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Different Labelings on Parallel Transformations of a Class of Trees

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Abstract: A graph $G = (V, E)$ with p vertices and q edges is said to be a mean graph if there exists an injective function $f : V \to \{0, 1, \cdots, q\}$ that induces an edge labeling $f^* : E \to \{1, 2, \cdots, q\}$ defined by

$$\begin{aligned} f^*(uv) &= \frac{f(u)+f(v)}{2} \text{ if } f(u)+f(v) \text{ is even} \\ &= \frac{f(u)+f(v)+1}{2} \text{ if } f(u)+f(v) \text{ is odd} \end{aligned}$$

for every edge uv of G.Further f is called a super-mean labeling if $f(V(G)) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, \cdots, p+q\}$. If the vertex labels are all even numbers in $\{2, 4, \cdots, 2q\}$ so that $f^*(e) = \frac{f(u)+f(v)}{2}$ then f is an even mean labeling of G and if the vertex labels are in $\{1, 3, \cdots, 2q-1\}$ so that $f^*(e) = \frac{f(u)+f(v)+1}{2}$, then G is an odd-mean graph. In this paper, we investigate a typical class of trees based on this definition.

Key Words: Mean labeling, super-mean labeling, even-mean labeling, odd-mean labeling, parallel transformation of trees.

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§1. Introduction

Throughout this paper ,by a graph we mean a simple finite undirected graph without isolated vertices.For basic notations and terminology in graph theory we follow [1]. The concept of mean labeling was introduced in [5], super-mean labeling in [4] and odd-mean labeling in [2].

§2. T_n Class of Trees

In [3], T_n class of trees are defined as follows.

Definition 2.1 *Let T be a tree and x and y be two adjacent vertices in T.Let there be two end vertices(non-adjacent vertices of degree 1)x' and y' in T such that the length of $x - x'$ is equal to the length of the path $y - y'$. If the edge xy is deleted and x' , y' are joined by an edge $x'y'$,*

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then such a transformation of edges from xy to $x'y'$ is called a parallel transformation of an edge in T.

Definition 2.2 *A tree is said to be a T_n tree if and only if a resultant parallel transformation of edges reduce T into a Hamiltonian path. Such Hamiltonian path is denoted as P_T.*

T_{27} is given below in figure 2.1(a). Here e_1, e_2, e_3, e_4 and e_5 are the edges to be deleted and e'_i ; $i = 1, \cdots, 5$ (shown in broken lines) the corresponding edges to be added to generate P_T from T_{27} (Figure 2.1(b)).

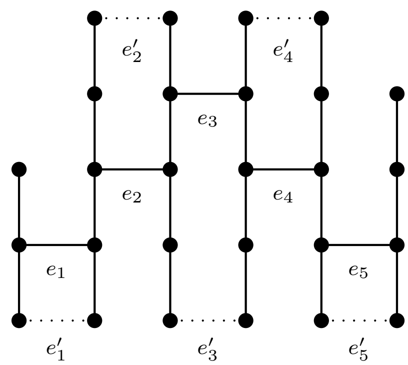

Fig 2.1 (a)

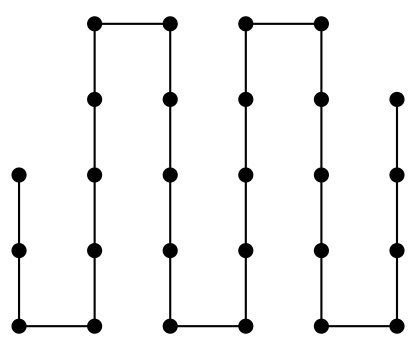

Fig 2.1 (b)

Theorem 2.1 *T_n is a mean graph.*

Proof Let T_n be a tree on n vertices and by definition there exist a path P_T corresponding to T_n. Let $E = \{e_1, e_2, \cdots, e_{n-1}\}$ be the edges of T_n. Let $E_S = \{e_1, e_2, \cdots, e_s\}$ be the set of edges to be deleted from and $E'_s = \{e'_1, e'_2, \cdots, e'_s\}$ be the edges to be added to T_n so as to obtain a Hamiltonian path P_T with $V(P_T) = V(T_n)$ and $E(P_T) = \{E(T_n) - E_S\} \cup E'_S$. Label the vertices of P_T as $x_1, x_2, \cdots, x_n$ starting from the initial pendant vertex.

Define an injective mapping $f : V(P_T) \to \{0, 1, \cdots, n-1\}$, as $f(x_i) = i-1$ for all i. Now f induces edge labeling f^* on $E(P_T)$ as

$$\begin{aligned} f^*(xy) &= \frac{f(x)+f(y)}{2} \text{ if } f(x) \text{ and } f(y) \text{ are of same parity} \\ &= \frac{f(x)+f(y)+1}{2} \text{ otherwise} \end{aligned}$$

where $xy \in E(P_T)$.

Since P_T is a path, every edge of P_T is of the form $x_i x_{i+1}$.

$$\begin{aligned} f^*(x_i x_{i+1}) &= \frac{f(x_i)+f(x_{i+1})+1}{2}, \text{ since } f(x_i) \text{ and } f(x_{i+1}) \text{ are of different parity} \\ &= \frac{i-1+i+1}{2} = i \text{ for } i = 1, 2, \cdots, n-1 \end{aligned}$$

Obviously f is injective and $f^*(G) = \{1, 2, \cdots, n-1\}$. So it is proved that f is a mean labeling on P_T. We have to prove that f is a mean labeling on T_n.
For this, it is enough to prove that $f^*(e_s) = f^*(e'_s)$ where $e_s = x_i x_j \in E(T_n)$ and $e'_s = x_{i+r} x_{j-r} \in E(T_n)$.

Now, e'_s must be of the form $x_{i+r} x_{i+r+1}$, since it is an edge of a path P_T. So

$$\begin{aligned} (x_{i+r}, x_{j-r}) &= (x_{i+r}, x_{i+r+1}) \\ \frac{f(x_{i+r})+f(x_{j-r})+1}{2} &= \frac{f(x_{i+r})+f(x_{i+r+1})+1}{2} \end{aligned}$$

Therefore

$$j = i + 2r + 1$$

So

$$f^*(e_s) = \frac{f(x_i)+f(x_{i+2r+1})+1}{2} = i + r$$

and

$$\begin{aligned} f^*(e'_s) &= f^*(x_{i+r}, x_{i+r+1}) \\ &= \frac{f(x_{i+r})+f(x_{i+r+1})+1}{2} = i + r \end{aligned}$$

Therefore,

$$f^*(e_s) = f^*(e'_s).$$

Thus, f admits mean labeling on T_n. Hence we get the theorem. □

Definition 2.3 *A graph with p vertices and q edges is said to be odd mean if there exists a function $f : V(G) \to \{0, 1, \cdots, 2q-1\}$ which is one-one and the induced map $f^* : E(G) \to \{1, 3, \cdots, 2q-1\}$ defined by $f^*(uv) = \frac{f(u)+f(v)}{2}$, if $f(u)+f(v)$ is even or $\frac{f(u)+f(v)+1}{2}$ if $f(u)+f(v)$ is odd, is a bijection. If a graph has an odd mean labeling, then we say that G is an odd mean graph.*

Definition 2.4 *A function f is called an even-mean labeling of a graph G with p vertices and q edges if f is an injection from the vertices of G to $\{2, 4, \cdots, 2q\}$ such that when each edge uv is assigned the label $\frac{f(u)+f(v)}{2}$, then the resulting edge labels are distinct.A graph which admits an even mean labeling is said to be an even-mean graph.*

Theorem 2.2 *T_n satisfies both even and odd mean labeling.*

Proof To prove T_n is an even-mean graph,we consider $f_e : V(G) \to \{2, 4, \cdots, 2q\}$ such that $f_e(x_i) = 2i$ for $i = 1, 2, \cdots, n$.

Now, to show T_n is odd-mean, we take another injective mapping $f_o : V(G) \to \{1, 3, \cdots, 2q+1\}$ as $f_o(x_i) = 2i - 1$ for $i = 1, 2, \cdots, n$. □

Theorem 2.3 *Parallel transformation of trees generate a class of super-mean graphs.*

Proof Consider a T_n tree on n vertices. By definition there exist a P_T corresponding to T_n. Let $E = \{e_1, \cdots, e_{n-1}\}$ be the edges of T_n. Let $E_r = \{e_1, e_2, \cdots, e_r\}$ be the edges to be deleted from $T_n, E_r \subset E$ and $E'_r = \{e'_1, \cdots, e'_r\}$ be the set of edges to be added to T_n to make a path P_T, such that if e_n is the deleted edge, e'_n is the corresponding edge added at a distance d_n by parallel transformation. Now we have $V(P_T) = V(T_n)$ and $E(P_T) = \{E(T_n) - E_r\} \cup E'_r$.

Now we label the vertices of P_T by $x_1, x_2, \cdots, x_n$ successively starting at one end vertex of the path P_T. Define a mapping $f : V(P_T) \to \{1, 2, \cdots, 2n-1\}$ such that $f(x_i) = 2i - 1$ for all $i = 1, \ldots, n$. Now, by the definition itself, f is one-one. Let f^* be the induced mapping defined on the edge set of P_T such that
$f \cup f^* = \{1, 2, \cdots, 2n-1\}$ as

$$\begin{aligned} f^*(xy) &= \frac{f(x)+f(y)}{2} \text{ if } f(x)+f(y) \text{ is even} \\ &= \frac{f(x)+f(y)+1}{2} \quad \text{ if } f(x)+f(y) \text{ is odd} \end{aligned}$$

where $xy \in E(P_T)$.

Since P_T is a path, every edge of P_T is of the form $x_i x_{i+1}$ for $i = 1, 2, \cdots, n-1$

$$\begin{aligned} f^*(x_i x_{i+1}) &= \frac{f(x_i)+f(x_{i+1})}{2} \\ &= 2i \quad ; \quad i = 1, 2, \cdots, n-1 \end{aligned}$$

Hence it is clear that f^* is one one and $f(G) \cup f^*(G) = \{1, 2, \cdots, 2n-1\}$. Hence f is a super mean labeling on P_T. Now it is to show f is super mean on T_n. It is enough to show that $f^*(e_k) = f^*(e'_k)$.

Let $e_k = x_i x_j$ where $x_i x_j \in E(T_n)$. To get P_T, we have to delete e_k and adjoin e'_k at a distance d from x_i such that $e'_k = x_{i+r} x_{j-r}$. Since e'_k is an edge of P_T, it must be of the form $e'_k = x_{i+r} x_{i+r+1}$.

Hence

$$\begin{aligned}
(x_{i+r}, x_{j-r}) &= (x_{i+r}, x_{i+r+1}) \\
\frac{f(x_{i+r}) + f(x_{j-r})}{2} &= \frac{f(x_{i+r}) + f(x_{i+r+1})}{2} \\
&\Longrightarrow j = i + 2r + 1 \\
f^*(e_k) &= f^*(x_i x_j) \\
&= \frac{f(x_i) + f(x_j)}{2} \\
&= \frac{f(x_i) + f(x_{i+2r+1})}{2} = 2(i+r) \\
f^*(e'_k) &= f^*(x_{i+r}, x_{i+r+1}) \\
&= \frac{f(x_{i+r}) + f(x_{i+r+1})}{2} = 2(i+r)
\end{aligned}$$

Therefore,

$$f^*(e_k) = f^*(e'_k).$$

Thus, f is super mean on T_n also. Hence, T_n is a super mean graph.

Example 2.1 In Figure 2.2, we show a super-mean labeling on tree T_{20}.

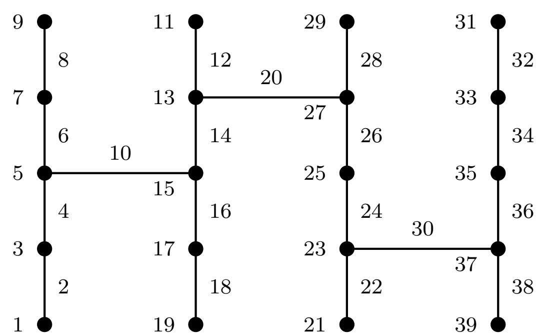

Fig 2.2

References

[1] F.Harary, *Graph Theory*, Addison-Wesley Publishing Company, 1972.

[2] K.Manickam and M.Marudai, Odd-mean labelings of graphs, *Bulletin of Pure and Applied Sciences*, 25E(1)(2006), 149-153.

[3] Mathew Varkey T.K. and Shajahan A., On labeling of parallel transformation of a class of trees, *Bulletin of Kerala Mathematics Association*, Vol.5, No.1(2009), 49-60.

[4] R.Ponraj and D.Ramya, Super mean labeling of graphs, *Preprint*.

[5] S.Somasundaram and R.Ponraj, Mean labelings of graphs, *National Academy Science Letter*, Vol.26(2003), 210-213.