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The Merrifield-Simmons Indices of Triangle-Trees with k Pendant-Triangles

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Abstract: Triangle-trees are a kind of graphs derived from Koch networks. The Merrifield-Simmons index of a graph is the total number of the independent sets of the graph. We prove that $P^{\Delta}_{k,n-k}$ is the triangle-tree with maximal Merrifield-Simmons index among all the triangle-trees with n triangles and k pendant triangles.

Key Words: Triangle-tree; Merrifield-Simmons index; pendant-triangle.

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§1. Introduction

The Koch networks (see [10], [13]) are derived from the Koch fractals (see [4], [9]) and are constructed iteratively. Let $K_{m,g}$ (m is a natural number) denote the Koch network after g iterations. Then, the family of Koch networks can be generated in the following way: initially ($g = 0$), $K_{m,0}$ consists of a triangle with three nodes labeled respectively by x, y, z, which have the highest degree among all nodes in the networks. For $g \geq 1$, $K_{m,g}$ is obtained from $K_{m,g-1}$ by performing the following operation. For each of the three nodes in every existing triangle in $K_{m,g-1}$, we add m groups of nodes. Each node group contains two nodes, both of which and their 'mother' node are connected to one another forming a new triangle. In other words, to get $K_{m,g}$ from $K_{m,g-1}$, we can replace each triangle in $K_{m,g-1}$ by a connected cluster on the right-hand side of the arrow in Fig.1.

Note that a Koch network does not have any cycle except for the triangles, we can call such a graph a triangle-tree.

Definition 1.1 *Let T^{Δ}_n (n is a natural number) denote a triangle-tree with n triangles. The family of triangle-trees can be generated in the following way: initially $n = 1$, T^{Δ}_1 consists of a triangle with three vertices labeled respectively by x, y, z. For $n \geq 2$, T^{Δ}_n is obtained from T^{Δ}_{n-1} by adding a pair of new vertices u, v, both of them are joined to a vertex of T^{Δ}_{n-1} and the edge uv is also added to form a new triangle. In other words, to get T^{Δ}_n from T^{Δ}_{n-1}, we add a new triangle to T^{Δ}_{n-1} by identifying a vertex of the new triangle with a vertex of T^{Δ}_{n-1}.*

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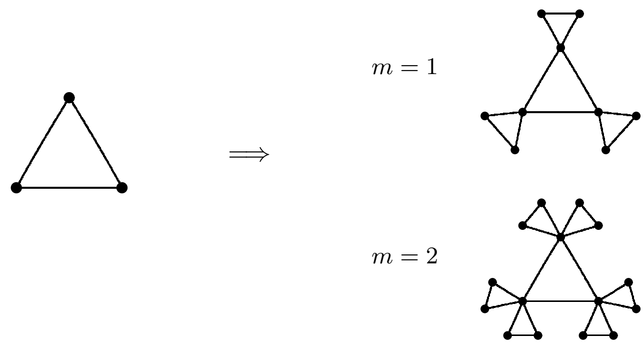

Fig.1

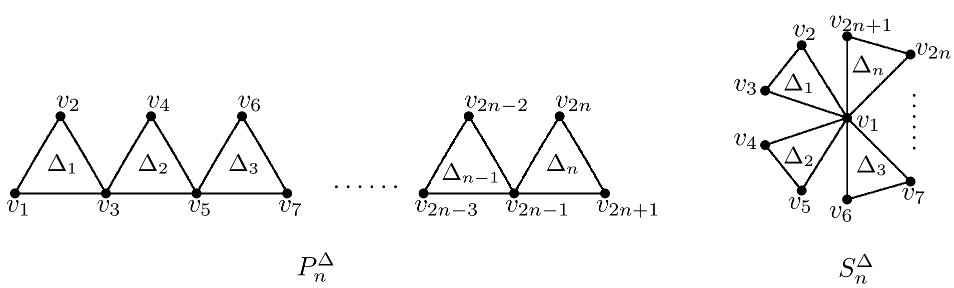

Fig.2

Obviously Koch networks are all triangle-trees. Suppose T^{Δ} is a triangle-tree, Δ is a triangle of T, if there are two vertices with degree two in Δ, we call the triangle Δ a pendant triangle of T^{Δ}. The triangle-path P_n^{Δ} (see Fig.2) is the only triangle-tree with only two pendant triangles and the triangle-star S_n^{Δ} (see Fig.2) is the only triangle-tree with n pendant triangles. For any two triangles Δ_1 and Δ_2 of T^{Δ}, if Δ_1 and Δ_2 have a common vertex, we say Δ_1 and Δ_2 are adjacent, and the distance between Δ_1 and Δ_2 is 1, denoted by $d(\Delta_1, \Delta_2) = 1$. If Δ_1 and Δ_2 do not have a common vertex, there is only one triangle-path between them. If the triangle-path between Δ_1 and Δ_2 contains d triangles, we say the distance between Δ_1 and Δ_2 is $d-1$, denoted by $d(\Delta_1, \Delta_2) = d-1$. The diameter of a triangle-tree is denoted by d^{Δ}, defined as

$$d^{\Delta}(T_n^{\Delta}) = \max\{d(\Delta, \Delta') \mid \Delta, \Delta' \text{are two triangles of } T_n^{\Delta}\}.$$

Throughout this paper $G = (V, E)$ is a finite simple undirected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The neighborhood of a vertex $v \in V$ is the set $N_G(v) = \{w : w \in V, vw \in E\}$, $d_G(v) = |N_G(v)|$, and $N_G[v] = N_G(v) \cup \{v\}$. For $S \subseteq V$, we use $G - S$ for the subgraph induced by $V(G) \setminus S$, $G[S]$ for the subgraph of G induced by S and $N_S(v) = \{w : w \in S, vw \in E(G)\}$. For $F \subseteq E(G)$, we use $G - F$ for the subgraph of G obtained by deleting F.

Let G be a graph on n vertices. Two vertices of G are said to be independent if they are not adjacent in G. A k-independent set of G is a set of k-mutually independent vertices. Denote by $f_k(G)$ the number of the k-independent sets of G. For convenience, we regard the empty vertex set as an independent set. Then $f_0(G) = 1$ for any graph G. Let $\alpha(G)$ denote the cardinality of a maximal independent set of G.

The *Merrifield-Sommons index* was introduced by Prodinger and Tichy in 1982, which is defined by

$$i(G) = \sum_{s=0}^{\alpha(G)} f_s(G),$$

although it is called Fibonacci number of a graph in [8]. It is one of the most popular topological indices in chemistry, which was extensively studied in monograph [7]. Now there have been many papers studying the Merrifield-Simmons index. In [8], Prodinger and Tichy showed that, for trees with order n, the star has the maximal Merrifield-Simmons index and the path has the minimal Merrifield-Simmons index. In [6], Li et al characterized the tree with the maximal Merrifield-Simmons index among the trees with given diameter. In [11], Yu and Lv characterized the trees with maximal Merrifield-Simmons indices, among the trees with given pendant vertices. For more results on Merrifield-Simmons index, see [1-3], [5] and [12].

Due to the similarity of triangle-trees and ordinary trees, it is very interesting to study the Merrifield-Simmons indices of triangle-trees. It is easy verify that, among all the triangle-trees with n triangles, S_n^{Δ} is the triangle-tree with maximal Merrifield-Simmons index and P_n^{Δ} is the triangle-tree with minimal Merrifield-Simmons index. As noting this result is similar to the result of ordinary trees, we consider all the triangle-trees with n triangles and k pendant triangles. It is very interesting to find that $P_{k,n-k}^{\Delta}$ (as shown in Fig. 3) is the triangle-tree with maximal Merrifield-Simmos index among all such triangle-trees, and this result is also similar to the result of ordinary trees.

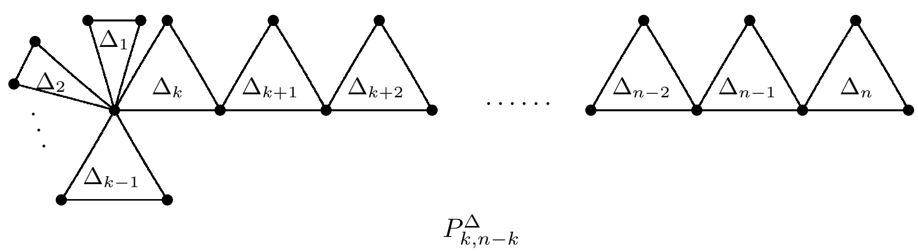

Fig.3

§2. Lemmas and Results

We first introduce the following lemma, which is obvious and well-known.

Lemma 2.1 *For a graph G, we have*

(1) $i(G) = i(G - v) + i(G - N[v])$ *for any* $v \in V(G)$;
(2) $i(G) = i(G - e) - i(G - N[e])$ *for any* $e \in E(G)$;
(3) *If* $G = G_1 \cup G_2$, *then* $i(G) = i(G_1)i(G_2)$.

Using the above lemma, we can derive some recursion formulas on the Merrifield-Simmons index of the triangle-path P_n^Δ. Denote $a_n = i(P_n^\Delta)$. It is easy to see that $a_1 = 4, a_2 = 10, a_3 = 24$. Let $Q_n = P_n^\Delta - v_1$, where v_1 is one of the vertices with degree two of the pendant-triangle of P_n^Δ(as shown in Fig 2) and $b_n = i(Q_n)$. It is easy to see that $b_1 = 3, b_2 = 7, b_3 = 17$. Let $R_n = Q_n - v_{2n+1}$, where v_{2n+1} is one of the vertices with degree two of another pendant-triangle of P_n^Δ (as shown in Fig.2). It is easy to see that $c_1 = 2, c_2 = 5, c_3 = 12$.

By Lemma 2.1, we know

$$\begin{aligned} a_n &= b_n + b_{n-1}, \\ b_n &= a_{n-1} + b_{n-1} = c_n + c_{n-1}, \\ c_n &= b_{n-1} + c_{n-1}. \end{aligned}$$

So we have

$$\begin{aligned} b_{n+1} &= 2b_n + b_{n-1}, \\ a_{n+1} &= 2a_n + a_{n-1}, \\ c_{n+1} &= 2c_n + c_{n-1}. \end{aligned}$$

Let $P_k^\Delta = \Delta_1\Delta_2\cdots\Delta_k$ be a path of a triangle-tree T^Δ, where $\Delta_i = v_{2i-1}v_{2i}v_{2i+1}$. If $d_{T^\Delta}(v_1) \geq 6,\ d_{T^\Delta}(v_{2k+1}) \geq 6,\ d_{T^\Delta}(v_{2i}) = 2\ (1 \leq i \leq k)$ and $d_{T^\Delta}(v_{2i+1}) = 4\ (1 \leq i \leq k-1)$, we call P_k^Δ an internal triangle-path of T^Δ. If the triangle $\Delta_1 = v_1v_2v_3$ is a pendant triangle of T^Δ, $d_{T^\Delta}(v_{2k+1}) \geq 6$, $d_{T^\Delta}(v_{2i}) = 2\ (1 \leq i \leq k)$ and $d_{T^\Delta}(v_{2i+1}) = 4\ (1 \leq i \leq k-1)$, we call P^Δ a pendant triangle-path of T^Δ. Let $s(T^\Delta)$ be the number of vertices in T^Δ with degree not less than 6 and $p(T^\Delta)$ be the number of pendant triangle-paths in T^Δ with length not less than 1.

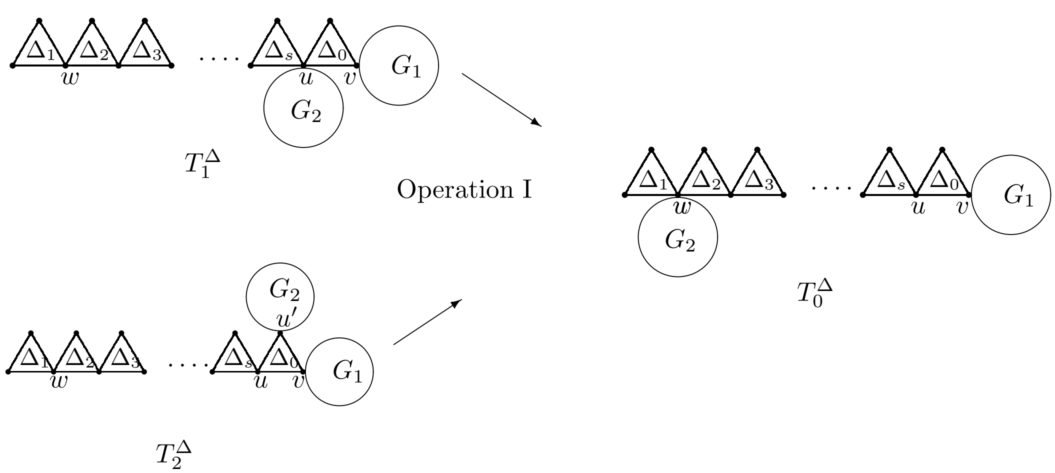

Fig.4

Denote $\mathcal{T}_{n,k}^\Delta$ $(3 \leq k \leq n-1)$ be the set of all triangle-trees with n triangles and k pendant triangles. In the following, we shall define two kinds of operations of $T^\Delta \in \mathcal{T}^\Delta_{n,k}$ and show that these two kinds of operations make the Merrifield-Simmons indices of the triangle-tree increase strictly.

If $T^\Delta \in \mathcal{T}^\Delta_{n,k}, T^\Delta \ncong P^\Delta_{k,n-k}$ and $p(T^\Delta) \neq 0$, then T^Δ can be seen as the triangle-trees T_1^Δ or T_2^Δ as shown in Fig.4, where $\Delta_1\Delta_2\cdots\Delta_s$ $(s \geq 2)$ is a pendant path of T^Δ with s triangles, G_1 and G_2 are two subtriangle-trees of T^Δ and $|V(G_1)| \geq 3, |V(G_2)| \geq 3$. If T_0^Δ is obtained from T_1^Δ or T_2^Δ by Operation I (as shown in Fig.4), it is easy to see that $T_0^\Delta \in \mathcal{T}^\Delta_{n,k}$.

Now we show that operation I makes the Merrifield-Simmons indices of the triangle-trees increase strictly.

Lemma 2.2 *If T_0^Δ is obtained from T_1^Δ or T_2^Δ by operation I, then $i(T_0^\Delta) > i(T_1^\Delta)$ and $i(T_0^\Delta) > i(T_2^\Delta)$.*

Proof Let $N_{G_1}[v] = V_1$, $N_{G_2}[u] = V_2$ in T_1^Δ, $N_{G_2}[u'] = V_2'$ in T_2^Δ and $N_{G_2}[w] = V_3$ in T_0^Δ.

If $s \geq 3$, by Lemma 2.1, we have

$$\begin{aligned} i(T_1^\Delta) &= i(T_1^\Delta - v) + i(T_1^\Delta - N_{T_1^\Delta}[v]) \\ &= i(G_1 - v)(2i(G_2 - u)b_s + i(G_2 - V_2)b_{s-1}) + i(G_1 - V_1)i(G_2 - u)b_s, \end{aligned}$$

$$\begin{aligned} i(T_2^\Delta) &= i(T_2^\Delta - v) + i(T_2^\Delta - N_{T_2^\Delta}[v]) \\ &= i(G_1 - v)(i(G_2 - u')a_s + i(G_2 - V_2')b_s) + i(G_1 - V_1)i(G_2 - u')b_s, \end{aligned}$$

$$\begin{aligned} i(T_0^\Delta) &= i(T_0^\Delta - v) + i(T_0^\Delta - N_{T_0^\Delta}[v]) \\ &= i(G_1 - v)(3i(G_2 - w)c_s + i(G_2 - V_3)c_{s-1}) \\ &\quad + i(G_1 - V_1)(3i(G_2 - w)c_{s-1} + i(G_2 - V_3)c_{s-2}). \end{aligned}$$

Obviously, $i(G_2 - w) = i(G_2 - u') = i(G_2 - u)$ and $i(G_2 - V_3) = i(G_2 - V_2') = i(G_2 - V_2)$, so we have

$$\begin{aligned} &i(T_0^\Delta) - i(T_1^\Delta) \\ &= i(G_1 - v)i(G_2 - u)(3c_s - 2b_s) + i(G_1 - v)i(G_2 - V_2)(c_{s-1} - b_{s-1}) \\ &\quad + i(G_1 - V_1)i(G_2 - u)(3c_{s-1} - b_s) + i(G_1 - V_1)i(G_2 - V_2)c_{s-2} \\ &= i(G_1 - v)i(G_2 - u)c_{s-2} - i(G_1 - v)i(G_2 - V_2)c_{s-2} \\ &\quad - i(G_1 - V_1)i(G_2 - u)c_{s-2} + i(G_1 - V_1)i(G_2 - V_2)c_{s-2} \\ &= c_{s-2}(i(G_1 - v) - i(G_1 - V_1))(i(G_2 - u) - i(G_2 - V_2)). \end{aligned}$$

Since $s \geq 3, c_{s-2} > 0$, $i(G_1 - v) - i(G_1 - V_1) > 0$ and $i(G_2 - u) - i(G_2 - V_2) > 0$, we know $i(T_0^\Delta) - i(T_1^\Delta) > 0$ when $s \geq 3$.

Similarly,

$$\begin{aligned}
&i(T_0^\Delta)-i(T_2^\Delta)\\
&=i(G_1-v)i(G_2-u')(3c_s-a_s)+i(G_1-v)i(G_2-V_2')(c_{s-1}-b_s)\\
&\quad+i(G_1-V_1)i(G_2-u')(3c_{s-1}-b_s)+i(G_1-V_1)i(G_2-V_2')c_{s-2}\\
&=i(G_1-v)i(G_2-u')(c_{s-2}+2c_{s-1})+i(G_1-v)i(G_2-V_2')(-c_{s-2}-2c_{s-1})\\
&\quad+i(G_1-V_1)i(G_2-u')(-c_{s-2})+i(G_1-V_1)i(G_2-V_2')c_{s-2}\\
&=c_{s-2}(i(G_1-v)-i(G_1-V_1))(i(G_2-u')-i(G_2-V_2'))\\
&\quad+2c_{s-1}i(G_1-v)(i(G_2-u')-i(G_2-V_2'))>0.
\end{aligned}$$

Therefore, $i(T_0^\Delta)-i(T_2^\Delta)>0$ when $s\geq 3$. If $s=2$, similarly, we have

$$\begin{aligned}
&i(T_0^\Delta)-i(T_1^\Delta)\\
&=i(G_1-v)i(G_2-u)(3c_2-2b_2)+i(G_1-v)i(G_2-V_2)(c_1-b_1)\\
&\quad+i(G_1-V_1)i(G_2-u)(3c_1-b_2)+i(G_1-V_1)i(G_2-V_2)\\
&=(i(G_1-v)-i(G_1-V_1))(i(G_2-u)-i(G_2-V_2)).
\end{aligned}$$

$$\begin{aligned}
&i(T_0^\Delta)-i(T_2^\Delta)\\
&=i(G_1-v)i(G_2-u')(3c_2-a_2)+i(G_1-v)i(G_2-V_2')(c_1-b_2)\\
&\quad+i(G_1-V_1)i(G_2-u')(3c_1-b_2)+i(G_1-V_1)i(G_2-V_2')\\
&=(i(G_1-v)-i(G_1-V_1))(i(G_2-u')-i(G_2-V_2'))\\
&\quad+4i(G_1-v)(i(G_2-u')-i(G_2-V_2'))>0.
\end{aligned}$$

Therefore, $i(T_0^\Delta)-i(T_1^\Delta)>0$ and $i(T_0^\Delta)-i(T_2^\Delta)>0$ when $s=2$. □

From Lemma 2.2, we can immediately get the following result.

Lemma 2.3 *Let $T^\Delta\in\mathcal{T}_{n,k}^\Delta(3\leq k\leq n-1)$, $T^\Delta\ncong P_{k,n-k}^\Delta$ and $p(T^\Delta)\geq 1$.*

(1) If $s(T^\Delta)=1$, we can finally get a triangle-tree T'^Δ by operation I with $i(T'^\Delta)>i(T^\Delta)$, and $p(T'^\Delta)=1$; it is easy to see that $T'^\Delta\cong P_{k,n-k}^\Delta$;

(2) If $s(T^\Delta)>1$, we can finally get a triangle-tree T'^Δ by operation I with $i(T'^\Delta)>i(T^\Delta)$ and $p(T'^\Delta)=0$.

If $T^\Delta\in\mathcal{T}_{n,k}^\Delta(3\leq k\leq n-1), T^\Delta\ncong P_{n,k}^\Delta$ and $p(T^\Delta)=0$, then we can find two pendant triangles Δ_1 and Δ_1' of T^Δ such that $d(\Delta_1,\Delta_1')=d^\Delta(T^\Delta)$. Suppose $\Delta_1=uu_1u_1'$ and $\Delta_1'=vv_1v_1'$, where u_1,u_1',v_1,v_1' are the vertices with degree 2 and $d(u)\geq 6$, $d(v)\geq 6$. Then the triangle-tree can be seen as the triangle-tree T^Δ shown in Fig 5, where $\Delta_1,\Delta_2,\cdots,\Delta_s$ are pendant triangles with common vertex u, $\Delta_1',\Delta_2',\cdots,\Delta_t'$ are pendant triangles with common vertex v, G_1 is the subgraph of T^Δ induced by $V(T^\Delta)\setminus\Big(\bigcup_{i=1}^{s}V(\Delta_i)\cup\bigcup_{i=1}^{t}V(\Delta_i')\Big)$.

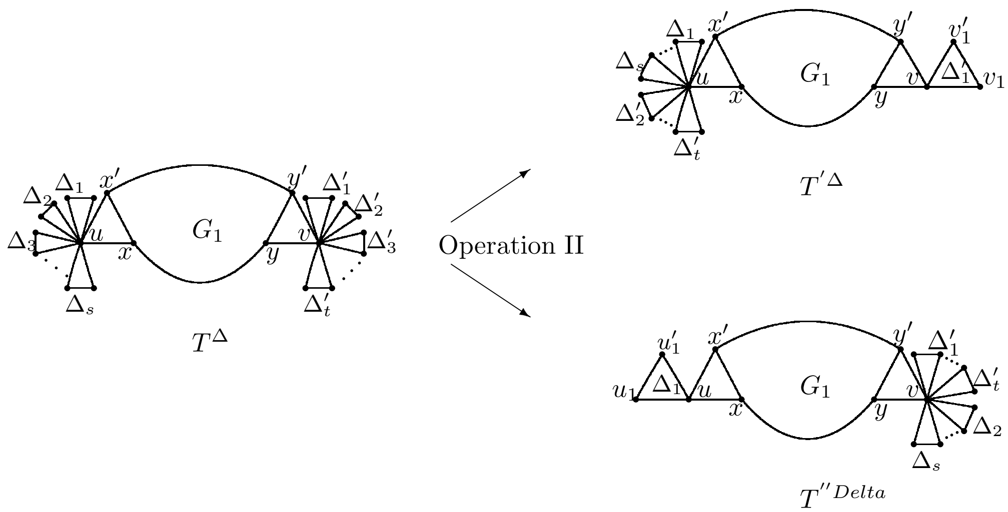

Fig.5

Note that if $d(\Delta_1,\Delta_2)=3$, then $x=y$; if $d(\Delta_1,\Delta_2)\geq 4$, then $|V(G_1)|\geq 5$. $T^{'\Delta}$ is a triangle-tree got from T^{Δ} by moving the pendant triangles $\Delta'_2,\Delta'_3,\cdots,\Delta'_t$ from v to u, and $T^{''\Delta}$ is a triangle-tree got from T^{Δ} by moving the pendant triangles $\Delta_2,\Delta_3,\cdots,\Delta_s$ from u to v. We say both of $T^{'\Delta}$ and $T^{''\Delta}$ are obtained from T^{Δ} by Operation II. It is easy to see that $T^{'\Delta},T^{''\Delta}\in\Gamma^{\Delta}_{n,k}$ $p(T^{'\Delta})=p(T^{''\Delta})=1$ and $s(T^{'\Delta})=s(T^{''\Delta})=s(T^{\Delta})-1$.

Lemma 2.4 *If $T^{'\Delta}$ and $T^{''\Delta}$ are obtained from T^{Δ} by Operation II, then either $i(T^{'\Delta})>i(T^{\Delta})$ or $i(T^{''\Delta})>i(T^{\Delta})$.*

Proof If $d(\Delta_1,\Delta'_1)\geq 3$, then $N_{G_1}(u)=\{x,x'\}$ and $N_{G_1}[v]=\{y,y'\}$. Note that if $d(\Delta_1,\Delta'_1)=3$, then $x=y$. By Lemma 2.2, we have

$$\begin{aligned}i(T^{\Delta}) &= i(T^{\Delta}-u)+i(T^{\Delta}-N_{T^{\Delta}}[u])\\ &= 3^s(3^t i(G_1)+i(G_1-\{y,y'\}))+3^t i(G_1-\{x,x'\}\})+i(G_1-\{x,x',y,y'\}),\end{aligned}$$

$$\begin{aligned}i(T^{'\Delta}) &= i(T^{'\Delta}-u)+i(T^{'\Delta}-N_{T^{'\Delta}}[u])\\ &= 3^{s+t-1}(3i(G_1)+i(G_1-\{y,y'\}))+3i(G_1-\{x,x'\})+i(G_1-\{x,x',y,y'\}),\end{aligned}$$

$$\begin{aligned}i(T^{''\Delta}) &= i(T^{''\Delta}-u)+i(T^{''\Delta}-N_{T^{''\Delta}}[u])\\ &= 3(3^{s+t-1}i(G_1)+i(G_1-\{y,y'\}))+3^{s+t-1}i(G_1-\{x,x'\})+i(G_1-\{x,x',y,y'\}).\end{aligned}$$

It is easy to see that

$$i(T^{'\Delta})-i(T^{\Delta})=3(3^{t-1}-1)(3^{s-1}i(G_1-\{y,y'\})-i(G_1-\{x,x'\})),$$

$$i(T^{''\Delta}) - i(T^{\Delta}) = 3(3^{s-1}-1)(3^{t-1}i(G_1 - \{x,x'\}) - i(G_1 - \{y,y'\})).$$

Note that $s,t \geq 2$. If $i(T^{'\Delta}) - i(T^{\Delta}) \leq 0$, we have $3^{s-1}i(G_1 - \{y,y'\}) \leq i(G_1 - \{x,x'\})$. Then we have

$$i(T^{''\Delta}) - i(T^{\Delta}) \geq 3(3^{s-1}-1)(3^{s-1}3^{t-1}-1)i(G_1 - \{y,y'\}) > 0.$$

If $d(\Delta_1, \Delta_1') = 2$, we have $T^{'\Delta} \cong T^{''\Delta}$. Suppose $N_{G_2}(u) = \{v,w\}$, $N_{G_2}(v) = \{u,w\}$, then

$$i(T^{'\Delta}) - i(T^{\Delta}) = 3(3^{t-1}-1)(3^{s-1}-1)i(G_1 - w) > 0.$$

Therefore, if $T^{'\Delta}$ and $T^{''\Delta}$ are obtained from T^{Δ} by operation II, then either $i(T^{'\Delta}) > i(T^{\Delta})$ or $i(T^{''\Delta}) > i(T^{\Delta})$. □

Theorem 2.5 *Let $T^{\Delta} \in \mathcal{T}^{\Delta}_{n,k}$. Then $i(T^{\Delta}) \leq 3^{k-1}b_{n-k+1} + b_{n-k}$, the equality holds if and only if $T^{\Delta} \cong P^{\Delta}_{k,n-k}$.*

Proof By Lemma 2.1, it is easy to see that

$$i(P^{\Delta}_{k,n-k}) = 3^{k-1}b_{n-k+1} + b_{n-k}.$$

Since $\mathcal{T}^{\Delta}_{n,2} = \{P^{\Delta}_n\}$ and $P^{\Delta}_n \cong P^{\Delta}_{n,0}, \mathcal{T}^{\Delta}_{n,n} = \{S^{\Delta}_n\}$ and $S^{\Delta}_n \cong P^{\Delta}_{2,n-2}$, we may assume $3 \leq k \leq n-1$. It is sufficient to show that $i(T^{\Delta}) < i(P^{\Delta}_{k,n-k})$ for any $T^{\Delta} \in \mathcal{T}^{\Delta}_{n,k}$ and $T^{\Delta} \ncong P^{\Delta}_{k,n-k}$.

For $T^{\Delta} \in \mathcal{T}^{\Delta}_{n,k}(3 \leq k \leq n-1)$ and $T^{\Delta} \ncong P^{\Delta}_{k,n-k}$, we know $1 \leq s(T^{\Delta}) \leq n-k$, we shall show $i(T^{\Delta}) \leq i(P^{\Delta}_{k,n-k})$ by induction on $s(T^{\Delta})$. When $s(T^{\Delta}) = 1$, since $T^{\Delta} \ncong P^{\Delta}_{k,n-k}$, we have $p(T^{\Delta}) \geq 2$. By (1) of Lemma 2.3, we have $i(T^{\Delta}) < i(P^{\Delta}_{k,n-k})$. Suppose the result holds for any triangle-tree $T^{'\Delta}$ with $s(T^{'\Delta}) = s-1$. Let $s(T^{\Delta}) = s \geq 2$. If $p(T^{\Delta}) \neq 0$, by (2) of Lemma 2.3, we can get a triangle-tree $T^{\Delta}_1 \in \mathcal{T}^{\Delta}_{n,k}$ such that $p(T^{\Delta}_1) = 0$, $s(T^{\Delta}_1) = s$ and $i(T^{\Delta}_1) > i(T^{\Delta})$. Then by Lemma 2.4, we can get a triangle-tree $T^{\Delta}_2 \in \mathcal{T}^{\Delta}_{n,k}$ from T^{Δ}_1 such that $p(T^{\Delta}_2) = 1, s(T^{\Delta}_2) = s-1$ and $i(T^{\Delta}_2) > i(T^{\Delta}_1)$. By the induction hypothesis, we have

$$i(T^{\Delta}) < i(T^{\Delta}_1) < i(T^{\Delta}_2) < i(P^{\Delta}_{k,n-k}).$$

Therefore, if $T^{\Delta} \in \mathcal{T}^{\Delta}_{n,k}$, then $i(T^{\Delta}) \leq 3^{k-1}b_{n-k+1} + b_{n-k} = i(P^{\Delta}_{k,n-k})$ and the equality holds if and only if $T^{\Delta} \cong P^{\Delta}_{k,n-k}$. □

Lemma 2.6 *For $3 \leq k \leq n$, $i(P^{\Delta}_{n-k+2,k-2}) > i(P^{\Delta}_{n-k+3,k-3})$.*

Proof By Lemma 2.1, it is easy to see that

$$i(P^{\Delta}_{n-k+2,k-2}) = 3^{k-1}b_{n-k+1} + b_{n-k},$$

$$i(P^{\Delta}_{n-k+3,k-3}) = 3^{k-2}b_{n-k+2} + b_{n-k+1}.$$

Since $b_{n+1} = 2b_n + b_{n-1}$, we have

$$\begin{aligned} i(P^{\Delta}_{k,n-k}) - i(P^{\Delta}_{n-k+3,k-3}) &= 3^{k-1}b_{n-k+1} + b_{n-k} - 3^{k-2}b_{n-k+2} + b_{n-k+1} \\ &= (3^{k-2}-1)(b_{n-k+1} - b_{n-k}) > 0. \end{aligned}$$

Hence $i(P^{\Delta}_{n-k+2,k-2}) > i(P^{\Delta}_{n-k+3,k-3})$ for $3 \le k \le n$. □

From Theorem 2.5 and Lemma 2.6, we can immediately get the following result.

Corollary 2.7 *Let T^{Δ} be a triangle-tree with $2n+1$ vertices and n triangles. Then*

(1) $i(T^{\Delta}) \le 3^n + 1$ *and the equality holds if and only if* $T^{\Delta} \cong S^{\Delta}_n$*;*
(2) *If* $T^{\Delta} \ncong S^{\Delta}_n$*, then* $i(T^{\Delta}) \le 7\times 3^{n-2}+3$ *and the equality holds if and only if* $T^{\Delta} \cong P^{\Delta}_{3,n-3}$.

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