

Nonsplit Geodetic Number of a Graph

Tejaswini K M, Venkanagouda M Goudar

Sri Gauthama Research Centre, (Affiliated to Kuvempu University, Shimoga)

Department of Mathematics, Sri Siddhartha Institute of Technology, Tumkur-572 105, Karnataka, India

Venkatesh

Department of mathematics, Kuvempu University, Shimoga, India

E-mail: tejaswini.ssit@gmail.com, vmgouda@gmail.com vensprema@gmail.com

Abstract: Let G be a graph. If $u, v \in V(G)$, a $u - v$ geodesic of G is the shortest path between u and v . The closed interval $I[u, v]$ consists of all vertices lying in some $u - v$ geodesic of G . For $S \subseteq V(G)$ the set $I[S]$ is the union of all sets $I[u, v]$ for $u, v \in S$. A set S is a geodetic set of G if $I(S) = V(G)$. The cardinality of a minimum geodetic set of G is the geodetic number of G , denoted by $g(G)$. In this paper, we study the nonsplit geodetic number of a graph $g_{ns}(G)$. The set $S \subseteq V(G)$ is a nonsplit geodetic set in G if S is a geodetic set and $\langle V(G) - S \rangle$ is connected, nonsplit geodetic number $g_{ns}(G)$ of G is the minimum cardinality of a nonsplit geodetic set of G . We investigate the relationship between nonsplit geodetic number and geodetic number. We also obtain the nonsplit geodetic number in the cartesian product of graphs.

Key Words: Cartesian products, distance, edge covering number, Smarandachely k -geodetic set, geodetic number, vertex covering number.

AMS(2010): 05C05, 05C12.

§1. Introduction

As usual $n = |V|$ and $m = |E|$ denote the number of vertices and edges of a graph G respectively. The graphs considered here are finite, undirected, simple and connected. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G . It is well known that this distance is a metric on the vertex set $V(G)$. For a vertex v of G , the eccentricity $e(v)$ is the distance between v and a vertex farthest from v . The minimum eccentricity among the vertices of G is radius, $\text{rad } G$ and the maximum eccentricity is the diameter, $\text{diam } G$. A $u - v$ path of length $d(u, v)$ is called a $u - v$ geodesic. We define $I[u, v]$ to the set (interval) of all vertices lying on some $u - v$ geodesic of G and for a nonempty subset S of $V(G)$, $I[S] = \cup_{u, v \in S} I[u, v]$. A set S of vertices of G is called a geodetic set in G if $I[S] = V(G)$, and a geodetic set of minimum cardinality is a minimum geodetic set, and generally, if there is a k -subset T of $V(G)$ such that $I(S) \cup T = V(G)$, where $0 \leq k < |G| - |S|$, then S is called a *Smarandachely k -geodetic set* of G . The cardinality of a minimum geodetic set in G is called

¹Received December 3, 2014, Accepted December 6, 2015.

the geodetic number and is denoted by $g(G)$. The concept of geodetic number of a graph was introduced in [1, 4, 7], further studied in [2, 3], and the split geodetic number of a graph was introduced in [10]. It was shown in [7] that determining the geodetic number of a graph is an NP-hard problem.

A set of vertices S in a graph G is a nonsplit geodetic set if S is a geodetic set and the subgraph $G[V - S]$ induced by $\langle V(G) - S \rangle$ is connected. The minimum cardinality of a nonsplit geodetic set, denoted $g_{ns}(G)$, is called the nonsplit geodetic number of G .

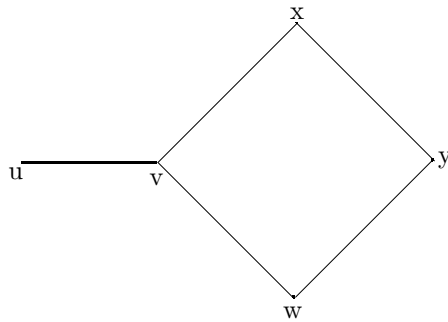


Figure 1.1

Consider the graph G of Figure 1.1. For the vertices u and y in G $d(u, y) = 3$ and every vertex of G lies on an $u - y$ geodesic in G . Thus $S = \{u, y\}$ is the geodetic set of G and so $g(G)$. Here the induced subgraph $\langle V(G) - S \rangle$ is connected. So that S is a minimum nonsplit geodetic set of G . Therefore nonsplit geodetic number $g_{ns}(G) = 2$.

A vertex v is an extreme vertex in a graph G , if the subgraph induced by its neighbours is complete. A vertex cover in a graph G is a set of vertices that covers all edges of G . The minimum number of vertices in a vertex cover of G is the vertex covering number $\alpha_0(G)$ of G . An edge cover of a graph G without isolated vertices is a set of edges of G that covers all the vertices of G . The edge covering number $\alpha_1(G)$ of a graph G is the minimum cardinality of an edge cover of G . For any undefined term in this paper, see [1, 6]

§2. Preliminary Notes

We need the following results to prove our results.

Theorem 2.1 *Every geodetic set of a graph contains its extreme vertices.*

Theorem 2.2 *For any tree T with k pendant vertices, $g(T) = k$.*

Theorem 2.3 *For any graph G of order n , $\alpha_1(G) + \beta_1(G) = n$.*

Theorem 2.4 For cycle C_n of order $n \geq 3$,

$$g(C_n) = \begin{cases} 2 & \text{if } n \text{ even,} \\ 3 & \text{if } n \text{ odd.} \end{cases}$$

Theorem 2.5 If G is a nontrivial connected graph, then $g(G) \leq g(G \times K_2)$.

§3. Nonsplit Geodetic Number

Theorem 3.1 For cycle C_n of order $n \geq 3$,

$$g_{ns}(C_n) = \begin{cases} \frac{n}{2} + 1 & \text{if } n \text{ is even,} \\ \lfloor \frac{n}{2} \rfloor + 2 & \text{if } n \text{ is odd.} \end{cases}$$

Proof Suppose C_n be cycle with $n \geq 3$, we have the following

Case 1. Let n be even. Consider $C_{2p} = \{v_1, v_2, \dots, v_{2p}, v_1\}$ be a cycle with $2p$ vertices. Then v_{p+1} is the antipodal vertex of v_1 . Suppose $S = \{v_1, v_{p+1}\}$ be the geodetic set of G . It is clear that $\langle V(G) - S \rangle$ is not connected. Thus S is not a nonsplit geodetic set. But $S' = \{v_1, v_2, \dots, v_{p+1}\}$ is a nonsplit geodetic set of G . So that $g_{ns}(G) \leq (p+1)$. If S_1 is any set of vertices of G with $|S_1| < |S'|$ then S_1 contains at most p -elements. Hence $V(G) - S_1$ is not connected. This follows that $g_{ns}(G) = p+1 = \frac{n}{2} + 1$.

Case 2. Let n be odd. Consider $C_{2p+1} = \{v_1, v_2, \dots, v_{2p+1}, v_1\}$ be a cycle with $2p+1$ vertices. Then v_{p+1} and v_{p+2} are the antipodal vertices of v_1 . Now consider $S = \{v_1, v_{p+1}, v_{p+2}\}$ be the geodetic set of G and it is clear that $\langle V(G) - S \rangle$ is not connected. Thus S is not a nonsplit geodetic set. But $S' = \{v_1, v_2, \dots, v_{p+1}, v_{p+2}\}$ is a nonsplit geodetic set of G so that $g_{ns}(G) \leq p+2$. If S_1 is any set of vertices of G with $|S_1| < |S'|$ then S_1 contains at most p -elements. Hence $\langle V(G) - S_1 \rangle$ is not connected. This follows that

$$g_{ns}(G) = p+2 = \left\lfloor \frac{n}{2} \right\rfloor + 2. \quad \square$$

Theorem 3.2 For any nontrivial tree T with k -pendant-vertices, then $g_{ns}(T) = k$.

Proof Let $S = \{v_1, v_2, \dots, v_k\}$ be the set containing pendant vertices of a tree T . By Theorem 2.2, $g(T) \geq |S|$. On the other hand, for an internal vertex v of T there exist pendant vertices x, y of T such that v lies on the unique x - y geodesic in T . Thus, $v \in I[S]$ and $I[S] = V(T)$. Then $g(T) \leq |S|$. Thus S itself a minimum geodetic set of T . Therefore $g(T) = |S| = k$ and $\langle V - S \rangle$ is connected. Hence $g_{ns}(T) = k$. $\square$

Theorem 3.3 For any integers $r, s \geq 2, g_{ns}(K_{r,s}) = r + s - 1$.

Proof Let $G = K_{r,s}$, such that $U = \{u_1, u_2, \dots, u_r\}$, $W = \{w_1, w_2, \dots, w_s\}$ are the partite

sets of G , where $r \leq s$ and also $V = U \cup W$.

Consider $S = U \cup W - x$ for any $x \in W$. Every $w_k \in W$, $1 \leq k \leq s - 1$ lies on $u_i - u_j$ geodesic for $1 \leq i \neq j \leq r$, so that S is a geodetic set of G . Since $\langle V(G) - S \rangle$ is connected and hence S itself a nonsplit geodetic set of G . Let S' be any set of vertices such that $|S'| < |S|$. If S' is not a subset of U then $\langle V(G) - S' \rangle$ is not connected and so S' is not a nonsplit geodetic set of G . If S' is not a subset of $W - x$, again S' is not a nonsplit geodetic set of G , by a similar argument. If $S' = U$ then S' is a geodetic set but $\langle V(G) - S' \rangle$ is not connected, so S' is not nonsplit geodetic set. If $S' = W - x$ then S' is not a nonsplit geodetic set of G . From the above argument, it is clear that S is a minimum nonsplit geodetic set of G . Hence $g_{ns}(Kr, s) = |S| = r + s - 1$. $\square$

Theorem 3.4 *If G is a star then $g_{ns}(G) = n - 1$.*

Proof Let $V(G) = \{v_1, v_2, \dots, v_{n-1}, v_n\}$ and let $S = \{v_1, v_2, \dots, v_{n-1}\}$ be the set of pendant vertices of G and is the geodetic set of G . Clearly, the subgraph induced by $\langle V(G) - S = v_n \rangle$ is connected. Hence $S = \{v_1, v_2, \dots, v_{n-1}\}$ is a minimum nonsplit geodetic set of G . Therefore $g_{ns}(G) = n - 1$. $\square$

Theorem 3.5 *For any nontrivial connected graph G different from star of order n and diameter d , $g_{ns}(G) \leq n - d + 1$.*

Proof Let u and v be the vertices of G for which $d(u, v) = d$ and let $u = v_0, v_1, \dots, v_d = v$ be a $u - v$ path of length d . Now $S = V(G) - \{v_1, v_2, \dots, v_{d-1}\}$ then $I[S] = V[G]$ and consequently $g_{ns}(G) \leq |S| \leq n - d + 1$. $\square$

Theorem 3.6 *For any tree T , $g_{ns}(T) + g(T) < 2m$.*

Proof Suppose $S = \{v_1, v_2, v_3, \dots, v_k\}$ be the set of all pendant vertices in T , forms a minimal geodetic set of $I[S] = V(T)$. Further $\{u_1, u_2, u_3, \dots, u_l\} \subset V(G) - S$ is the set of internal vertices in T . Then $\langle V(G) - S \rangle$ forms a minimal non split geodetic set of T , it follows that $|S| + |S| < 2m$. Hence $g_{ns}(T) + g(T) < 2m$. $\square$

Theorem 3.7 *For any graph G of order n , $g_{ns}(G) \leq g_s(G)$, where G is not a cycle..*

Proof Let G be any graph with n vertices. Consider a nonsplit geodetic set $S = \{v_1, v_2, \dots, v_k\}$ of a graph G . Since $\langle V(G) - S \rangle$ is connected, the set S is not a split geodetic set of G . Now, we consider a set $S' = S \cup \{a, b\}$ for any $a, b \in V(G)$ such that $\langle V(G) - S' \rangle$ is disconnected. Therefore S' is the split geodetic set of G with minimum cardinality. Thus $|S| < |S'|$. Clearly $g_{ns}(G) \leq g_s(G)$. $\square$

Theorem 3.8 *Let G be a cycle of order n then $g_s(G) \leq g_{ns}(G)$.*

Proof Let G be a cycle of order n , we discuss the following cases.

Case 1. Suppose n is even. Let $S = \{v_i, v_j\}$ be the split geodetic set of G where v_i, v_j are the two antipodal vertices of G . The $v_i - v_j$ geodesic includes all the vertices of G and $\langle V(G) - S \rangle$

is disconnected. But $S' = \{v_i, v_{i+1}, \dots, v_j\}$ is a nonsplit geodetic set of G and the induced subgraph $\langle V(G) - S' \rangle$ is connected. Thus $|S| \leq |S'|$. Clearly $g_s(G) \leq g_{ns}(G)$.

Case 2. Suppose n is odd. Let $S = \{v_i, v_j, v_k\}$ be the split geodetic set of G . By the Theorem 2.4, no two vertices of S form a non split geodetic set and $\langle V(G) - S \rangle$ is disconnected. But $S' = \{v_i, v_{i+1}, \dots, v_j, v_k\}$ is a nonsplit geodetic set of G and the induced subgraph $\langle V(G) - S' \rangle$ is connected. Thus $|S| \leq |S'|$. Clearly $g_s(G) \leq g_{ns}(G)$. $\square$

Theorem 3.9 For the wheel $W_n = K_1 + C_{n-1}$ ($n \geq 5$),

$$g_{ns}(W_n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

Proof Let $W_n = K_1 + C_{n-1}$ and let $V(W_n) = \{x, u_1, u_2, \dots, u_{n-1}\}$, where $\deg(x) = n-1 > 3$ and $\deg(u_i) = 3$ for each $i \in \{1, 2, \dots, n-1\}$. We discuss the following cases.

Case 1. Let n be even. Consider geodesic

$$P : \{u_1, u_2, u_3\}, Q : \{u_3, u_4, u_5\}, \dots, R : \{u_{2n-1}, u_{2n}, u_{2n+1}\}.$$

It is clear that the vertices $u_2, u_4, \dots, u_{2n}$ lies on the geodesic P, Q and R . Also $u_1, u_3, u_5, \dots, u_{2n-1}, u_{2n+1}$ is a minimum nonsplit geodetic set such that $\langle V(G) - S \rangle$ is connected and it has $\frac{n}{2}$ vertices. Hence $g_{ns}(W_n) = \frac{n}{2}$.

Case 2. Let n be odd. Consider geodesic

$$P : \{u_1, u_2, u_3\}, Q : \{u_3, u_4, u_5\}, \dots, R : \{u_{2n-1}, u_{2n}, u_{2n+1}\}.$$

It is clear that the vertices $u_2, u_4, \dots, u_{2n}$ lies on the geodesic P, Q and R . Also $u_1, u_3, u_5, \dots, u_{2n-1}, u_{2n+1}$ is a minimum nonsplit geodetic set such that $\langle V(G) - S \rangle$ is connected and it has $\frac{n-1}{2}$ vertices. Hence $g_{ns}(W_n) = \frac{n-1}{2}$. $\square$

Theorem 3.10 Let G be a graph such that both G and $\overline{G}$ are connected then $g_{ns}(G) + g_{ns}(\overline{G}) \leq n(n-3) + 2$.

Proof Since both G and $\overline{G}$ are connected, we have $\Delta(G) \cdot \Delta(\overline{G}) < n-1$. Thus $\beta_0(G), \beta_0(\overline{G}) \geq 2$. Hence,

$$g_{ns} \leq n-1 \Rightarrow g_{ns}(G) \leq 2(n-1) - n + 1 \Rightarrow g_{ns}(G) \leq 2m - n + 1.$$

Similarly, $g_{ns}(\overline{G}) \leq 2\overline{m} - n + 1$. Thus,

$$\begin{aligned} g_{ns}(G) + g_{ns}(\overline{G}) &\leq 2(m + (\overline{m})) - 2n + 2 \Rightarrow g_{ns}(G) + g_{ns}(\overline{G}) \leq n(n-1) - 2n + 2 \\ &\Rightarrow g_{ns}(G) + g_{ns}(\overline{G}) \leq n^2 - n - 2n + 2 \\ &\Rightarrow g_{ns}(G) + g_{ns}(\overline{G}) \leq n^2 - 3n + 2 \\ &\Rightarrow g_{ns}(G) + g_{ns}(\overline{G}) \leq n(n-3) + 2. \end{aligned} \quad \square$$

Theorem 3.11 For any nontrivial tree T , $g_{ns}(T) \geq \alpha_0(T)$.

Proof Let S be a minimum cover set of vertices in T . Then S has at least one vertex and every vertex in S is adjacent to some vertices in $\langle V(G) - S \rangle$. This implies that S is a nonsplit geodetic set of G . Thus $g_{ns}(T) \geq \alpha_0(T)$. $\square$

Theorem 3.12 For any nontrivial tree T with m edges, $g_{ns}(T) \leq m - \lceil \frac{\alpha_1(T)}{2} \rceil + 2$, where $\alpha_1(T)$ is an edge covering number.

Proof Suppose $S' = \{e_1, e_2, \dots, e_i\}$ be the set of all end edges in T and $J \subseteq E(T) - S'$ be the minimal set of edges such that $|S' \cup J| = \alpha_1(T)$. By the Theorem 2.2 S' is the minimal geodetic set of G . Also it follows that $\langle V(G) - S' \rangle$ is connected. Clearly,

$$g_{ns}(T) \leq |E(T)| - \left\lceil \frac{|S' \cup J|}{2} \right\rceil + 2 \Rightarrow g_{ns}(T) \leq m - \left\lceil \frac{\alpha_1(T)}{2} \right\rceil + 2. \quad \square$$

Theorem 3.13 For a cycle C_n of order n , $g_{ns}(G) = \alpha_0(C_n) + 1$.

Proof Consider a cycle C_n of order n . We discuss the following cases.

Case 1. Suppose that n is even and $\alpha_0(C_n)$ is the vertex covering number of C_n . We have by Theorem 3.1, $g_{ns}(G) = \frac{n}{2} + 1$ and also for an even cycle, vertex covering number $\alpha_0(C_n) = \frac{n}{2}$. Hence,

$$g_{ns}(G) = \frac{n}{2} + 1 = \alpha_0(C_n) + 1.$$

Case 2. Suppose that n is odd and $\alpha_0(C_n)$ is the vertex covering number of C_n . We have by Theorem 3.1, $g_{ns}(G) = \lfloor \frac{n}{2} \rfloor + 2$ and also for an odd cycle, vertex covering number $\alpha_0(C_n) = \lfloor \frac{n}{2} \rfloor + 1$. Hence,

$$g_{ns}(G) = \lfloor \frac{n}{2} \rfloor + 2 \Rightarrow g_{ns}(G) = \alpha_0(C_n) + 1. \quad \square$$

Theorem 3.14 If G is a connected noncomplete graph of order n , $g_{ns} \leq (n - \kappa(G)) + 1$, where $\kappa(G)$ is vertex connectivity.

Proof Let $\kappa(G) = k$. Since G is connected but not complete $1 \leq \kappa(G) \leq n - 2$. Let $U = \{u_1, u_2, \dots, u_k\}$ be a minimum cut set of G , let $G_1, G_2, \dots, G_r$ ($r \geq 2$) be the components of $G - U$ and let $W = V(G) - (U - 1)$ then every vertex u_i ($1 \leq i \leq k$) is adjacent to at least one vertex of G_j for every $(i \leq j \leq r)$. Therefore, every vertex u_i belongs to a W geodesic path. Thus

$$g_{ns}(G) = |W| \leq (V(G) - U) + 1 \leq (n - \kappa(G)) + 1. \quad \square$$

§4. Corona Graph

Let G and H be two graphs and let n be the order of G . The corona product $G \circ H$ is defined as the graph obtained from G and H by taking one copy of G and n copies of H and then joining by an edge, all the vertices from the i^{th} -copy of H with the i^{th} -vertex of G .

Theorem 4.1 Let G be a connected graph of order n and H be any graph of order m then $g_{ns}(G \circ H) = nm$.

Proof Let S be a nonsplit geodetic set in $G \circ H$, $v_i \in V(G)$, $1 \leq i \leq n$ and $u_j \in V(H)$, $1 \leq j \leq m$. For each v_i there is a copy Hv_i which contains u_j vertices. Clearly $V(Hu_j) \cap S$ is a geodetic set of $G \circ H$ and $\langle V(G) - S \rangle$ is connected. Further every $w_k \in (G \circ H)$ lies on the geodesic path in S . Therefore S is the minimum nonsplit geodetic set. Thus, $|S| = g_{ns}(G \circ H) = nm$. $\square$

§5. Adding a Pendant Vertex

An edge $e = (u, v)$ of a graph G with $\deg(u) = 1$ and $\deg(v) > 1$ is called an *pendant edge* and u an *pendant vertex*.

Theorem 5.1 *Let G' be the graph obtained by adding a pendant edge (u, v) to a cycle $G = C_n$ of order $n > 3$, with $u \in G$ and $v \notin G$, then*

$$g_{ns}(G') = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$$

Proof Let $\{u_1, u_2, u_3, \dots, u_n, u_1\}$ be a cycle with n vertices. Let G' be the graph obtained from $G = C_n$ by adding an pendant edge (u, v) such that $u \in G$ and $v \notin G$. We discuss the following cases.

Case 1. For $G = C_{2n}$, let $S = \{v, u_i\}$ be a non split geodetic set of G' , where v is the pendant vertex of G' and $\text{diam}(G') = v - u_i$ path, clearly $I[S] = V[G']$. Also for all $x, y \in V(G') - S$, $\langle V(G') - S \rangle$ is connected. Hence, $g_{ns}(G') = 2$.

Case 2. For $G = C_{2n+1}$, let $S = \{v, a, b\}$ be a non split geodetic set of G' , where v is the pendant vertex of G' and $a, b \in G$ such that $d(v, a) = d(v, b)$. Thus $I[S] = V[G']$ and $\langle V(G') - S \rangle$ is connected. Hence, $g_{ns}(G') = 3$. $\square$

Theorem 5.2 *Let G' be the graph obtained by adding a pendant vertex (u_i, v_i) for $i = 1, 2, 3, \dots, n$ to each vertex of $G = C_n$ such that $u \in G, v_i \notin G$, then $g_{ns}(G') = k$.*

Proof Let $G = C_n = \{u_1, u_2, u_3, \dots, u_n, u_1\}$ be a cycle with n vertices. Let G' be the graph obtained by adding an pendant vertex $\{u_i, v_i\}$, $i = 1, 2, 3, \dots, n$ to each vertex of G such that $u_i \in G$ and $v_i \notin G$. Let $S = \{v_1, v_2, v_3, \dots, v_k\}$ be a non split geodetic set of G' . Clearly $I[X] \neq V(G')$. Also, $x, y \in V(G') - S$ with $V(G') - S$ connected. Thus, $g_{ns}(G') = k$. $\square$

Theorem 5.3 *Let G' be the graph obtained by adding k pendant vertices $\{(u, v_1), \dots, (u, v_k)\}$ to a cycle $G = C_n$ of order $n > 3$, with $u \in G$ and $\{v_1, v_2, \dots, v_k\} \notin G$. Then*

$$g_{ns}(G') = \begin{cases} k + 1 & \text{if } n \text{ is even} \\ k + 2 & \text{if } n \text{ is odd} \end{cases}$$

Proof Consider $\{u_1, u_2, u_3, \dots, u_n, u_1\}$ be a cycle with n vertices. Let G' be the graph

obtained from $G = C_n$ by adding k pendant edges $\{u_i v_1, u_i v_2, \dots, u_i v_k\}$ such that u_i a single vertex of G and $\{v_1, v_2, v_3, \dots, v_k\}$ does not belongs to G . We discuss the following cases.

Case 1. Let $G = C_{2n}$. Consider $X = \{v_1, v_2, v_3, \dots, v_k\} \cup u_i$, for any vertex u_i of G . Now $S = \{X\}$ be a non split geodetic set, such that $\{v_1, v_2, v_3, \dots, v_k\}$ are the pendant vertices of G' and u_j is the antipodal vertex of u_i in G . Thus $I[X] = V[G']$. Consider $P = \{v_1, v_2, v_3, \dots, v_k\}$ as a set of pendant vertices such that $|P| < |S|$ is not a non split geodetic set i.e for some vertex $u_j \in V_{G'}$, $u_j \notin I[P]$. If $P = X$, then P is not nonsplit geodetic set. Thus S is a minimum non split geodetic set of G' and $\langle V(G') - S \rangle$ is connected. Thus, $g_{ns}(G') = k + 1$.

Case 2. Let $G = C_{2n+1}$. Consider $S = \{v_1, v_2, v_3, \dots, v_k, a, b\}$ be a non split geodetic set, where $\{v_1, v_2, \dots, v_k\} \notin G$ are k pendant vertices of G' not in G and $a, b \in G$ such that $d(u, a) = d(u, b)$. Thus $I[S] = V[G']$. Also $x, y \in V(G') - S$ it follows that $\langle V(G') - S \rangle$ is connected. Therefore, $g_{ns}(G') = k + 2$. $\square$

§6. Cartesian Products

The cartesian product of the graphs H_1 and H_2 written as $H_1 \times H_2$, is the graph with vertex set $V(H_1) \times V(H_2)$, two vertices u_1, u_2 and v_1, v_2 being adjacent in $H_1 \times H_2$ iff either $u_1 = v_1$ and $(u_2, v_2) \in E(H_2)$, or $u_2 = v_2$ and $(u_1, v_1) \in E(H_1)$.

Theorem 6.1 Let K_2 and $G = C_n$ be the graphs then

$$g_{ns}(K_2 \times G) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n > 5 \text{ is odd} \\ 4 & \text{if } n=3 \end{cases}$$

Proof Consider $G = C_n$, let $K_2 \times G$ be graphs formed from two copies G_1 and G_2 of G . Let $V = \{v_1, v_2, \dots, v_n\}$ be the vertices of G_1 and $W = \{w_1, w_2, \dots, w_n\}$ be the vertices of G_2 and $U = V \cup W$. We consider the following cases.

Case 1. Let n be even. Consider $S = \{v_i, w_j\}$ be the non split geodetic of $K_2 \times G$ such that v_i to w_j path is equal to $\text{diam}(K_2 \times G)$ which includes all the vertices of $K_2 \times G$. Hence $\langle U - S \rangle$ is connected. Therefore, $g_{ns}[K_2 \times G] = 2$.

Case 2. Let n be odd. Consider $S = \{v_i, w_j, v_k\}$ be the non split geodetic set of $K_2 \times G$ such that v_i to w_j path is equal to $\text{diam}(K_2 \times G)$ and is equal to $w_j - v_k$ path and also $v_i - w_j \cup w_j - v_k$ path includes all the vertices of $K_2 \times G$. Hence $\langle U - S \rangle$ is connected. Therefore, $g_{ns}[K_2 \times G] = 3$.

Case 3. For $n = 3$, let $S = \{v_i, w_j, v_k\}$ be the geodetic set of $K_2 \times G$, that is $v_i - w_j$ is equal to $\text{diam}(K_2 \times G)$ and is equal to $w_j - v_k$ and also $I[S] = U(K_2 \times G)$. But $\langle U - S \rangle$ is not connected. Let $S' = S \cup \{v_n\} = \{v_i, w_j, v_k, v_n\}$ be the non split geodetic set of $K_2 \times G$. Hence, $\langle U - S' \rangle$ is connected. Therefore, $g_{ns}[K_2 \times G] = 4$. $\square$

Theorem 6.2 For any complete graph K_n of order n , $g_{ns}[K_2 \times K_n] = n + 1$.

Proof Consider $K_2 \times K_n$ be graph formed from two copies of G_1 and G_2 of G . Now, let us prove the result by mathematical induction,

For $n = 2$, $g_{ns}[K_2 \times K_2] = 3$, since $K_2 \times K_2 = C_4$ by Theorem 3.1 we have $g_{ns}[C_4] = 3$ the result is true.

Let us assume that the result is true for $n=m$, that is $g_{ns}[K_2 \times K_m] = m + 1$.

Now, we shall prove the result for $n = m + 1$. Let $S = \{v_1, v_2, v_3, \dots, v_{m+2}\}$ be the nonsplit geodetic set formed from some elements in G_1 and the elements which are not corresponds to elements in G_1 of $K_2 \times K_{m+1}$. Clearly $I[S] = V(K_2 \times K_n)$. Let P be any set of vertices such that $|P| < |S|$. Suppose $P = \{v_1, v_2, v_3, \dots, v_m\}$ which is not non split geodetic set, because $I[P] \neq V(K_2 \times K_{m+1})$. So S itself a minimum geodetic set of $K_2 \times K_{m+1}$. Hence, $g_{ns}[K_2 \times K_{m+1}] = m + 1 + 1$. Thus, $g_{ns}(K_2 \times K_n) = n + 1$. $\square$

Theorem 6.3 For any complete graph of order $n \geq 3$, $g_{ns}(K_n \times K_n) = n$.

Proof We shall prove the result by mathematical induction, For $n \geq 3$, let us assume that the result is true for $n = m$, that is $g_{ns}(K_m \times K_m) = m$.

Now, we shall prove the result for $n = m + 1$. Let $S = \{v_1, v_2, v_3, \dots, v_{m+1}\}$ be the non split geodetic set formed from some elements in G_1 and the elements which are not corresponds to elements in G_1 of $K_{m+1} \times K_{m+1}$. Clearly $I[S] = V(K_2 \times K_n)$. Now, consider P be any set of vertices such that $|P| < |S|$. Suppose $P = \{v_1, v_2, v_3, \dots, v_m\}$ which is not non split geodetic set, because $I[P] \neq V(K_{m+1} \times K_{m+1})$. So S itself a minimum geodetic set of $K_{m+1} \times K_{m+1}$. Hence, $g_{ns}(K_{m+1} \times K_{m+1}) = m + 1$. Thus $g_{ns}(K_n \times K_n) = n$. $\square$

Theorem 6.4 Let G and H be graphs then $g_{ns}(G \times H) \geq \max\{g(G), g(H)\}$. Equality holds when G, H are complete graphs and $n \geq 3$.

Proof If S is a minimum geodetic set in $G \times H$ then we have $I[S] = \cup_{a,b \in S} I[a, b] = \cup_{a,b \in S} I[a_1, b_1] \times I[a_2, b_2] \subseteq (\cup_{a_1, b_1 \in S} I[a_1, b_1]) \times (\cup_{a_2, b_2 \in S} I[a_2, b_2]) = I[S_1] \times I[S_2]$, $V(G \times H) = I[S] \subseteq I[S_1] \times I[S_2]$. Therefore S_1 and S_2 geodetic set in G, H respectively, so $g_{ns}(G \times H) = |S| \geq \max\{|S_1|, |S_2|\} \geq \max\{g(G), g(H)\}$, proving the lower bound.

Consider complete graphs G, H with vertex sets $V(G) = \{u_1, u_2, \dots, u_p\}$ and $V(H) = \{v_1, v_2, \dots, v_q\}$ where without loss of generality $p \geq q$. Then $g(G) = p$ and $g(H) = q$. Let $S = \{(u_1, v_2), (u_2, v_2), \dots, (u_q, v_q), (u_{q+1}, v_q), (u_{q+2}, v_q), \dots, (u_p, v_q)\}$.

It is straight forward to verify that S is a non split geodetic set for $G \times H$. Hence, $g_{ns}(G \times H) \leq |S| \leq p = \max\{g(G), g(H)\} \leq g_{ns}(G \times H)$, so equality holds. $\square$

Theorem 6.5 Let $G = T$ and $H = K_2$ be the graphs with $g(G) = p \geq g(H) = q \geq 2$ then $g_{ns}(G \times H) \leq pq - q$.

Proof Set $X = G \times H$. Let $S = \{g_1, g_2, \dots, g_p\}$ and $T = \{h_1, h_2, \dots, h_q\}$ be the geodetic sets of G and H respectively, and $U = \{(S \times T) / \cup_{i,j=1}^{p,q} \{(g_i, h_j)\}\}$.

We claim that $I_X[U] = V(X)$. Let (g, h) be an arbitrary vertex of X . Then there exists indices i and i' such that $g \in I_G[g_i, g_{i'}]$ and there are indices j and j' such that $h \in I_H[h_j, h_{j'}]$. Since $p, q \geq 2$ we may assume that $i = i'$ and $j = j'$. Indeed, if say $g = g_i$ then i' to be an

arbitrary index from $\{1, 2, \dots, p\}$ different from i . Set $B = \{(g_i, h_j), (g_i, h_{j'}), (g_{i'}, h_j), (g_{i'}, h_{j'})\}$.

Suppose that one of the vertices from B is not in U . We may without loss of generality assume $(g_i, h_j) \notin U$. This means that $i = j$. Therefore $i' \neq j$ and $i \neq j'$. Then we infer that $(g, h) \in I_X[(g_i, h_{j'}), (g_{i'}, h_j)]$. Otherwise, all vertices from B are in U , then $(g, h) \in I_X[(g_i, h_j), (g_{i'}, h_{j'})]$. Hence, $g_{ns}[G \times H] \leq pq - q$. $\square$

Theorem 6.6 *Let K_2 and T be the graphs then $g_{ns}(K_2 \times T) = g_{ns}(T)$.*

Proof Consider a tree T . Let $K_2 \times T$ be a graph formed from two copies T_1 and T_2 of T and S be a minimum non split geodetic set of $K_2 \times T$. Now, we define S' to be the union of those vertices of S in T_1 and the vertices of T_1 corresponding to vertices of T_2 belonging to S . Let $v \in V(T_1)$ lies on some $x - y$ geodesic, where $x, y \in S$. Since S is a non split geodetic set by Theorem 3.2, i.e., $g_{ns}(T) = k$ at least one of x and y belongs to V_1 . If both $x, y \in V_1$ then $x, y \in S'$. Hence, we may assume that $x \in V_1, y \in V_2$. If y corresponds to x then $v = x \in S'$. Hence, we assume that y corresponds to $y' \in S'$, where $y' \neq x$. Since $d(x, y) = d(x, y') + 1$ and the vertex v lies on an $x - y$ geodesic in $K_2 \times G$. Hence, v lies on $x - y$ geodesic in G that is $g_{ns}(G) \leq g_{ns}(K_2 \times G)$.

Conversely, let S contains a vertex with the property that every vertex of T_1 lies on $x - w$ geodesic T_1 for some $w \in S$. Let S' consists of x together with those vertices of T_2 corresponding to those $S - \{x\}$. Thus, $|S'| = |S|$. We show that S' is a non split geodetic set of $K_2 \times T$. Hence $g_{ns}(K_2 \times T) \leq g_{ns}(T)$. Thus, $g_{ns}(K_2 \times T) = g_{ns}(T)$. $\square$

Theorem 6.7 *Let K_2 and $G = P_n$ be the two graphs,*

$$g_{ns}(K_2 \times G) = \begin{cases} 2 & \text{if } n \geq 3 \\ 3 & \text{if } n = 2 \end{cases}$$

Proof Consider a trivial graph K_1 as a connected graph. Let G_1 and G_2 be the two copies of G and also $V(G_1) = \{a_1, a_2, \dots, a_n\}$, $V(G_2) = \{b_1, b_2, \dots, b_n\}$. Let $S = \{a_1, b_n\}$ be the non split geodetic set of $K_2 \times G$ and also $d(a_1, b_n) = \text{diam}(a_1, b_n)$. Thus, $V - S = \{a_2, a_3, \dots, a_n, b_1, b_2, \dots, b_{n-1}\}$ is the induced subgraph and it is connected. Hence $g_{ns}[K_2 \times G] = 2$.

Similarly, the result is obvious for $n = 2$ that is $g_{ns}[K_2 \times G] = 3$. $\square$

§7. Block Graphs

A block graph has a subgraph G_1 of G (not a null graph) such that G_1 is non separable and if G_2 is any other graph of G , then $G_1 \cup G_2 = G_1$ or $G_1 \cup G_2$ is separable. For any graph G a complete subgraph of G is called clique of G . The number of vertices in a largest clique of G is called the clique number of G and denoted by $\omega(G)$.

Theorem 7.1 *For any block graph G , $g_{ns}(G) = n - c_i$ where n be the number of vertices and c_i be the number of cut vertices.*

Proof Let $V = \{v_1, v_2, \dots, v_n\}$ be the number of vertices of G . Consider S be the geodetic set of G and $\langle V(G) - S \rangle$ is connected. Thus S itself a nonsplit geodetic set of G . Since every geodetic set does not contain any cut vertices. Hence, $g_{ns}(G) = n - c_i$. $\square$

Theorem 7.2 For any block graph G , $g_{ns}(G) \leq \omega(G) + 2c_i$ where $\omega(G)$ be the clique number and c_i be the number of cut vertices.

Proof Let $V = \{v_1, v_2, \dots, v_n\}$ be the number of vertices of G . In a block graph, every geodetic set is a nonsplit geodetic set. Consider S be the geodetic set of G and $\langle V(G) - S \rangle$ is connected. Thus S itself a nonsplit geodetic set of G . By the definition, the number of vertices in a largest clique of G is $\omega(G)$ and also every geodetic set does not contain any cut vertices of G . It follows that $g_{ns}(G) \leq \omega(G) + 2c_i$. $\square$

Theorem 7.3 For any block graph G , $g_{ns}(G) = \alpha_0(G) + 1$ where $\alpha_0(G)$ be the vertex covering number.

Proof Let G be a block graph of order n . Now, we prove the result by mathematical induction.

For $c_i = 1$, the vertex covering number of G is

$$\alpha_0(G) = n - c_i - 1 \Rightarrow \alpha_0(G) = n - 1 - 1 \Rightarrow \alpha_0(G) + 1 = n - 1,$$

by Theorem 7.1, we have

$$g_{ns}(G) = n - c_i \Rightarrow g_{ns}(G) = n - 1.$$

Therefore, $g_{ns}(G) = \alpha_0(G) + 1$. Thus the result is true for $c_i = 1$. Let us assume that the result is true for $c_i = m$ that is $g_{ns}(G) = \alpha_0(G) + 1$.

Now, we shall prove the result for $c_i = m + 1$, where $m+1$ is the number of cut vertices. Let $S = \{v_1, v_2, \dots, v_n\}$ be the minimum nonsplit geodetic set of G . Since every geodetic set does not contain any cut vertex, by Theorem 7.1 we have $g_{ns}(G) = n - m - 1$. Therefore,

$$\alpha_0(G) = n - c_i - 1 \Rightarrow \alpha_0(G) = (n - m - 1) - 1 \Rightarrow \alpha_0(G) + 1 = n - m - 1.$$

Thus, $g_{ns}(G) = \alpha_0(G) + 1$. $\square$

References

- [1] F.Buckley, F.Harary, *Distance in Graphs*, Addison-Wesley, Redwood City,CA, 1990.
- [2] F.Buckley, F.Harary, L V.Quintas, Extremal results on the geodetic number of a graph, *Scientia A2*, Redwood City,CA, (1988), 17-26.
- [3] G.Chartrand, F.Harary and P.Zhang, On the geodetic number of a graph, *Networks*, 39(1988), 1-6.
- [4] G.Chartrand, E.M.Palmer and P.Zhang, The geodetic number of a graph, A Survey, *Congr.Numer.*, 156(2002) 37-58.

- [5] G.Chartrand and P.Zhang, The forcing geodetic number of a graph, *Discuss.Math, Graph Theory*, 19(1999), 45-58.
- [6] F.Buckley, F.Harary, *Graph Theory*, Addison-Wesley, 1969.
- [7] F.Harary, E.Loukakis, C.Tsouros, The geodetic number of a graph, *Math.Comput.Modelling*, 17(11)(1993), 89-95.
- [8] P.A.Ostrand, Graphs with specified radius and diameter, *Discrete Math.*, 4(1973), 71-75.
- [9] A.P.Santhakumaran,P.Titus and J.John, On the connected geodetic number of a graph, *Congr.Numer.*, (in press).
- [10] Venkanagouda M Goudar, Ashalatha K S and Venkatesh, Split geodetic number of a graph, *Advances and Applications in Discrete Mathematics*, Vol.13,(2014), 9-22.