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Folding of Cayley Graphs

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Abstract: The aim of this paper is to discuss the folding of Cayley graphs of finite group.We prove that, for any finite group $G, |G\ | = n$ and H is a subgroup of G. Then Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ of G with respect to $S = H \setminus \{1_G\}$ can be folded into a complete graph K_r where $r = |H|$. Hence every Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ of valency $n-1$ can not be folded. Also every Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ of valency one can be folded and $\Gamma = \mathrm{Cay}(G, S)$, where S is generating set, every elements in it is self inverse and $|S| = \frac{1}{2}|G|$, can be folded to an edge. Theorems governing these types of foldings are achieved.

Key Words: Cayley graph, folding, graph folding.

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§1. Introduction

It was Robertson, S.A. [7] who in 1977 introduced the idea of folding on manifolds. Following this first paper there has been huge progress in the folding theory. All are focusing on topology and manifolds .Many other studies on the folding of different types of manifolds introduced by many others [5], [6], [8]. Also a graph folding has been introduced by E. El-Kholy [4]. But EL-Ghoul in [3], turns this idea to algebras branch by giving a definition of the folding of abstract rings and studying its properties. Zeen El-Deen in [9] introduced the folding of groups and studying its properties. Some applications on the folding of a manifold into itself was introduced by P. Di.Francesco [2].

Graph Theory began with Leonhard Euler in his study of the Bridges of Könignsburg problem. Since Euler solved this very first problem in Graph Theory, the field has exploded, becoming one of the most important areas of applied mathematics we currently study. Generally speaking, Graph Theory is a branch of Combinatorics but it is closely connected to Applied Mathematics, Topology and Computer Science.

There are frequent occasions for which graphs with a lot of symmetry are required. One such family of graphs is constructed using groups. The study of graphs of groups is innovative because through this description one can immediately look at the graph and deduce many properties of this group. Cayley graphs are an example where graphs theory can be applied to groups. These graphs are useful for studying the structure of groups and the relationships

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between elements with respect to subsets of these groups (for example, generating sets, inverse closed sets,$\cdots$, etc.).

Cayley (1878) used graphs to draw picture of groups as we will see in the following definitions.

§2. Definitions and Notations.

We will start putting down some definitions which are needed in this paper. We begin with a short review of some basic definitions and properties of graphs. *A graph* Γ consists of a set of elements called vertices $V(\Gamma)$, and a set of unordered pairs of these elements, called edges $E(\Gamma)$. We will write (x,y) for directed edge, and xy or $\{x,y\}$ for an undirected edge. we will only deal with *simple graphs*; that is, graph with no loops and no multiple edges and we will define all graphs to have a nonempty vertex set. A graph with no edges, but at least one vertex, is called *empty graph*.

It is important to note that a graph may have many different geometric representation, but we just use these as a visualization tools and focus on $V(\Gamma)$ and $E(\Gamma)$ for our analysis.

A graph is said to be *connected* if every pair of vertices has a path connecting them. Otherwise the graph is disconnected. *The valency* of a vertex is the number of edges with the vertex as an end point.If all the vertices of a graph have the same valency then it called *a regular* graph. A graph is *complete* if every vertex is connected to every other vertex, and we denote the complete graph on n vertices by K_n. A graph is said to be *bipartite* if its vertex set can be partitioned into two sets, V_1 and V_2 , such that there are no edges of the form $\{x,y\}$ where $x,y \in V_1$ *or* $x,y \in V_2$. The *complete bipartite* graph $K_{m,n}$ is a bipartite graph with vertex set $V_1 \bigcup V_2$, such that V_1 and V_2 have size m and n respectively, and edge set $\{\{x,y\},\ x \in V_1\ , y \in V_2\}$. A clique of a graph is its maximal complete subgraph.

Definition 2.1 *Let S be a subset of a finite group G . The Cayley digraph $\Gamma = Cay(G,S)$ of G with respect to S is the directed graph given as follows. The vertices of $\Gamma = Cay(G,S)$ are the elements of the group G . There is an arc between two vertices g and h if and only if $g^{-1}h \in S$. In other words, for every vertex $g \in G$ and element $s \in S$, there is an arc from g to gs .*

Notice that if the identity 1 of G is in S , then there is a loop at every vertex, while if $1 \notin S$, the digraph has no loops. For convenience we will assume that the latter case holds; it makes no difference to the results. Also notice that since S is a set, it contains no multiple entries and hence there are no multiple arcs.

Definition 2.2 *A Cayley digraph can be consider to be a* Cayley graph *if whenever S is closed inverse, that is ; if $s \in S$, we also have $s^{-1} \in S$, since in this case every arc is a part of a digon, and we can replace a digons with undirected edges*

Definition 2.3 *A non empty subset S of a group G is called a* Cayley subset *if $S = S^{-1}$ and $1_G \notin S$.*

It should be noted that, the Cayley graph depends very much on the given Cayley subset as well as on the group. Also Cayley graph $\Gamma = Cay(G,S)$ has valency $|S|$ and that $\Gamma = Cay(G,S)$ is connected if and only if S is generating set for G i.e., $\langle S\rangle = G$.

The *complement* $\overline{S}$ of Cayley subset S with respect to $G^* = G\setminus\{1_G\}$ is also a Cayley subset .Because if $x\in\overline{S}$ then $x\notin S$ and since S is a Cayley subset then $x^{-1}\notin S$ .Hence $x^{-1}\in\overline{S}$,i.e., $\overline{S}$ is a Cayley subset. It is clear that The $\overline{\Gamma} = Cay(G,\overline{S})$ and $\Gamma = Cay(G,S)$ have the same vertex set as G , where vertex g and h are adjacent in $\overline{\Gamma} = Cay(G,\overline{S})$ if and only if they are not adjacent in $\Gamma = Cay(G,S)$.

Definition 2.4([4]) *A graph map* $f:\Gamma_1\longrightarrow\Gamma_2$ *between two graphs* Γ_1 *and* Γ_2 *is a* graph folding *if and only if* f *maps vertices to vertices and edges to edges, i.e., if,*

(1) *for each* $v\in V(\Gamma_1), f(v)$ *is a vertex in* $V(\Gamma_2)$*;*

(2) *for each* $e\in E(\Gamma_1), f(e)$ *is an edge in* $E(\Gamma_2)$*.*

Note that if the vertices of an edge $e = (u,v)\in E(\Gamma_1)$ are mapped to the same vertex, then the edge e will collapse to this vertex and hence we cannot get a graph folding.

In the case of a graph folding f the set of singularities, $\sum f$, consists of vertices only. The graph folding is non trivial iff $\sum f \neq\phi$. In this case the no. $V(f(\Gamma_1))\leq$ no. $V(\Gamma_1)$, also no. $E(f(\Gamma_1))\leq$ no. $E(\Gamma_1)$.

§3. Folding of Cayley Graphs

In this section we will discuss the folding of Cayley graph $\Gamma = Cay(G,S)$ to a subgraph Γ^* of it. We notice that not every Cayley graph $\Gamma = Cay(G,S)$ can be folded into a subgraph of it, for example, Let $G = S_3$ be the Symmetric group of order 6, $G = \{(),(1\ 2),(1\ 3),(2\ 3),(1\ 2\ 3),(1\ 3\ 2)\}$ and let $S = \{(1\ 2),(1\ 2\ 3),(1\ 3\ 2)\}$ since S is generating set and $|S| = 3$, so $\Gamma = Cay(S_3,S)$ is connected graph of valency 3 . This graph cannot be folded into the induced subgraph Γ^* which shown in Figure 1. Because the vertex (2 3) is adjacent with the vertices (1 2 3) , (1 2) and (1 3) then it can not mapped by any folding to these vertices. Also the vertex (2 3) can not mapped to the vertex ( ) because the edge $\{(1\ 3),(2\ 3)\}$ has no image in Γ^*. Finally the vertex (2 3) can not mapped to the vertex (1 3 2) because the edge $\{(1\ 2),(2\ 3)\}$ has no image in Γ^*.

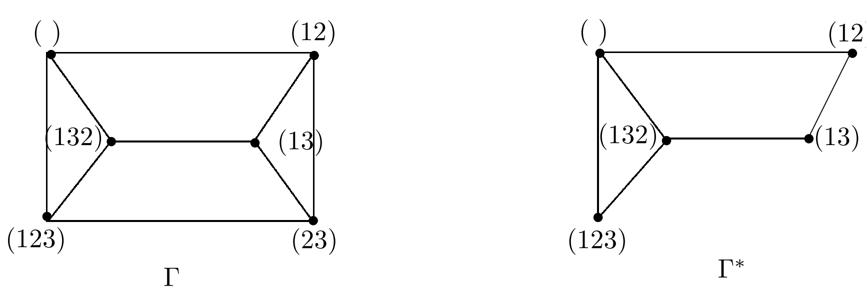

Figure 1 $\Gamma = Cay(G,S)$ can not be folded into Γ^*

It is known that, up to isomorphism , for any finite group G , there is 1 Cayley graph of G of valency zero. which is a trivial graph and this graph has a trivial foldings since there are no edges, then any graph map on the vertices is a graph folding.

Theorem 3.1 *Let G be a finite group and H is a subgroup of G. Then Cayley graph $\Gamma = Cay(G,S)$ of G with respect to $S = H \setminus \{1_G\}$ can be folded and the end of these foldings is a complete subgraph (clique) K_r where $r = |H|$ i.e.,*

$$\textit{the map } \phi : \Gamma = Cay(G,S) \longrightarrow K_r \textit{ is graph folding.}$$

Proof Let G be a finite group and H is a subgroup of G. Define the set S to be the subgroup H with the identity removed $S = H \setminus \{1_G\}$, so S is closed inverse. Then we can define Cayley graph $\Gamma^* = Cay(H,S)$ as follows, from the definition, there exist an edge between $\{1_G\}$ and every element $x \in S$. Also for all $x, y \in S$ there exist an edge between them, because, since S is inverse closed, then

$$x^{-1}, y^{-1} \in S \ , \ xy^{-1} \in H \ \text{ and } \ yx^{-1} \in H \implies xy^{-1} \neq 1 \ \text{ or } \ yx^{-1} \neq 1$$

$$\implies xy^{-1} \in S \ \text{ or } \ yx^{-1} \in S$$

Then Cayley graph of H with respect to S is a complete graph $K_{|H|}$.

Also we can define Cayley graph $\Gamma = Cay(G,S)$ such that $\{g,h\}$ be an edge of $\Gamma = Cay(G,S)$ if $gh^{-1} \in S$ and hence gh^{-1} an element of H . From the properties of cosets $gh^{-1} \in H$ implies that $Hg = Hh$. This means that, two vertices are adjacent if and only if they are in the same cosets .Thus $\Gamma = Cay(G,S)$ is a graph depicting or describing the cosets of H in G .

Since the number of the cosets of H in G is the index $m = |G : H|$, thus there are m components in $\Gamma = Cay(G,S)$ each of which is a clique of size $|H|$ i.e., $K_{|H|}$. Then $\Gamma = Cay(G,S) = \{H, Hx_1, Hx_2, ..., Hx_{m-1}\}$ where $x_i \notin H, \ i = 1, 2, \cdots, m-1$

We can define a graph maps $\phi_i : \Gamma = Cay(G,S) \longrightarrow \Gamma^* = Cay(H,S)$ by

$$\phi_i : V(Hx_i) \longrightarrow V(H) \quad , \qquad \text{where,} \qquad \phi_i(ax_i) = a \ \text{ for all } \ a \in H \quad .$$

These are graph foldings since any edge in Hx_i be in the form $e = \{ax_i, \ bx_i\}$ where $a, \ b \in H$ will mapped under ϕ into the edge $e' = \{a, b\}$ in H . Then ϕ_i preserves the edges between vertices. The end of these foldings is the folding $\phi : \Gamma = Cay(G,S) \longrightarrow K_r$ where K_r is a complete subgraph (clique) and $r = |H|$. □

Example 3.1 Let $G = D_8 = \{ \alpha, \ \beta \mid \alpha^4 = \beta^2 = (\alpha\beta)^2 = 1 \ ; \ \alpha\beta\alpha = \beta \}$ be the dihedral group of order 8, $G = \{1, \ \alpha, \ \alpha^2, \ \alpha^3, \ \beta, \ \alpha\beta, \ \alpha^2\beta, \ \alpha^3\beta \}$.

(1) Let $H = \langle \alpha \rangle = \{1, \ \alpha, \ \alpha^2, \ \alpha^3 \}$ be a subgroup of G . Since H is closed inverse but not generating set of G and $|G : H| = 2$ .So there exist two cosets of H in G i.e., H , $H\beta = \{\beta, \ \alpha\beta, \ \alpha^2\beta, \ \alpha^3\beta \}$. Let $S = H \setminus \{1\}$, then $\Gamma = Cay(D_8, S)$ has two disjoint component $\{ H \ , \ H\beta \}$ each of which is a clique of size $r = |H| = 4$, see Figure 2.

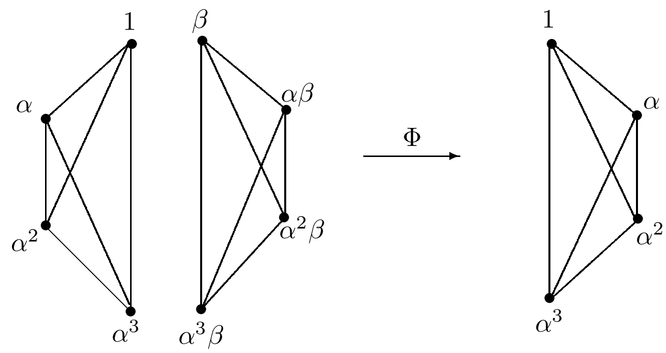

Figure 2 Folding of Cayley graph $\Gamma = Cay(D_8, \{\alpha,\ \alpha^2,\ \alpha^3\ \}\ )$

Then, the map $\phi : \Gamma = Cay(G, S) \longrightarrow\ \Gamma^* = Cay(H, S)\ \cong\ K_4$ defined by

$$\phi : V(H\beta) \longrightarrow V(H)\ ,\ \text{where}\ \phi\ (\alpha^i\ \beta) =\ \alpha^i\quad , i = 0, 1, 2, 3\ \ \text{where}\ \ \alpha^i \in H$$

This is a graph folding since it preserves the edges between vertices and the limit of the foldings is a clique of order 4.

(2) Let $H\ = \langle\ \beta\ \rangle\ = \{1, \beta\}$ be a subgroup of G , H is closed inverse but not generating set of G . Since $|G : H| = 4$, there exist four cosets of H in G i.e., $\{\ H\ , \alpha H, \alpha^2 H, \alpha^3 H\ \}$ .Let $S\ = H \setminus \{1\}$, then $\Gamma = Cay(D_8, S)$ has four disjoint component $\{\ H\ , \alpha H, \alpha^2 H, \alpha^3 H\ \}$ each of which is a clique of size $r = 2$, see Figure 3.

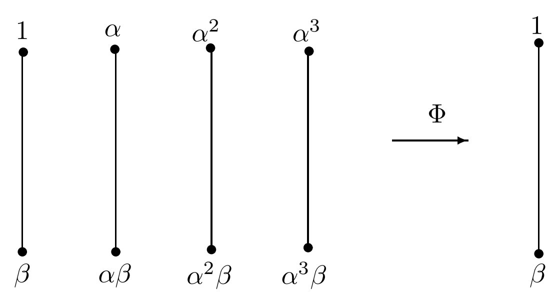

Figure 3 Folding of Cayley graph $\Gamma = Cay(D_8, \{\beta\ \}\ )$

Then the maps $\phi_i : \Gamma = Cay(G, S) \longrightarrow\ \Gamma^* = Cay(H, S)\ \cong\ K_2$ defined by

$$\phi_i : V(\alpha^i H) \longrightarrow V(H)\ \ \text{where}\ \phi_i\ (\alpha^i a) =\ a\ \ ,\ i = 0, 1, 2, 3\quad \text{where}\ \ a \in H$$

are graph foldings and the end of these foldings is a clique of order 2.

Theorem 3.2 *For any finite group G of order n, $|G\ | = n$. Every Cayley graph $\Gamma = Cay(G, S)$ of valency $n - 2$ can be folded to a clique of order $\frac{n}{2}$.*

Proof Let G be a finite group of order n , $G = \{1, a_1, a_2, \cdots, a_{n-1}\ \}$, let S be a Cayley subset of G , since the identity element $1\ \notin S$ and valency of Cayley graph $\Gamma = Cay(G, S)$

is the valency of S , if Γ has valency $n-2$ so $|S| = n-2$. From the definition of Cayley graph, the identity element is adjacent to all the elements in S and since $\Gamma = Cay(G,S)$ has no loop, then there exists exactly one element $y \in G$, $y \notin S$ not adjacent with the identity element. this means that the two elements which is not in S is not adjacent and they adjacent to all elements in S . For any element $a_i \in S$, since $|a_i| = n-2$, $\{1, a_i\} \in E(\Gamma)$ and $\{y, a_i\} \in E(\Gamma)$ then there exist one element $b \in S$ such that $\{b, a_i\} \notin E(\Gamma)$ and a_i must adjacent to all other elements in S. Then, the vertices of $\Gamma = Cay(G,S)$ can be partitioned into $\frac{n}{2}$ sets $\{A_1, A_2, \cdots, A_{\frac{n}{2}}\}$ each set has two vertices which are not adjacent, for example $A_1 = \{1, y\}$, $y \notin S$ and $A_i = \{a_k, a_r\}$, $a_k, a_r \in S$ such that a_k and a_r not adjacent and each elements in A_i are adjacent to all elements in A_j , $i \neq j$, $i, j = 1, 2, \cdots, \frac{n}{2}$. Then we have a complete $\frac{n}{2}$ partite graph $K_{2,2,\cdots,2}$, so we can define $\frac{n}{2}$ foldings on Γ as follows $\phi_k : \Gamma \longrightarrow \Gamma^*$ defined by

$$\phi_k\ (x) = y\ ,\ if\ x,\ y \in A_i\ ,\quad i, k = 1, 2, \cdots, \frac{n}{2}$$

are graph foldings and the end of these foldings is a clique of order $\frac{n}{2}$. □

Example 3.2 Let $G = D_{2.3} = \{\ \alpha,\ \beta \mid \alpha^3 = \beta^2 = (\alpha\beta)^2 = 1\ ;\ \alpha\beta\alpha = \beta\ \}$ be the dihedral group of order 6 , $G = \{1,\ \alpha,\ \alpha^2,\ \beta,\ \alpha\beta,\ \alpha^2\beta\ \}$, and $S = \{\ \alpha,\ \alpha^2,\ \beta,\ \alpha\beta\ \}$ be a Cayley subset of G . The Cayley graph $\Gamma = Cay(D_6, S\ )$ is shown in Figure 4.

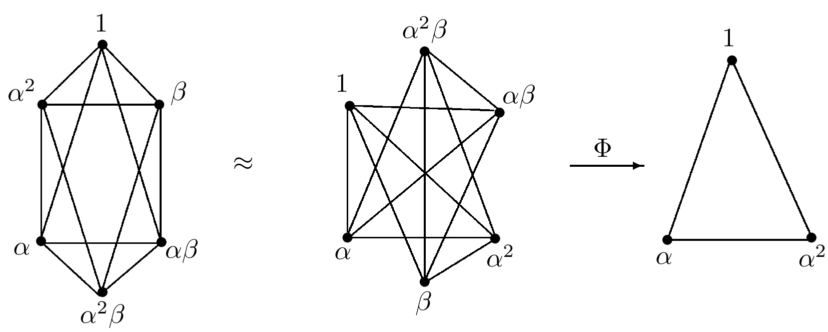

Figure 4 Folding of Cayley graph $\Gamma = Cay(D_6, \{\ \alpha,\ \alpha^2,\ \beta,\ \alpha\beta\ \}\ )$

The vertices of the graph $\Gamma = Cay(D_6, S\ )$ can divides into $\frac{n}{2} = 3$ sets, each set has two elements which are not adjacent in $\Gamma = Cay(D_6, S\ )$ .So we have $A_1 = \{1,\ \alpha^2\beta\ \}$, $A_2 = \{\alpha,\ \beta\ \}$, $A_3 = \{\alpha^2,\ \alpha\beta\ \}$. Then we can define three foldings $\phi_1(\ \alpha^2\beta) = 1$, $\phi_2(\ \beta) = \alpha^2$ and $\phi_3(\ \alpha\beta) = \alpha^2$. The composition of these foldings is the map $\phi : V(\Gamma = Cay(D_6, S\ )) \rightarrow V(\Gamma = Cay(D_6, S\ ))$ defined by

$$\phi\ (x) = \begin{cases} 1 & \text{if} \quad x \in\ A_1 \\ \alpha & \text{if} \quad x \in\ A_2 \\ \alpha^2 & \text{if} \quad x \in\ A_3 \end{cases}$$

Since the image of any edge of $E\ (\Gamma)$ will be the edge, then ϕ is a graph folding.

Proposition 3.1 *For any finite group G , $|G\ | = n$. Every Cayley graph $\Gamma = Cay(G,S)$ of valency $n-1$ can not be folded.*

Proof Let G be a finite group $|G| = n$, let S be a Cayley subset of G, since valency of Cayley graph $\Gamma = Cay(G,S)$ is the valency of S, if Γ has valency $n-1$ so $|S| = n-1$. Then $S = G\setminus\{1_G\}$ is a generating and closed inverse which implies that Cayley graph $\Gamma = Cay(G,S)$ is connected. Let $H = S\cup\{1\} = G$, so H is a subgroup of G and the index $|G:H| = 1$. Then we have one component which is a clique of order $|S| = n-1$ i.e., $\Gamma = Cay(G,S) \cong K_{n-1}$, so every vertex of Γ is adjacent to all other vertices. Then we can not define any folding on Γ since mapping any vertex to another will collapse the edge between them. □

Proposition 3.2 *For any finite group G. Every Cayley graph $\Gamma = Cay(G,S)$ of valency one can be folded.*

Proof Let G be a finite group $|G| = n$. Up to isomorphism, there is one Cayley graph $\Gamma = \mathrm{Cay}(G,S)$ of valency one. This graph $\Gamma = \mathrm{Cay}(G,S)$ is disconnected graph consists of $\frac{n}{2}$ disconnected components each component is an edge between two vertices. Since if $S = \{a\}$, $a\in G$ we have two cases

(i) If a is self inverse, $a = a^{-1}$, then $H = \{1,a\}$ is subgroup of G. Then from Theorem 3.1 $\Gamma = \mathrm{Cay}(G,S)$ consists of disjoint components $\{H, Hx_i\}$, $x_i \notin H$, $i = 1,2,\ldots,\frac{n}{2}-1$, where H is a clique of order two, i.e., there is an edge between the two vertices on $Hx_i = \{x_i, ax_i\}$. so each Hx_i is a clique of order two. This graph $\Gamma = \mathrm{Cay}(G,S)$ can be folded into H.

(ii) If a is not self inverse $a \neq a^{-1}$, then H is not subgroup of G. Let $S = H\setminus\{1_G\} = \{a\}$, then $\Gamma = \mathrm{Cay}(G,S)$ consists of an edge between elements of $H = \{1,a\}$. For any other vertex $x \in G$, then $x^{-1}\in G$ so there exists only one vertex $y\in G$ such that $xy^{-1} = a \in H$, which implies an edge between x and y. Then the graph $\Gamma = Cay(G,S)$ is disconnected graph consists of disconnected components each component is an edge between two vertices, if $|G| = n$ then Γ consists of $\frac{n}{2}$ disconnecting edges, which can be folded into one component. . □

§4. Folding Cayley Graph of Non-Abelian Group

In this section we will discuss the folding of Cayley graph of finite non-abelian groups.

Theorem 4.1 *For any finite non-abelian group G. Every Cayley graph of G with respect to a Cayley subset S, $\Gamma = Cay(G,S)$, where S is generating set, every elements in it is self inverse and $|S| = \frac{1}{2}|G|$, can be folded to an edge.*

Proof Let G be a finite non-abelian group, $|G| = n$ and $S \subseteq G$ such that S is generating set, every elements in it is self inverse and $|S| = \frac{1}{2}|G|$. Since the valency of Cayley graph $\Gamma = Cay(G,S)$ is equal to the valency of S then $|\Gamma| = |S| = \frac{1}{2}|G|$. If x and $y\in S$ then $x^{-1} = x \in S$ and $y = y^{-1}\in S$ but there is no edge between x and y in $\Gamma = Cay(G,S)$, since if there exist an edge between them this must implies that $x^{-1}y \in S$ and $x^{-1}y = xy$, but $xy \notin G$ because $(xy)^{-1} = y^{-1}x^{-1} = yx \neq xy$. So to have a graph $\Gamma = Cay(G,S)$ of valency $\frac{1}{2}|n|$, every element in S must connected to every element in $G-S$. Then we have a complete bipartite graph $\Gamma = Cay(G,S) = K_{\frac{n}{2},\frac{n}{2}}$.

Let $G = \{a_1, a_2, \cdots, a_{\frac{n}{2}}, b_1, b_2, \cdots, b_{\frac{n}{2}}\} = V(\Gamma)$ and let $S = \{a_1, a_2, \cdots, a_{\frac{n}{2}}\}$, then each vertex of S is joined to each vertex of $(G-S)$ by exactly one edge, thus

$E(\Gamma) = \{ (a_1, b_1), (a_1, b_2), \cdots, (a_1, b_{\frac{n}{2}}), (a_2, b_1), (a_2, b_2), \cdots, (a_2, b_{\frac{n}{2}})$ $, \cdots, (a_{\frac{n}{2}}, b_1), (a_{\frac{n}{2}}, b_2), \cdots, (a_{\frac{n}{2}}, b_{\frac{n}{2}}) \}$

Now, let $\phi : V(\Gamma) \longrightarrow V(\Gamma)$ be a graph map defined by

$$\phi(x) = \begin{cases} a_1 & \text{if } x \in S \\ b_1 & \text{if } x \in G-S \end{cases}$$

Thus the image of any edge of $E(\Gamma)$ will be the edge (a_1, b_1), then ϕ is a graph folding.□

Example 4.1 Let $G = S_3$ be the Symmetric group of order 6, $G = \{(), (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$ and let $S = \{(1\ 2), (1\ 3), (2\ 3)\}$ since S is generating set, every elements in it is self inverse and $|S| = \frac{1}{2}|G| = 3$, so $\Gamma = Cay(S_3, S)$ has valency 3 and the vertices of Γ can be partitioned into two sets S and $G-S$ such that there are no edges between any two vertices on the same set, see Figure 5. Then $\Gamma \cong K_{3,3}$ which can be folded by the function

$$\phi(x) = \begin{cases} (1\ 2) & \text{if } x \in S \\ (\,) & \text{if } x \in G-S \end{cases}$$

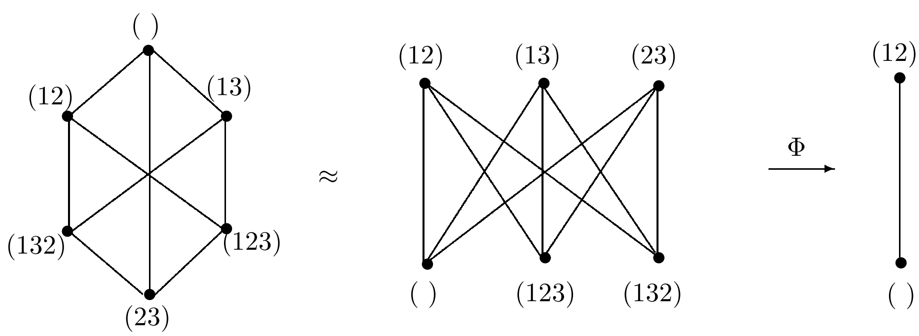

Figure 5 Folding of Cayley graph $\Gamma = Cay(S_3, \{(1\ 2), (1\ 3), (2\ 3)\})$

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