \cdots, n-2$ tokens. We prove this by induction method, here $(k+1)^{th}$ term is n-2 token.

If k=2 token then,

$$\begin{aligned} V(F_2(G)) &= \{(v_1v_2), (v_1v_3), (v_1v_4), \cdots, (v_1v_n) \cup (v_2v3), (v_2v_4), (v_2v_5), \\ &\quad \cdots, (v_2v_n)\cup, \cdots, \cup(v_{n-1}v_n)\}. \end{aligned}$$

Here we consider two vertices $A = \{v_1v_n\}$ and $B = \{v_2v_n\}$. By the symmetric difference we get v_1v_2. That is,$|A \triangle B| = (A \cup B) - (A \cap B)$ =$v_1v_2v_n - v_n$= v_1v_2. Therefore v_1v_2 are adjacent in G then A and B are also adjacent in $F_2(G)$.

Similarly v_2v_n is adjacent with v_3v_n and the same follows for all vertices.

Now if k=3 token, then

$$\begin{aligned} V(F_3(G)) &= \{(v_1v_2v_3),(v_1v_2v_4),\cdots,(v_1v_2v_3),(v_1v_3v_4),(v_1v_3v_5),\\ &\quad \cdots,(v_1v_3v_n),\cdots,(v_{n-2}v_{n-1}v_n)\}. \end{aligned}$$

Here also $(v_1v_2v_n)$ is adjacent with $(v_1v_3v_n)$,$(v_1v_3v_n)$ with $(v_1v_4v_n)$,$\cdots$ and $(v_{n-3}v_{n-2}v_{n-1})$ with $(v_{n-2}v_{n-1}v_n)$. Hence, we get spanning cycle in $F_3(G)$ as $\{v_1v_{n-1}v_n, v_2v_{n-1}v_n, v_3v_{n-1}v_n,\cdots, v_{n-2}v_{n-1}v_n, v_1v_{n-1}v_n\}$. Therefore, $F_3(G)$ is hamiltonian graph. Thus the result is true for all k=n.

Similarly, If k=n-2 token, then $V(F_{n-2}(G)) = V(F_2(G))$. By the Lemma 2.3 and Remark 1.2, $F_{n-1}(G) \cong F_1(G)$. That is,

$$F_k(G) \cong F_{n-k}(G), \tag{1}$$

$$F_2(G) \cong F_{n-k}(G), \tag{2}$$

and if $k = 1$, then

$$F_1(G) \cong F_{n-1}(G). \tag{3}$$

Then $G \cong F_1(G) \cong F_{n-1}(G)$. Thus $F_{n-1}(G)$ is hamiltonian.

For the converse, assume $F_k(G)$ is hamiltonian, we have to prove G is complete. Suppose G is not complete graph then by the symmetric difference the vertices in $F_k(G)$; k= 2,3,...,n-2, form a sub graph homiomorphic to $K_{2,3}$ a contradiction.

Theorem 2.5 *If G is wheel, then $F_k(G)$ is hamiltonian graph.*

Proof Let G be wheel, hence it contains spanning cycle and let $\{v_1, v_2, v_3,\cdots, v_n\}$ be the vertex of graph G. Here v_n is a vertices of maximum degree in G. Let $V_1, V_2, V_3,\cdots, V_{\binom{n}{k}}$ be the vertices in graph $F_k(G)$. By the lemma 2.3, we know that

$$G \cong F_1(G) \cong F_{n-1}(G).$$

Then G is hamiltonian then $F_1(G)$ and $F_{n-1}(G)$ are also hamiltonian.

Now we prove for $k = 2, 3, 4,\cdots, n-2$ tokens. We know that $F_k(G) \cong F_{n-1}(G)$. If k=2 then,

$$\begin{aligned} V(F_2(G)) &= \{V_1, V_2, V_3,\cdots, V_{\binom{n}{k}}\}\\ &= \{(v_1v_2),(v_1v_3),\cdots,(v_1v_n),(v_2v_3),\cdots,(v_2v_n),(v_3v_4),\\ &\quad (v_3v_5),...,(v_3v_n),...,(v_{n-1}v_n)\}. \end{aligned}$$

In graph G, the n^{th} vertex is adjacent with remaining all the vertices. Therefore by the symmetric difference we get spanning cycle as $\{v_1v_n, v_1v_{n-1}, v_1v_{n-2},\cdots, v_1v_2, v_2v_n, v_2v_{n-1},\cdots, v_2v_3, v_3v_n,\cdots, v_3v_4, v_4v_n,\cdots, v_{n-2}v_{n-1}, v_{n-1}v_n, v_1v_n\}$. Thus $F_2(G)$ contains spanning cycle then $F_{n-2}(G)$ also contains spanning cycle. Clearly, by Remark 2.2, $F_2(G)$ and $F_{n-2}(G)$ are hamiltonian.

Similarly for all tokens we get spanning cycle. Hence $F_k(G)$ is hamiltonian. □

§3. Covering Invariants Of Token Graphs

In the following section, we determine the point covering number $\alpha_0(G)$,line covering number $\alpha_1(G)$, point independence number $\beta_0(G)$ and line independence number $\beta_1(G)$ of token graph of complete graph.

Theorem 3.1 *For any complete graph $K_n; n > 1$,*

$$\alpha_1(F_k(K_n)) = \left\lceil \frac{\binom{n}{k}}{2} \right\rceil, \quad \beta_1(F_k(K_n)) = \left\lfloor \frac{\binom{n}{k}}{2} \right\rfloor.$$

Proof Let K_n be the complete graph with n-vertices and $F_k(K_n)$ be the token graph of complete graph with $\binom{n}{k}$ number of vertices. $\left\lceil \frac{\binom{n}{k}}{2} \right\rceil$ lines are required cover all the points in $F_k(K_n)$. By Remark 1.2, $F_1(K_n) \cong K_n$ and $F_{n-1}(K_n) \cong K_n$, i.e., $F_k(K_n) \cong K_n$ when $k = 1$ or $n - 1$.

For $k = 2, 3, 4, \cdots, n-2$, the vertices $F_k(K_n)$ are adjacent but not mutually and by Remark 1.1, it contains $\binom{n}{k}$ number of vertices. Hence $\left\lceil \frac{\binom{n}{k}}{2} \right\rceil$ number of lines are require to cover all the points. $\alpha_1(F_k(K_n)) = \left\lceil \frac{\binom{n}{k}}{2} \right\rceil$.

From the Gallai result, we know that

$$\alpha_1(G) + \beta_1(G) = |G|$$

In $F_k(K_n)$,

$$\begin{aligned}
\alpha_1(F_k(K_n)) + \beta_1(F_k(K_n)) &= \binom{n}{k} \Rightarrow \lceil \frac{\binom{n}{k}}{2} \rceil + \beta_1(F_k(K_n)) = \binom{n}{k}, \\
\beta_1(F_k(K_n)) &= \binom{n}{k} - \left\lceil \frac{\binom{n}{k}}{2} \right\rceil \\
&= \binom{n}{k} - \frac{\binom{n}{k}}{2} \quad (\text{Note that } \frac{\binom{n}{k}}{2} \cong \left\lceil \frac{\binom{n}{k}}{2} \right\rceil.) \\
&= \frac{\binom{n}{k}}{2}.
\end{aligned}$$

But $F_k(G)$ contains odd number of vertices then,

$$\beta_1(K_n) = \frac{\binom{n}{k}}{2} \cong \lfloor \frac{\binom{n}{k}}{2} \rfloor, \quad \beta_1(K_n) = \lfloor \frac{\binom{n}{k}}{2} \rfloor.$$

□

Theorem 3.2 *For any complete graph $K_n; n > 1$,*

$$\beta_0(F_k(K_n)) = \begin{cases} \left\lceil \frac{\binom{n}{k}}{\triangle(K_n)} \right\rceil & if \ \binom{n}{k} \ is \ even, \\ \\ \left\lfloor \frac{\binom{n}{k}}{\triangle(K_n)} \right\rfloor + 1 & if \ \binom{n}{k} \ is \ odd \end{cases}$$

and $\alpha_0(F_k(K_n)) = \binom{n}{k} - \beta_0(F_k(K_n))$.

Proof Let K_n be the complete graph with n-vertices and $F_k(K_n)$ be the token graph of complete graph with $\binom{n}{k}$ number of vertices. By the definition of complete graph, $\triangle(K_n) = n-1$ and $\alpha_0(K_n) = n-1$, $\beta_0(K_n) = 1$. Therefore by the Remark 1.1, $\alpha_0(F_k(K_n)) = n-1$ and $\beta_0(F_k(K_n)) = 1$ when $k = 1$ or $n-1$.

Now we have to prove for $k = 2, 3, 4, \cdots, n-2$ tokens. $F_k(K_n)$ contains $\binom{n}{k}$ number of vertices, and $\left\lceil \frac{\binom{n}{k}}{\triangle(K_n)} \right\rceil$ are required to cover the vertices in $F_k(K_n)$. $\left\lceil \frac{\binom{n}{k}}{\triangle(K_n)} \right\rceil$ number of vertices are non-adjacent to each other and adjacent with remaining $\binom{n}{k} - \left\lceil \frac{\binom{n}{k}}{\triangle(K_n)} \right\rceil$ number of vertices in $F_k(K_n)$, when $\binom{n}{k}$ is even. if $\binom{n}{k}$ is odd, then $\binom{n}{k}$ is covered by $\left\lceil \frac{\binom{n}{k}}{\triangle(K_n)} \right\rceil + 1$ vertices, which are non-adjacent to each other and are adjacent with remaining $\binom{n}{k} - \left\lceil \frac{\binom{n}{k}}{\triangle(K_n)} \right\rceil + 1$ vertices in $F_k(K_n)$. Thus independence number in $F_k(K_n)$ is $\left\lceil \frac{\binom{n}{k}}{\triangle(K_n)} \right\rceil$ or $\left\lceil \frac{\binom{n}{k}}{\triangle(K_n)} \right\rceil + 1$. So,

$$\beta_0(F_k(K_n)) = \begin{cases} \left\lceil \frac{\binom{n}{k}}{\triangle(K_n)} \right\rceil & \text{if } \binom{n}{k} \text{ is even,} \\ \\ \left\lfloor \frac{\binom{n}{k}}{\triangle(K_n)} \right\rfloor + 1 & \text{if } \binom{n}{k} \text{ is odd.} \end{cases}$$

From the Gallai result, we know that

$$\alpha_1(G) + \beta_1(G) = |G|.$$

In $F_k(K_n)$, when $\binom{n}{k}$ is even then,

$$\begin{aligned} \alpha_0(F_k(K_n)) + \beta_0(F_k(K_n)) &= \binom{n}{k} \\ \Rightarrow \alpha_0(F_k(K_n)) + \left\lceil \frac{\binom{n}{k}}{\triangle(G)} \right\rceil &= \binom{n}{k} \\ \alpha_0(F_k(K_n)) &= \binom{n}{k} - \left\lceil \frac{\binom{n}{k}}{\triangle(G)} \right\rceil \\ \alpha_0(F_k(K_n)) &= \binom{n}{k} - \beta_0(F_k(K_n)). \end{aligned}$$

If $\binom{n}{k}$ is odd then,

$$\begin{aligned} \alpha_0(F_k(K_n)) + \beta_0(F_k(K_n)) &= \binom{n}{k} \\ \Rightarrow \alpha_0(F_k(K_n)) + \left\lceil \frac{\binom{n}{k}}{\triangle(G)} \right\rceil + 1 &= \binom{n}{k} \end{aligned}$$

$$\alpha_0(F_k(K_n)) = \binom{n}{k} - \left\lceil \frac{\binom{n}{k}}{\triangle(G)} \right\rceil + 1$$
$$\alpha_0(F_k(K_n)) = \binom{n}{k} - \beta_0(F_k(K_n)).$$

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