International J.Math. Combin. Vol.2(2016), 51-64

Radio Mean Number of Certain Graphs

R.Ponraj and S.Sathish Narayanan

(Department of Mathematics, Sri Paramakalyani College, Alwarkurichi-627412, India)

E-mail: ponrajmaths@gmail.com, sathishrvss@gmail.com

Abstract: A *radio mean labeling* of a connected graph G is a one to one map f from the vertex set $V(G)$ to the set of natural numbers N such that for each distinct vertices u and v of G, $d(u,v)+\left\lceil\frac{f(u)+f(v)}{2}\right\rceil \geq 1+\text{diam }(G)$. The radio mean number of f, $rmn(f)$, is the maximum number assigned to any vertex of G. The radio mean number of G, $rmn(G)$ is the minimum value of $rmn(f)$ taken over all radio mean labeling f of G. In this paper we find the radio mean number of Jelly fish, subdivision of jelly fish, book with n pentagonal pages and $\langle K_{1,n} : m\rangle$.

Key Words: Radio mean number, subdivision of a graph, complete bipartite graph.

AMS(2010): 05C78.

§1. Introduction

For standard terminology and notion we follow Harary [6] and Gallian [4]. Unless or otherwise mentioned, $G=(V(G),E(G))$ is a simple, finite, connected and undirected graph. A graph labeling is an assignment of integers to the vertices, or edges, or both, subject to certain conditios. Graph labeling used for several areas of science and few of them are communication network, coding theory, database management ete. In particular, radio labeling applied for channel assignment problem. The concept of radio labeling was introduced by Chatrand et al. [1] in 2001. Also in [2, 3], radio number of several graphs were found. Motivated by the above labeling, Ponraj et al. [7] introduced the notion of radio mean labeling of G. A *radio mean labeling* is a one to one mapping f from $V(G)$ to N satisfying the condition

$$d(u,v)+\left\lceil\frac{f(u)+f(v)}{2}\right\rceil \geq 1+\text{diam }(G) \tag{1.1}$$

for every $u,v \in V(G)$. The span of a labeling f is the maximum integer that f maps to a vertex of Graph G. The radio mean number of G, $rmn(G)$ is the lowest span taken over all radio mean labelings of the graph G. The condition 1.1 is called radio mean condition. In [7, 8, 9], they have found the radio mean number of some graphs like three diameter graphs, lotus inside a circle, gear graph, Helms, Sunflower graphs, subdivision of complete bipartite, corona of complete graph with path, one point union of cycle C_6 and wheel related graphs. In

[1]Received October 26, 2015, Accepted May 15, 2016.

this article we find the radio mean number of Jelly fish, subdivision of jelly fish, book with n pentogonal pages and $\langle K_{1,n} : m\rangle$. We write $d(u,v)$ for the distance between the vertices u and v in G. The maximum distance between any pair of vertices is called the diameter of G and denoted by $diam(G)$. Let x be any real number. Then $\lceil x\rceil$ stands for smallest integer greater than or equal to x.

§2. Main Results

First we look into the Jelly fish graphs. Jelly fish graphs $J(m,n)$ obtained from a cycle C_4 : $uvxyu$ by joining x and y with an edge and appending m pendent edges to u and n pendent edges to v.

Theorem 2.1 *The radio mean number of a jelly fish graph $J(m,n)$ is $m+n+4$.*

Proof Let $V(J(m,n)) = \{u,v,x,y\} \cup \{u_i, v_j : 1 \leq i \leq m; 1 \leq j \leq n\}$ and $E(J(m,n)) = \{uy, yv, vx, xu, xy\} \cup \{uu_i, vv_j : 1 \leq i \leq m; 1 \leq j \leq n\}$. It is clear that $diam(J(m,n)) = 4$. The vertex labeling of $J(1,1)$, $J(1,2)$ given in Figure 1 shows that their radio mean numbers are 6, 7 respectively.

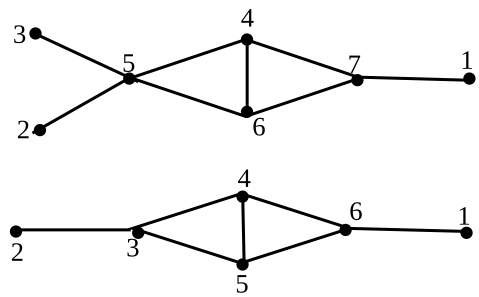

Figure 1

Assume $m \geq 2$ and $n \geq 3$. We define a vertex labeling f as follows. Assign the label 1 to u_1. Then put the label 2 to v_1, 3 to v_2 and so on. In this sequence v_n received the label $n+1$. Then assign the label $n+2$ to u_2, $n+3$ to u_3 and so on. Clearly label of u_m is $m+n$. Then assign the labels $m+n+3$, $m+n+1$, $m+n+2$, $m+n+4$ respectively to the vertices u, v, x, y. Now we check the radio mean condition

$$d(u,v) + \left\lceil \frac{f(u)+f(v)}{2} \right\rceil \geq 1 + \text{diam } (J(m,n))$$

for all $u,v \in V(J(m,n))$. It is easy to verify that the vertices u, v, x, y are mutually satisfies the radio mean condition.

Case 1. Check the pair (u, u_i).

$$d(u,u_i) + \left\lceil \frac{f(u)+f(u_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{m+n+3+1}{2} \right\rceil \geq 6.$$

Case 2. Consider the pair (u, u_i).

$$d(u, v_j) + \left\lceil \frac{f(u) + f(v_j)}{2} \right\rceil \geq 3 + \left\lceil \frac{m+n+3+2}{2} \right\rceil \geq 8.$$

Case 3. Check the pair (x, u_i).

$$d(x, u_i) + \left\lceil \frac{f(x) + f(u_i)}{2} \right\rceil \geq 2 + \left\lceil \frac{m+n+2+1}{2} \right\rceil \geq 6.$$

Case 4. Verify the pair (x, v_j).

$$d(x, v_j) + \left\lceil \frac{f(x) + f(v_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{m+n+2+2}{2} \right\rceil \geq 7.$$

Case 5. Consider the pair (y, u_i).

$$d(y, u_i) + \left\lceil \frac{f(y) + f(u_i)}{2} \right\rceil \geq 2 + \left\lceil \frac{m+n+4+1}{2} \right\rceil \geq 7.$$

Case 6. Check the pair (y, v_j).

$$d(y, v_j) + \left\lceil \frac{f(y) + f(v_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{m+n+4+2}{2} \right\rceil \geq 8.$$

Case 7. Check the pair (v, v_j).

$$d(v, v_j) + \left\lceil \frac{f(v) + f(v_j)}{2} \right\rceil \geq 1 + \left\lceil \frac{m+n+1+2}{2} \right\rceil \geq 5.$$

Case 8. Verify the pair (v, u_i).

$$d(v, u_i) + \left\lceil \frac{f(v) + f(u_i)}{2} \right\rceil \geq 3 + \left\lceil \frac{m+n+1+1}{2} \right\rceil \geq 7.$$

Case 9. Consider the pair (u_i, v_j).

$$d(u_i, v_j) + \left\lceil \frac{f(u_i) + f(v_j)}{2} \right\rceil \geq 4 + \left\lceil \frac{1+2}{2} \right\rceil \geq 6.$$

Hence $rmn(J(m,n)) = m + n + 4$. □

Now, we find the radio mean number of subdivision of jelly fish graph. If $x = uv$ is an edge of G and w is not a vertex of of G, then x is subdivided when it is replaced by the lines uw and wv. If every edges of G is subdivided, the resulting graph is the *subdivision graph* $S(G)$.

Theorem 2.2 *For a subdivision of graph* $J_{m,n}$,

$$rmn(S(J_{m,n})) = \begin{cases} 16 & \text{if} \quad m = n = 1 \\ 2m + 2n + 11 & \text{otherwise} \end{cases}$$

Proof Let $V(S(J_{m,n})) = \{z_i : 1 \leq i \leq 9\} \cup \{u_i, u_i' : 1 \leq i \leq m\} \cup \{v_j, v_j' : 1 \leq j \leq n\}$ and $E(S(J_{m,n})) = \{z_iz_{i+1} : 1 \leq i \leq 7\} \cup \{z_8z_1, z_7z_9, z_9z_3\} \cup \{z_1u_i, u_iu_i', z_5v_j, v_jv_j' : 1 \leq i \leq m, 1 \leq j \leq n\}$. Clearly $diam(S(J_{m,n})) = 8$.

Case 1. $m = n = 1$.

In this case 1 should be a label of the vertex u_1 or u_1' or v_1 or v_1'. If not, 1 is a label of any one of the remaining vertices , say x, and suppose 2 is a label of any other vertex, say x'. Then

$$d(x, x') + \left\lceil \frac{f(x) + f(x')}{2} \right\rceil \leq 6 + \left\lceil \frac{1+2}{2} \right\rceil \leq 8,$$

a contradiction.

Subcase 1. u_1 receives the label 1.

Then 2 should be a label of v_1' otherwise we get a contradiction as previously. For satisfying the radio mean condition, 3 should be a label of a vertex which is at least at a distance 6 from the vertex v_1' and 7 from u_1, such a vertex doesn't exists. Therefore, in this case, $rmn(S(J_{1,1})) \geq 14$.

Subcase 2. u_1' receives the label 1.

Then 2 should be a label of either v_1 or v_1'. Otherwise as in subcase a, the radio mean condition is not satisfied. If v_1 or v_1' receives the label 2 then 3 can not be a label of any of the remaining vertices. Suppose 3 is a label of any other vertices , say x, then

$$d(u_1', x) + \left\lceil \frac{f(u_1') + f(x)}{2} \right\rceil \leq 8.$$

or

$$d(v_1', x) + \left\lceil \frac{f(v_1') + f(x)}{2} \right\rceil \leq 8.$$

or

$$d(v_1, x) + \left\lceil \frac{f(v_1) + f(x)}{2} \right\rceil \leq 8,$$

a contradiction. Thus, here also $rmn(S(J_{1,1})) \geq 14$. By symmetry, the same case arises when v_1 or v_1' receives the label 1. Therefore in all the cases $rmn(S(J_{1,1})) \geq 14$. Now we will try to label the vertices of $S(J_{1,1})$ with the property that the sum of the distance between the any pair of vertices and the mean value of labels of that pair of vertices exceeds the integer 9. We drop the label 1 from the set of integers $\{1, 2, \cdots, 13\}$ and add a new label 14 . Thus the labels are $\{2, 3, \cdots, 14\}$. Suppose l, m, n are any three vertices of $S(J_{1,1})$ with their respective labels are 2, 3, 4. Then $d(l, m) \geq 6$, $d(l, n) \geq 6$ and $d(m, n) \geq 5$. It is clear that, such type of vertices l, m, n doesn't exists. So $rmn(S(J_{1,1})) \geq 15$. Now consider the labels from the set $\{3, 4, \cdots, 15\}$. Since the vertices with labels 3 and 4 are at least at a distance 5, any one of the vertices with these label should be a pendent vertex and the other is either z_6 or z_9 or z_4. Now suppose either 3 or 4 is a label of z_6 or z_4 then 5 can not be a label of any of the rest vertices. So 3 or

4 should be a label of z_9. Suppose 3, 4 are the labels of u_1', z_9 then 5 should be the label of v_1'. Then 6 can not be a label of any of the remaining vertices. The same problem arises when 4, 3 are the labels of u_1', z_9. By symmetry, if we assign the label 3 or 4 to the vertex v_1' , then 6 can not be a label of any other vertices as discussed above. Hence $rmn(S(J_{1,1})) \geq 16$. Consider the labeling given in Figure 2.

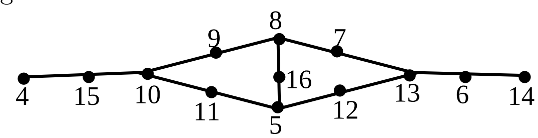

Figure 2

From Figure 2, it is clear that $rmn(S(J_{1,1})) \leq 16$. Hence $rmn(S(J_{1,1})) = 16$.

Case 2. $m \neq 1, n \neq 1$.

Subcase 1. $m + n \leq 4$.

As discussed in case 1, clearly it is not possible to label the vertices of $S(J_{m,n})$ from the sets $\{1, 2, \cdots, 2m+2n+9\}$ and $\{1, 2, \cdots, 2m+2n+10\}$. That is $rmn(S(J_{m,n})) \geq 2m+2n+11$. The following Figure 3 shows that $rmn(S(J_{m,n})) \leq 2m + 2n + 11$ where $m + n \leq 4$.

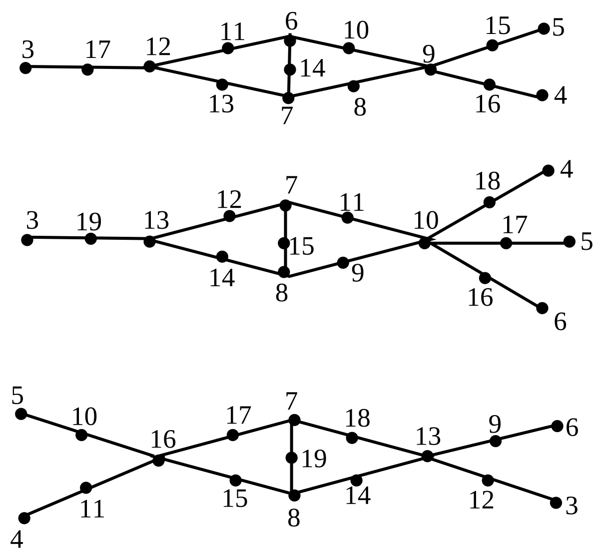

Figure 3

Hence $rmn(S(J_{m,n})) = 2m + 2n + 11$ for $m + n \leq 4$ and $m \neq 1$, $n \neq 1$.

Subcase 2. $m + n > 4$.

Define an injective map $f : V(S(J_{m,n})) \rightarrow \{1, 2, \cdots, 2m + 2n + 11\}$ by $f(v_1') = 3$, $f(v_1) =$

$2m+2n+2$,

$$\begin{aligned} f(u_i') &= i+3, & 1 \le i \le m \\ f(v_i') &= m+2+i, & 2 \le i \le n \\ f(v_{n-i+1}) &= m+n+2+i, & 1 \le i \le n-1 \\ f(u_{m-i+1}) &= m+2n+1+i, & 1 \le i \le m \end{aligned}$$

$f(z_3) = 2m+2n+3,\ f(z_2) = 2m+2n+4,\ f(z_1) = 2m+2n+5,\ f(z_8) = 2m+2n+6,$ $f(z_7) = 2m+2n+7,\ f(z_6) = 2m+2n+8,\ f(z_5) = 2m+2n+9,\ f(z_4) = 2m+2n+10$ and $f(z_9) = 2m+2n+11$. Now we check the radio mean condition that

$$d(u,v) + \left\lceil \frac{f(u)+f(v)}{2} \right\rceil \ge 9.$$

for every pair of vertices $u, v \in V(S(J_{m,n}))$.

Case 1. Consider the pair (z_i, z_j).

$$d(z_i, z_j) + \left\lceil \frac{f(z_i)+f(z_j)}{2} \right\rceil \ge 1 + \left\lceil \frac{2m+2n+3+2m+2n+4}{2} \right\rceil \ge 15.$$

Case 2. Check the pair (u_i, u_i').

$$d(u_i, u_i') + \left\lceil \frac{f(u_i)+f(u_i')}{2} \right\rceil \ge 1 + \left\lceil \frac{2m+2n+5}{2} \right\rceil \ge 9.$$

Case 3. Check the pair (u_i', u_j), $i \ne j$.

$$d(u_i', u_j) + \left\lceil \frac{f(u_i')+f(u_j)}{2} \right\rceil \ge 3 + \left\lceil \frac{4+m+2n+2}{2} \right\rceil \ge 9.$$

Case 4. Examine the pair (u_i, u_j).

$$d(u_i, u_j) + \left\lceil \frac{f(u_i)+f(u_j)}{2} \right\rceil \ge 2 + \left\lceil \frac{m+2n+2+m+2n+3}{2} \right\rceil \ge 11.$$

Case 5. Verify the pair (u_i', u_j').

$$d(u_i', u_j') + \left\lceil \frac{f(u_i')+f(u_j')}{2} \right\rceil \ge 4 + \left\lceil \frac{4+5}{2} \right\rceil \ge 9.$$

Case 6. Check the pair (u_i', z_j).

$$d(u_i', z_j) + \left\lceil \frac{f(u_i')+f(z_j)}{2} \right\rceil \ge 2 + \left\lceil \frac{2m+2n+3+4}{2} \right\rceil \ge 11.$$

Case 7. Examine the pair (u_i, z_j).

$$d(u_i, z_j) + \left\lceil \frac{f(u_i)+f(z_j)}{2} \right\rceil \ge 1 + \left\lceil \frac{2m+2n+3++m+2n+2}{2} \right\rceil \ge 12.$$

Case 8. Verify the pair (u_i, v_j).

$$d(u_i, v_j) + \left\lceil \frac{f(u_i) + f(v_j)}{2} \right\rceil \geq 6 + \left\lceil \frac{m + 2n + 2 + m + n + 3}{2} \right\rceil \geq 14.$$

Case 9. Consider the pair (u_i, v'_j).

$$d(u_i, v'_j) + \left\lceil \frac{f(u_i) + f(v'_j)}{2} \right\rceil \geq 7 + \left\lceil \frac{m + 2n + 3 + 3}{2} \right\rceil \geq 13.$$

Case 10. Examine the pair (u'_i, v'_j).

$$d(v'_i, v'_j) + \left\lceil \frac{f(u'_i) + f(v'_j)}{2} \right\rceil \geq 8 + \left\lceil \frac{4 + 3}{2} \right\rceil \geq 12.$$

Case 11. Verify the pair (u'_i, v_j), $i \neq j$.

$$d(u'_i, v_j) + \left\lceil \frac{f(u'_i) + f(v_j)}{2} \right\rceil \geq 7 + \left\lceil \frac{4 + m + n + 3}{2} \right\rceil \geq 13.$$

Case 12. Check the pair (v_i, v_j), $i \neq j$.

$$d(v_i, v_j) + \left\lceil \frac{f(v_i) + f(v_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{m + n + 3 + m + n + 4}{2} \right\rceil \geq 11.$$

Case 13. Verify the pair (v'_i, v'_j), $i \neq j$. In this case, obviously $m \geq 2$.

$$d(v'_i, v'_j) + \left\lceil \frac{f(v'_i) + f(v'_j)}{2} \right\rceil \geq 4 + \left\lceil \frac{3 + m + 4}{2} \right\rceil \geq 9.$$

Case 14. Consider the pair (v_i, v'_i).

$$d(v_i, v'_i) + \left\lceil \frac{f(v_i) + f(v'_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{2m + 2n + 5}{2} \right\rceil \geq 9.$$

Case 15. Check the pair (v_i, v'_j).

$$d(v_i, v'_j) + \left\lceil \frac{f(v_i) + f(v'_j)}{2} \right\rceil \geq 3 + \left\lceil \frac{3 + m + n + 3}{2} \right\rceil \geq 9.$$

Case 16. Verify the pair (v_i, z_j), $i \neq j$.

$$d(v_i, z_j) + \left\lceil \frac{f(v_i) + f(z_j)}{2} \right\rceil \geq 1 + \left\lceil \frac{m + n + 3 + 2m + 2n + 3}{2} \right\rceil \geq 12.$$

Case 17. Check the pair (v'_i, z_j), $i \neq j$.

$$d(v'_i, z_j) + \left\lceil \frac{f(v'_i) + f(z_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{3 + m + 2n + 3}{2} \right\rceil \geq 10.$$

Hence $rmn(S(J_{m,n})) \leq 2m + 2n + 11$ where $m + n > 4$. As in argument in case 1, $rmn(S(J_{m,n})) \geq 2m + 2n + 11$ for this case also. Hence $rmn(S(J_{m,n})) = 2m + 2n + 11$ when $m + n > 4$. □

Next investigation is about book with n pentogonal pages. n copies of the cylce C_5 with one edge common is called book with n pentogonal pages.

Theorem 2.3 *The radio mean number of a book with n pentogonal pages, BP_n, is $3n + 2$.*

Proof Let $V(BP_n) = \{u, v\} \cup \{u_i, v_i, w_i : 1 \leq i \leq n\}$ and $E(BP_n) \cup \{u, v\} \cup \{uu_i, u_iw_i, w_iv_i, v_iv : 1 \leq i \leq n\}$. Note that

$$diam(BP_n) = \begin{cases} 2 & \text{if } \ n = 1 \\ 4 & \text{otherwise} \end{cases}$$

For n = 1,2, the labeling given in Figure 4 satisfies the radio mean condition.

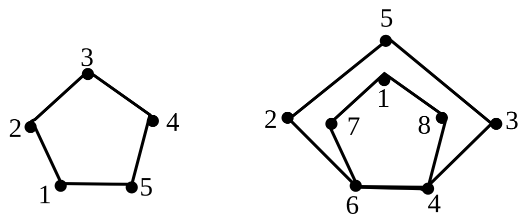

Figure 4

For $n \geq 3$, define an injective map $f : V(BP_n) \to \{1, 2, \cdots, 3n + 2\}$ by

$$\begin{aligned} f(w_i) &= i, & 1 \leq i \leq n \\ f(v_{n-i+1}) &= n + i, & 1 \leq i \leq n \\ f(u_{n-i+1}) &= 2n + i & 1 \leq i \leq n \end{aligned}$$
$$f(u) = 3n + 1, \text{ and } \quad f(v) = 3n + 2$$

Now we check the condition

$$d(u, v) + \left\lceil \frac{f(u) + f(v)}{2} \right\rceil \geq 5$$

for every pair of vertices $u, v \in V(BP_n)$.

Case 1. Check the pair (u_i, w_i).

$$d(u_i, w_i) + \left\lceil \frac{f(u_i) + f(w_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{3n + 1}{2} \right\rceil \geq 6.$$

Case 2. Verify the pair (v_i, w_i).

$$d(v_i, w_i) + \left\lceil \frac{f(v_i) + f(w_i)}{2} \right\rceil \geq 1 + \left\lceil \frac{2n + 1}{2} \right\rceil \geq 5.$$

Case 3. Examine the pair (u_i, v_i).

$$d(u_i, v_i) + \left\lceil \frac{f(u_i) + f(v_i)}{2} \right\rceil \geq 2 + \left\lceil \frac{n+1+2n+1}{2} \right\rceil \geq 8.$$

Case 4. Consider the pair (w_i, u_j).

$$d(w_i, u_j) + \left\lceil \frac{f(w_i) + f(u_j)}{2} \right\rceil \geq 3 + \left\lceil \frac{1+2n+1}{2} \right\rceil \geq 7.$$

Case 5. Consider the pair (v_i, w_j).

$$d(v_i, w_j) + \left\lceil \frac{f(v_i) + f(w_j)}{2} \right\rceil \geq 3 + \left\lceil \frac{n+1+1}{2} \right\rceil \geq 6.$$

Case 6. Verify the pair (u_i, v_j).

$$d(u_i, v_j) + \left\lceil \frac{f(u_i) + f(v_j)}{2} \right\rceil \geq 3 + \left\lceil \frac{2n+1+n+2}{2} \right\rceil \geq 9.$$

Case 7. Check the pair (w_i, w_j).

$$d(w_i, w_j) + \left\lceil \frac{f(w_i) + f(w_j)}{2} \right\rceil \geq 4 + \left\lceil \frac{1+2}{2} \right\rceil \geq 6.$$

Case 8. Examine the pair (u_i, u_j).

$$d(u_i, u_j) + \left\lceil \frac{f(u_i) + f(u_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{2n+1+2n+2}{2} \right\rceil \geq 10.$$

Case 9. Consider the pair (v_i, v_j).

$$d(v_i, v_j) + \left\lceil \frac{f(v_i) + f(v_j)}{2} \right\rceil \geq 2 + \left\lceil \frac{n+1+n+2}{2} \right\rceil \geq 7.$$

Since $\left\lceil \frac{f(u)+f(x)}{2} \right\rceil = \left\lceil \frac{3n+1+f(x)}{2} \right\rceil \geq 6$, the pair (u, x) for every $x \in V(BP_n)$ satisfy the radio mean condition. Similarly the pair (v, y) for every $y \in V(BP_n)$ also satisfy the condition. Hence $rmn(BP_n) \leq 3n+2$. Since f is injective, $rmn(BP_n) = 3n+2$. □

The following result is used for the next theorem.

Theorem 2.4([7]) *Let G be a (p,q)-connected graph with diameter $= 2$. Then $rmn(G) = p$.*

Let $\langle K_{1,n} : m\rangle$ denotes the graph obtained by taking m disjoint copies of $K_{1,n}$ and joining a new vertex to the centers of the m copies of $K_{1,n}$. Let $V(\langle K_{1,n} : m\rangle) = \{v\} \cup \{v_i : 1 \leq i \leq m\} \cup \{u_j^i : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(\langle K_{1,n} : m\rangle) = \{vv_i : 1 \leq i \leq m\} \cup \{v_i u_j^i : 1 \leq i \leq m, 1 \leq j \leq n\}$.

Theorem 2.5 *For integers $m, n \geq 1$,*

$$rmn(\langle K_{1,n} : m\rangle) = \begin{cases} 6 & \text{if} \quad m = 2, n = 1 \\ mn + m + 1 & \text{otherwise} \end{cases}$$

Proof First we observe that

$$diam(\langle K_{1,n} : m\rangle) = \begin{cases} 2 & \text{if} \ \ m = 1 \\ 4 & \text{otherwise} \end{cases}$$

Case 1. $m = 1$.

In this case $\langle K_{1,n} : 1\rangle \cong K_{1,n+1}$, which is a 2-diameter graph and hence by Theorem 2.4, $rmn(\langle K_{1,n} : 1\rangle) = n + 2$.

Case 2. $m = 2$.

Subcase 1. $n = 1$.

Since 1 and 2 are labels of the vertices which are at least at a distance 3, either 1 or 2 is a label of a pendent vertex. Assume that the label of u_1^1 is 1. Then 2 is a label of either v_2 or u_1^2. Then 3 can not be a label of the remaining vertices. Similarly we can show that if 2 is a label of u_1^1, 1 is a label of either v_2 or u_1^2 and then 3 can not be a label of the remaining vertices. Hence $rmn(\langle K_{1,1} : 2\rangle) \geq 6$. Obviously, Figure 5 shows that $rmn(\langle K_{1,1} : 2\rangle) \leq 6$.

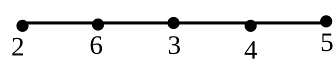

Figure 5

Hence $rmn(\langle K_{1,1} : 2\rangle) = 6$.

Subcase 2. $n \geq 2$.

Define an injective function $f : V(\langle K_{1,n} : 2\rangle) \to \{1, 2, \cdots, 2n + 3\}$ by $f(u_1^1) = 1$, $f(v) = 2n + 1$, $f(v_1) = 2n + 3$, $f(v_2) = 2n + 2$,

$$\begin{aligned} f(u_i^2) &= i + 1, \quad 1 \leq i \leq n \\ f(u_i^1) &= n + i, \quad 1 \leq 2 \leq n. \end{aligned}$$

We now check whether the labeling f is a required labeling. It is easy to check that the pairs (v_1, v_2), (v_1, v) and (v_2, v) satisfy the radio mean condition.

Subcase 1. Check the pair (u_i^2, u_j^2), $i \neq j$.

$$d(u_i^2, u_j^2) + \left\lceil \frac{f(u_i^2) + f(u_j^2)}{2} \right\rceil \geq 2 + \left\lceil \frac{2 + 3}{2} \right\rceil \geq 5.$$

Subcase 2. Verify the pair (u_i^2, u_j^1).

$$d(u_i^2, u_j^1) + \left\lceil \frac{f(u_i^2) + f(u_j^1)}{2} \right\rceil \geq 4 + \left\lceil \frac{2+1}{2} \right\rceil \geq 6.$$

Subcase 3. Consider the pair (u_i^1, u_j^1), $i \neq j$.

$$d(u_i^1, u_j^1) + \left\lceil \frac{f(u_i^1) + f(u_j^1)}{2} \right\rceil \geq 2 + \left\lceil \frac{1+n+2}{2} \right\rceil \geq 5.$$

Subcase 4. Examine the pair (u_i^1, v_1).

$$d(u_i^1, v_1) + \left\lceil \frac{f(u_i^1) + f(v_1)}{2} \right\rceil \geq 1 + \left\lceil \frac{1+2n+3}{2} \right\rceil \geq 5.$$

Subcase 5. Check the pair (u_i^1, v).

$$d(u_i^1, v) + \left\lceil \frac{f(u_i^1) + f(v)}{2} \right\rceil \geq 2 + \left\lceil \frac{1+2n+1}{2} \right\rceil \geq 5.$$

Subcase 6. Consider the pair (u_i^1, v_2).

$$d(u_i^1, v_2) + \left\lceil \frac{f(u_i^1) + f(v_2)}{2} \right\rceil \geq 3 + \left\lceil \frac{1+2n+2}{2} \right\rceil \geq 7.$$

Subcase 7. Verify the pair (u_i^2, v_2).

$$d(u_i^2, v_2) + \left\lceil \frac{f(u_i^2) + f(v_2)}{2} \right\rceil \geq 1 + \left\lceil \frac{2+2n+2}{2} \right\rceil \geq 5.$$

Subcase 8. Check the pair (u_i^2, v).

$$d(u_i^2, v) + \left\lceil \frac{f(u_i^2) + f(v)}{2} \right\rceil \geq 2 + \left\lceil \frac{2+2n+1}{2} \right\rceil \geq 6.$$

Subcase 9. Examine the pair (u_i^2, v_1).

$$d(u_i^2, v_1) + \left\lceil \frac{f(u_i^2) + f(v_1)}{2} \right\rceil \geq 3 + \left\lceil \frac{2+2n+3}{2} \right\rceil \geq 8.$$

Therefore, $rmn(\langle K_{1,n} : 2\rangle) \leq 2n+3$. But $rmn(\langle K_{1,n} : 2\rangle) \geq 2n+3$ and hence

$$rmn(\langle K_{1,n} : 2\rangle) = 2n+3.$$

Case 3. $m \geq 3$.

For $n = 1$, $m = 3$, Figure 6 shows that $rmn(\langle K_{1,1} : 3\rangle) = 7$. Now we consider the cases $n \geq 2$, $m = 3$ and $n \geq 1$, $m \geq 4$. Define a function $f : V(\langle K_{1,n} : m\rangle) \to \{1, 2, \cdots, mn+m+1\}$

by $f(v) = mn + 1$,

$$\begin{aligned} f(u_j^i) &= (j-1)m + i, \quad 1 \le i \le m, 1 \le j \le n \\ f(v_i) &= mn + 1 + i, \quad 1 \le i \le m. \end{aligned}$$

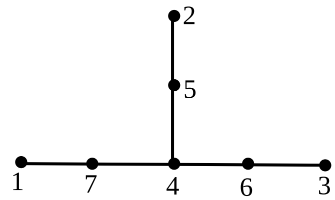

Figure 6

We show that f is a valid radio mean labeling.

Subcase 1. Check the pair (u_i^j, u_k^j).

$$d(u_i^j, u_k^j) + \left\lceil \frac{f(u_i^j) + f(u_k^j)}{2} \right\rceil \ge 2 + \left\lceil \frac{1 + m + 1}{2} \right\rceil \ge 5.$$

Subcase 2. Consider the pair (u_i^j, u_k^r), $j \ne r$.

$$d(u_i^j, u_k^r) + \left\lceil \frac{f(u_i^j) + f(u_k^r)}{2} \right\rceil \ge 4 + \left\lceil \frac{1 + 2}{2} \right\rceil \ge 6.$$

Subcase 3. Verify the pair (v_i, v_j).

$$d(v_i, v_j) + \left\lceil \frac{f(v_i) + f(v_j)}{2} \right\rceil \ge 2 + \left\lceil \frac{mn + 2 + mn + 3}{2} \right\rceil \ge 8.$$

Subcase 4. Examine the pair (v, u_i^j).

$$d(v, u_i^j) + \left\lceil \frac{f(v) + f(u_i^j)}{2} \right\rceil \ge 2 + \left\lceil \frac{mn + 1 + 1}{2} \right\rceil \ge 5.$$

Subcase 5. Verify the pair (v, v_i).

$$d(v, v_i) + \left\lceil \frac{f(v) + f(v_i)}{2} \right\rceil \ge 1 + \left\lceil \frac{mn + 1 + mn + 2}{2} \right\rceil \ge 6.$$

Subcase 6. Check the pair (v_i, u_j^i).

For $n \ge 2$ and $m = 3$,

$$d(v_i, u_j^i) + \left\lceil \frac{f(v_i) + f(u_j^i)}{2} \right\rceil \ge 1 + \left\lceil \frac{mn + 2 + 1}{2} \right\rceil \ge 6.$$

If $n \geq 1$ and $m \geq 4$ then,

$$d(v_i, u_j^i) + \left\lceil \frac{f(v_i) + f(u_j^i)}{2} \right\rceil \geq 1 + \left\lceil \frac{mn + 2 + 1}{2} \right\rceil \geq 5.$$

Subcase 7. Consider the pair (v_i, u_j^k), $i \neq k$.

$$d(v_i, u_j^k) + \left\lceil \frac{f(v_i) + f(u_j^k)}{2} \right\rceil \geq 3 + \left\lceil \frac{mn + 2 + 2}{2} \right\rceil \geq 7.$$

Hence $rmn(\langle K_{1,n} : m\rangle) = mn + m + 1$. □

Example 2.6 A radio mean labeling of $\langle K_{1,5} : 4\rangle$ is given in Figure 7.

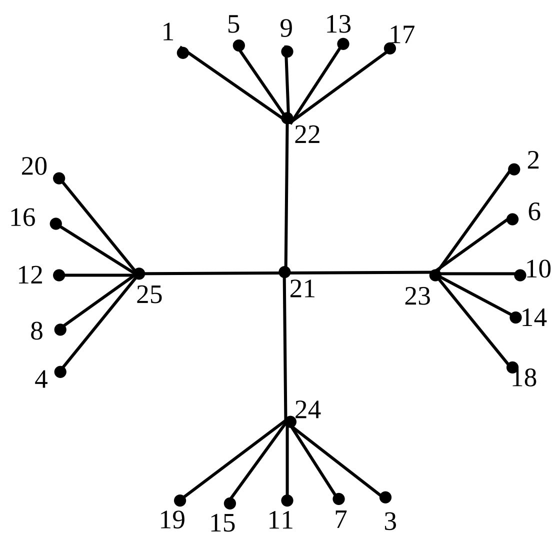

Figure 7

References

[1] Chartrand, Gray and Erwin, David and Zhang, Ping and Harary, Frank, Radio labeling of graphs, *Bull. Inst. Combin. Appl.,* **33**(2001), 77-85.

[2] Daphne Der-Fen Liu, Radio number for trees. *Discrete Math.,* 3**08(7)**(2008), 1153-1164.

[3] Daphne Der-Fen Liu and Xuding Zhu. Multilevel distance labelings for paths and cycles. *SIAM J. Discrete Math.,* **19(3)**(2005), 610-621.

[4] J.A.Gallian, A Dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, 19 (2012) #Ds6.

[5] Gutman, Distance of thorny graphs, *Publ. Inst. Math. (Beograd)*, 63 (1998), 31-36.

[6] F.Harary, *Graph Theory*, Addision Wesley, New Delhi (1969).

[7] R.Ponraj, S.Sathish Narayanan and R.Kala, Radio mean labeling of a graph, *AKCE International Journal of Graphs and Combinatorics*, 12 (2015), 224 - 228.

[8] R.Ponraj, S.Sathish Narayanan and R. Kala, On radio mean number of some graphs, *International journal of Mathematical Combinatorics*, 3(2014), 41-48.

[9] R.Ponraj, S.Sathish Narayanan and R. Kala, Radio mean number of some wheel related graphs, *Jordan Journal of Mathematics and Statistics (JJMS)*, 7(4)(2014), 273 - 286.

[10] R.Ponraj, and S.Sathish Narayanan, Radio mean number of some subdivision graphs, *Jordan Journal of Mathematics and Statistics (JJMS)*, 9(1)(2016), 45 - 64.