

Entire Equitable Dominating Graph

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Abstract: The entire equitable dominating graph $EE_qD(G)$ of a graph G with vertex set $V \cup S$, where S is the collection of all minimal equitable dominating sets of G and two vertices $u, v \in V \cup S$ are adjacent if u, v are not disjoint minimal equitable dominating sets in S or $u, v \in D$, where D is the minimal equitable dominating set in S or $u \in V$ and v is a minimal equitable dominating set in S containing u . In this paper, we initiate a study of this new graph valued function and also established necessary and sufficient conditions for $EE_qD(G)$ to be connected and complete. Other properties of $EE_qD(G)$ are also obtained.

Key Words: Dominating set, equitable dominating set, entire equitable dominating graph, Smarandachely dominating set.

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§1. Introduction

All graphs considered here are finite, undirected with no loops and multiple edges. We denote by p the order (i.e number of vertices) and by q the size (i.e number of edges) of such a graph G . Any undefined term and notation in this paper may be found in Harary [5].

A set of vertices which covers all the edges of a graph G is called *vertex cover* for G . The smallest number of vertices in any vertex cover for G is called its *vertex covering number* and is denoted by $\alpha_0(G)$ or α_0 . A set of vertices in G is *independent* if no two of them are adjacent. The largest number of vertices in such a set is called the *vertex independence number* of G and is denoted by $\beta_0(G)$ or β_0 . The *connectivity* $\kappa = \kappa(G)$ of a graph G is the minimum number of vertices whose removal results a disconnected or trivial graph. Analogously the *edge-connectivity* $\lambda = \lambda(G)$ is the minimum number of edges whose removal results a disconnected or trivial graph. The *diameter* of a connected graph is the maximum distance between two vertices in G and is denoted by $diam(G)$. If G and H are graphs with the property that the identification of any vertex of G with an arbitrary vertex of H results in a unique graph (up to

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isomorphism), then we write as $G \bullet H$ for this graph.

A subset D of V is called a *dominating set* of G if every vertex in $V - D$ is adjacent to at least one vertex in D . The *domination number* $\gamma(G)$ of G is the minimum cardinality taken over all minimal dominating sets of G . (See Ore [12]).

A subset D of V is called an *equitable dominating set* if for every $v \in V - D$, there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$. The minimum cardinality of such a dominating set is called the *equitable domination number* of G and is denoted by $\gamma^e(G)$. For more details about graph valued functions, domination number and their related parameters we refer [1-4, 6 - 10, 12]. The opposite of equitable dominating set is the *Smarandachely dominating set* with $|deg(u) - deg(v)| \leq 1$ for $\forall uv \in E(G)$.

The purpose of this paper is to introduce a new graph valued function in the field of domination theory in graphs.

§2. Entire Equitable Dominating Graph

Definition 2.1 The entire equitable dominating graph $EE_qD(G)$ of a graph G with vertex set $V \cup S$, where S is the collection of all minimal equitable dominating sets of G and two vertices $u, v \in V \cup S$ adjacent if u, v are not disjoint minimal equitable dominating sets in S or $u, v \in D$, where D is the minimal equitable dominating set in S or $u \in V$ and v is a minimal equitable dominating set in S containing u .

In Fig.1, a graph G and its entire equitable dominating graph $EE_qD(G)$ are shown. Here $D_1 = \{1, 3\}$, $D_2 = \{1, 4\}$, $D_3 = \{2, 3\}$ and $D_4 = \{2, 4\}$ are minimal equitable dominating sets of G .

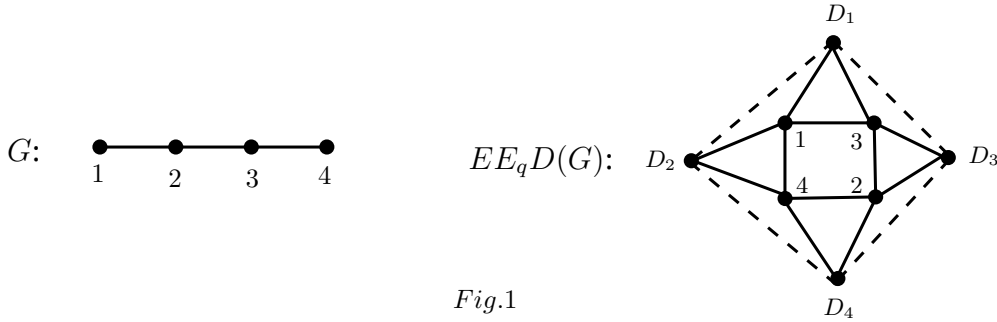


Fig.1

§3. Preliminary Results

The following will be useful in the proof of our results.

Theorem 3.1([5]) For any nontrivial graph G , $\alpha_0 + \beta_0 = p = \alpha_1 + \beta_1$.

Theorem 3.2([5]) A connected graph G is Eulerian if and only if every vertex of G has even degree.

§4. Results

First we obtain a necessary and sufficient condition on a graph G such that the entire equitable dominating graph $EE_qD(G)$ is connected.

Theorem 4.1 *For any graph G with at least three vertices, the entire equitable dominating graph $EE_qD(G)$ is connected if and only if $\Delta(G) < p - 1$.*

Proof Let $\Delta(G) < p - 1$ and u, v be any two vertices in G . We consider the following cases:

Case 1. If u and v are adjacent vertices in G , then there exist two not disjoint minimal equitable dominating sets D_1 and D_2 containing u and v respectively. Therefore by the definition 2.1, u and v are adjacent in $EE_qD(G)$.

Case 2. Suppose there exist two vertices $u \in D_1$ and $v \in D_2$ such that u and v are not adjacent in G . Then there exists a minimal equitable dominating set D_3 containing both u and v and by definition 2.1, D_1 and D_2 are connected in $EE_qD(G)$.

Conversely, suppose $EE_qD(G)$ is connected. Suppose $\Delta(G) = p - 1$ and u is a vertex of degree $p - 1$. Then the degree of u in $EE_qD(G)$ is minimum. If every vertex of G has degree $p - 1$, then every vertex of G forms a minimal equitable dominating set. Therefore $EE_qD(G)$ has at least two components, a contradiction. Thus $\Delta(G) < p - 1$. $\square$

Proposition 4.1 *$EE_qD(G) = pK_2$ if and only if $G = K_p; p \geq 2$.*

Proof Suppose $G = K_p; p \geq 2$. Then clearly each vertex of G will form a minimal equitable dominating set. Hence by definition 2.1, $EE_qD(G) = pK_2$.

Conversely, suppose $EE_qD(G) = pK_2$ and $G \neq K_p$. Then there exists at least one minimal equitable dominating set D containing two vertices of G . Then D will form C_3 in $EE_qD(G)$, a contradiction. Hence $G = K_p; p \geq 2$. $\square$

Theorem 4.2 *For any graph G , $EE_qD(G)$ is either connected or it has at least one component which is K_2 .*

Proof If $\Delta(G) < p - 1$, then by Theorem 4.1, $EE_qD(G)$ is connected. If G is complete graph $K_p; p \leq 2$ and by Proposition 4.1, then each component of $EE_qD(G)$ is K_2 .

Next, we must prove that $\delta(G) < \Delta(G) = p - 1$. Let $v_1, v_2, \dots, v_n$ be the set of vertices in G such that $\deg(v_i) = p - 1$, then it is clear that $\{v_i\}$ forms a minimal equitable dominating set and which forms a component isomorphic to K_2 . Hence $EE_qD(G)$ has at least one component which is K_2 . $\square$

In the next theorem, we characterize the graphs G for which $EE_qD(G)$ is complete.

Theorem 4.3 *$EE_qD(G) = K_{p+2}$ if and only if G is $K_{1,p}; p \geq 3$.*

Proof Suppose $G = K_{1,p}; p \geq 3$. Then there exists a minimal equitable dominating set D

contains all the vertices of G i.e $|D| = |\{u, v_1, v_2, v_3, \dots, v_p\}| = p+1$. Hence $EE_qD(G) = K_{p+2}$.

Conversely, $EE_qD(G) = K_{p+2}$, then we prove that G is $K_{1,p}; p \geq 3$. Let us suppose that, $G \neq K_{1,p}; p \geq 3$. Then there exists a minimal equitable dominating set D of cardinality is maximum p i.e $|D| = |\{v_1, v_2, v_3, \dots, v_p\}| = p$, a contradiction. Therefore G must be $K_{1,p}; p \geq 3$. $\square$

Theorem 4.4 *Let G be a nontrivial connected graph of order p and size q . The entire equitable dominating graph is a graph with order $2p$ and size p if and only if $G = K_p; p \geq 2$.*

Proof Let G be a complete graph with $p \geq 2$, then by Proposition 4.1, $G = K_p; p \geq 2$.

Conversely, suppose $EE_qD(G)$ be a $(2p, p)$ graph. Then pK_2 is the only graph with order $2p$ and size p . $\square$

In the next results, we obtain the bounds on the order and size of $EE_qD(G)$.

Theorem 4.5 *For any graph G , $2p \leq p' \leq \frac{p(p-1)}{2} + 1$, where p' denotes the number of vertices in $EE_qD(G)$. Further, the lower bound is attained if and only if G is either P_4 or $K_p; p \geq 2$ and upper bound is attained if and only if G is $K_3 \cup K_2$, $K_3 \bullet K_2$ or $C_4 \cup K_1$.*

Proof The lower bound follows from the fact that the twice the number of vertices in G and the upper bound follows that the maximum number of edges in G .

Suppose the lower bound is attained. Then every vertex of G forms a minimal equitable dominating set or every vertex of G is in exactly two minimal equitable dominating sets. This implies that the necessary condition.

Conversely, suppose G is P_4 or $K_p; p \geq 2$. Then by definition of entire equitable dominating graph, $V(EE_qD(G)) = 2p$. If the upper bound is attained. Then G must be one of the following graphs are $K_3 \cup K_2$, $K_3 \bullet K_2$ or $C_4 \cup K_1$.

If $G = K_3 \cup K_2$, then every vertex of G is in exactly two minimal equitable dominating sets hence

$$V(EE_qD(G)) = \frac{p(p-1)}{2} + 1 = \frac{pq}{2} + 1.$$

Suppose $G = K_3 \bullet K_2$. Then the pendant vertex of G is in all the minimal equitable dominating sets and forms $(p-1)$ minimal equitable dominating sets. Therefore the upper bound holds.

Now if G is $C_4 \cup K_1$. Then every equitable dominating sets contains an isolated vertex and they are not disjoint sets and by definition 2.1. Therefore upper bound holds.

Conversely, suppose G is one of the following graphs $K_3 \cup K_2$, $K_3 \bullet K_2$ or $C_4 \cup K_1$. Then it is obvious that $V(EE_qD(G)) = \frac{p(p-1)}{2} + 1$. $\square$

Theorem 4.6 *For any graph G , $p \leq q' \leq \frac{p(p+1)}{2} + 1$, where q' denotes the number of edges in $EE_qD(G)$. Further, the lower bound is attained if and only if $G = K_p \geq 2$ and the upper bound is attained if and only if G is $K_3 \cup K_1$.*

Proof The proof follows from Theorem 4.5. $\square$

In the next result, we find the diameter of $EE_qD(G)$.

Theorem 4.7 *Let G be any graph with $\Delta(G) < p - 1$, then $\text{diam}(EE_qD(G)) \leq 2$, where $\text{diam}(G)$ is the diameter of G .*

Proof Let G be any graph with $\Delta(G) < p - 1$, then by Theorem 4.1, $EE_qD(G)$ is connected. Let u, v be any arbitrary vertices in $EE_qD(G)$. We consider the following cases.

Case 1. Suppose $u, v \in V$, u and v are nonadjacent in G . Then there exists a minimal equitable dominating set containing u and v and by definition 2.1, $d_{EE_qD(G)}(u, v) = 1$. If u and v are adjacent in G and there is no minimal equitable dominating set containing u and v , then there exists another vertex $w \in V$ which is not adjacent to both u and v . Let D_1 and D_2 be two minimal equitable dominating sets containing (u, w) and (w, v) respectively. This implies that $d_{EE_qD(G)}(u, v) = 2$.

Case 2. Suppose $u \in V$ and $v \in S$. Then $v = D$ is a minimal equitable dominating set of G . If $u \in S$, then u and v are adjacent in $EE_qD(G)$. Otherwise, there exists another vertex $w \in D$. This implies that

$$d_{EE_qD(G)}(u, v) \leq d_{EE_qD(G)}(u, w) + d_{EE_qD(G)}(w, v) = 2.$$

Case 3. Suppose $u, v \in S$. Then $u \in D_1$ and $v \in D_2$ are two minimal equitable dominating sets of G and by Definition 2.1, $d_{EE_qD(G)}(u, v) = 1$. $\square$

We now characterize graphs G for which $SE_qD(G) = EE_qD(G)$. A *semientire equitable dominating graph* $SE_qD(G)$ of a graph G is the graph with vertex set $V \cup S$ and two vertices $u, v \in V \cup S$ adjacent if $u, v \in D$, where D is a minimal equitable dominating set or $u \in V$ and $v = D$ is a minimal equitable dominating set containing u ([1]).

Proposition 4.2([3]) *The semientire equitable dominating graph $SE_qD(G)$ is pK_2 if and only if $G = K_p$; $p \geq 2$.*

Remark 4.1([3]) For any graph G , $SE_qD(G)$ is a subgraph of $EE_qD(G)$.

Theorem 4.8 *For any graph G , $SE_qD(G) \subseteq EE_qD(G)$. Further, equality G , $SE_qD(G) = EE_qD(G)$ if and only if G has exactly one minimal equitable dominating set containing all vertices of G .*

Proof By Remark 4.1, $SE_qD(G) \subseteq EE_qD(G)$. Suppose $SE_qD(G) = EE_qD(G)$. Then by Theorem 4.3, D is the only minimal equitable dominating set contains all the vertices of G . Therefore G must be $K_{1,n}$; $n \geq 3$.

The converse is obvious. $\square$

In the next results, we discuss about α_0 and β_0 of $EE_qD(G)$.

Theorem 4.9 *For any graph G with no isolated vertices,*

(1) $\alpha_0(EE_qD(G)) = |S| + 1$, where S is the collection of all minimal equitable dominating

sets of G ;

$$(2) \beta_0(EE_qD(G)) = \gamma(G).$$

Proof (i) Let G be graph of order p . Let $S = \{s_1, s_2, \dots, s_i\}$ be the set of all minimal equitable dominating sets. Then by definition 2.1 and Theorem ?? . Therefore the minimum number of vertices in $EE_qD(G)$ which covers all the edges. Hence $\alpha_0(EE_qD(G)) = |S| + 1$.

(ii) By definition of $EE_qD(G)$, for any vertex v_i ; $1 \leq i \leq p$ of $EE_qD(G)$ are not adjacent. Hence these vertices forms a maximum independent set of $EE_qD(G)$. Hence (ii) follows. $\square$

In the next two results, we prove the vertex connectivity and edge- connectivity of $EE_qD(G)$.

Theorem 4.10 For any graph G , $\kappa(EE_qD(G)) = \min\{\min(deg_{EE_qD(G)} v_i), \min_{1 \leq j \leq n} |S_j|\}$, where S_j 's is the collection of all minimal equitable dominating sets of G .

Proof Let G be any graph with order p and size q . We consider the following cases.

Case 1. Let $u \in v'_i(EE_qD(G))$ for some i , having the minimum degree among all v'_i in $EE_qD(G)$. If the degree of u is less than any other vertex in $EE_qD(G)$, then by deleting the vertices which are adjacent to u , results a disconnected graph.

Case 2. Let $v \in S_j$ for some j , having the minimum degree among all S_j 's in $EE_qD(G)$. If degree of v is less than any other vertex in $EE_qD(G)$, then by deleting all the vertices which are adjacent to v . This results the graph is disconnected. Hence the result follows. $\square$

Theorem 4.11 For any graph G , $\lambda(EE_qD(G)) = \min\{\min(deg_{EE_qD(G)} v_i), \min_{1 \leq j \leq n} |S_j|\}$, where S_j 's is the collection of all minimal equitable dominating sets of G .

Proof Let G be any (p, q) graph. We consider two cases.

Case 1. Let $u \in v'_i(EE_qD(G))$, having minimum degree among all v'_i in $EE_qD(G)$. If the degree of u is less than any other vertex in $EE_qD(G)$, then by deleting those edges of $EE_qD(G)$ which are incident with u , results a disconnected graph.

Case 2. Let $v \in S_j$, having the minimum degree among all vertices of S_j . If degree of v is less than any other vertex in $EE_qD(G)$, then by deleting those edges which are adjacent to v , results in a disconnected. Hence the result follows. $\square$

Next, we prove the necessary and sufficient condition for $EE_qD(G)$ to be Eulerian.

Theorem 4.12 For any graph G , $EE_qD(G)$ is Eulerian if and only if one of the following conditions are satisfied:

- (1) There exists a vertex $u \in V$ is in all minimal equitable dominating sets and cardinality of every minimal equitable dominating set D of G is even;
- (2) If $v \in V$ is a vertex of odd degree, then it is in odd number of minimal equitable dominating sets, otherwise it is in even number of minimal equitable dominating sets of G .

Proof Suppose $\Delta < p-1$ and by Theorem 4.1, $EE_qD(G)$ is connected. Suppose $EE_qD(G)$

is Eulerian. on the contrary if condition (i) is not satisfied, then there exists a minimal equitable dominating set contains odd number of vertices and does not contains a vertex of odd degree, a contradiction. Therefore by Theorem 3.2, $EE_qD(G)$ is Eulerian. Hence condition (1) holds.

Suppose (2) does not hold. Then there exists $v \in V$ of even degree which is in odd number of minimal equitable dominating sets, a contradiction. Hence (ii) hold.

Conversely, suppose the conditions (1) and (2) are satisfied. Then every vertex of $EE_qD(G)$ has even degree and hence $EE_qD(G)$ is Eulerian. $\square$

§5. Domination in $EE_qD(G)$

We calculate the domination number of $EE_qD(G)$ of some standard class of graphs.

Theorem 5.1 *For any graph G with no isolated vertices.*

- (1) *If $G = K_p; p \geq 2$, then $\gamma(EE_qD(K_p)) = p$;*
- (2) *If $G = K_{1,p}; p \geq 3$, then $\gamma(EE_qD(K_{1,p})) = 1$;*
- (3) *If $G = C_p, p \geq 3$, then $\gamma(EE_qD(C_p)) = 2$.*

Theorem 5.2 *For any graph G , $\gamma(EE_qD(G)) = 1$, if and only if G is $K_{1,p}; p \geq 3$.*

Proof If G is $K_{1,p}; p \geq 3$, then there exists a minimal equitable dominating set D contains all the vertices of G and by Theorem ??, it is clear that, $EE_qD(G)$ is complete. Hence $\gamma(EE_qD(G)) = 1$.

Conversely, suppose $\gamma(EE_qD(G)) = 1$ and $G \neq K_{1,p}; p \geq 3$. Then there exists a minimal dominating set D in $EE_qD(G)$ of cardinality greater than or equal to 2, a contradiction. Therefore G must be $K_{1,p}; p \geq 3$. $\square$

We conclude this paper by exploring one open problem on $EE_qD(G)$.

Problem 1. *Give necessary and sufficient condition for a given graph G is entire equitable dominating graph of some graph.*

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