International J.Math. Combin. Vol.2(2015), 88-105

The Moving Coordinate System and Euler-Savary's Formula for the One Parameter Motions On Galilean (Isotropic) Plane

Mücahit AKBIYIK and Salim YÜCE

Department of Mathematics, Facuty of Arts and Sciences

Yildiz Technical University, Esenler, 34210, Istanbul, Turkey

E-mail: makbiyik@yildiz.edu.tr, sayuce@yildiz.edu.tr

Abstract: In this article, one Galilean (or called Isotropic) plane moving relative to two other Galilean planes (or Isotropic Planes), one moving and the other fixed, was taken into consideration and the relation between the absolute, relative and sliding velocities of this movement and pole points were obtained. Also a canonical relative system for one-parameter Galilean planar motion was defined. In addition, Euler-Savary formula, which gives the relationship between the curvature of trajectory curves, was obtained with the help of this relative system.

Key Words: Kinematics, moving coordinate system, Euler Savary formula, Galilean plane (isotropic plane).

AMS(2010): 53A17, 53A35, 53A40

§1. Introduction

Galilean Geometry, is described by Yaglom, [1]. So far, many researcher has done a lot of studies as [2-4], etc in the Galilean Plane (or Isotropic Plane) and Galilean Space. Also, Euler-Savary's formula is very famous theorem. It gives relation between curvature of roulette and curvatures of these base curve and rolling curve, [14]. It takes place in a lot of studies of engineering and mathematics. A few of them are studies worked by Alexander and Maddocks,[5], Buckley and Whitfield, [6], Dooner and Griffis, [7], Ito and Takahaski, [8], Pennock and Raje, [9].

In 1959, Müller, [10]; defined one-parameter planar motion in the Euclidean plane E^2. He studied the moving coordinate system and Euler-Savary's formula during one parameter planar motions. Then, Ergin in 1991 and 1992, [11-[12]; considering the Lorentzian Plane L^2, instead of the Euclidean plane E^2, introduced the one parameter planar motion in the Lorentzian plane L^2 and gave the relations between both the velocities and accelerations and also defined the moving coordinate system. Furthermore, in 2002 Aytun [13] studied the Euler Savary formula for the one parameter Lorentzian motions as using Müller's Method [10]. And in 2003, Ikawa [14] gave the Euler-Savary formula on Minkowski without using Müller's Method [10].

[1]Received December 26, 2014, Accepted May 28, 2015.

In 1983, Otto Röschel, [15]; studied kinematics in the isotropic plane. He investigated fundamental properties of the point-paths, developed a formula analog to the wellknown formula of Euler-Savary and studied special motions: An isotropic elliptic motion and an isotropic four-bar-motion. And in 1985, he [16]; studied motions $\sum / \sum_0$ in the isotropic plane. Given C^2 -curve k in the moving frame $\sum$ he found the enveloped curve k_0 in the fixed frame $\sum_0$ and considered the correspondance between the isotropic curvatures A and A_0 of k and k_0. Then he investigated third - order properties of the point-paths. And then in 2013, Yüce, [17], considering the Galilean Plane G^2, instead of the Euclidean Plane E^2 or instead of the Lorentzian Plane L^2, defined one parameter planar Galilean motion in Galilean Plane G^2 analog [10] or [11]. Moreover, they analyzed the relationships between the absolute, relative and sliding velocities of one-parameter Galilean Planar motion as well as the related pole lines.

Now we investigate the moving coordinate system and Euler Savary's Formula during the one parameter planar Galilean motion in Galilean Plane G^2 analog [10] or [11] by using Müller's method.

§2. Preliminaries

In this section, the basic information about Galilean geometry which is described by Yaglom, [1], will be given.

Let $\{x\}$ and $\{x'\}$ be two relative frames and origin point O with velocity v on a line o move according to relative frame $\{x'\}$, that is, $b(t) = b + vt$ where t is time and b is coordinate of point O with respect to coordinate system $\{x'\}$ at the moment $t = 0$ (see, Figure 1).

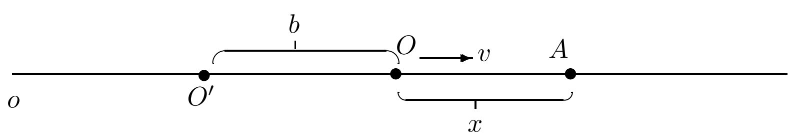

Figure 1 The rectilinear motion

Then, relation between coordinates of x and x' is

$$x' = x + b(t) \quad (1)$$

$$= x + b + vt. \quad (2)$$

Also, since time would be $t' = t + a$ (example, t is Gregorian calendar, t' is Hijra calendar), we can write

$$\begin{cases} x' = x + vt + b \\ t' = t + a. \end{cases} \quad (3)$$

This transformations (1.1) are called *Galilean Transformations for rectilinear motions.* If point $A(x,t)$ with coordinate x and t of a (two-dimensional) plane xOt (see, Figure 2) represents position of point $A(x)$ on a line o at time t, then two-dimensional Geometry which is invariant under the Galilean Transformations for rectilinear motions is obtained.

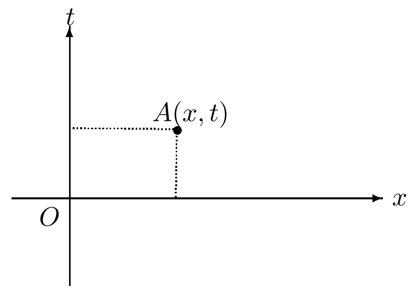

Figure 2 xOt plane

So, this geometry is called *the geometry of Galileo's principle of relativity for rectilinear motions* or *two-dimensional Galilean geometry* and is represented by G^2. Since we shall only talk about the two-dimensional Galilean geometry in this work, we shall shortly call *Galilean plane*. If transformation (3) is arranged as x instead of t and y instead of x, we get

$$\begin{aligned} x' &= x + a \\ y' &= y + vx + b. \end{aligned} \tag{4}$$

This transformation (4) composed of the shear transformation

$$\begin{aligned} x_1 &= x \\ y_1 &= y + vx \end{aligned} \tag{5}$$

and the translation transformation

$$\begin{aligned} x' &= x_1 + a \\ y' &= y_1 + b. \end{aligned} \tag{6}$$

Theorem 2.1([1]) *Transformation* (4) *maps*

(1) *lines onto lines;*

(2) *parallel lines onto parallel lines;*

(3) *collinear segments* AB, CD *onto collinear segments* $A'B'$,$C'D'$ *with* $\frac{C'D'}{A'B'} = \frac{CD}{AB}$*;*

(4) *a figure* F *onto a figure* F' *of the same area.*

In the Galilean plane, the vectors $\{\mathbf{g}_1 = (1,0), \mathbf{g}_2 = (0,1)\}$ are called *orthogonal basis vectors* of G^2, and also a vector which is parallel to vector $\mathbf{g}_2$ is called *special vector*. If $\{\mathbf{g}_1, \mathbf{g}_2\}$ are orthogonal basis vectors and $\mathbf{a}, \mathbf{b} \in G^2$ whose coordinates are (x_1, x_2) and (y_1, y_2) according to this basis vectors $\{\mathbf{g}_1, \mathbf{g}_2\}$, respectively, then the Galilean inner product of vectors $\mathbf{a}, \mathbf{b} \in G^2$ with respect to bases $\{\mathbf{g}_1, \mathbf{g}_2\}$ is defined by

$$\langle \mathbf{a}, \mathbf{b} \rangle_G = x_1 y_1 \tag{7}$$

(also you can see in [4]). If $\mathbf{a}, \mathbf{b}$ are special vectors, then the Galilean special inner product of special vectors is defined by

$$\langle \mathbf{a}, \mathbf{b} \rangle_\delta = x_2 y_2. \tag{8}$$

Hence, the norm of every vector $\mathbf{a} = (x_1, x_2) \in G^2$ on the Galilean plane is denoted by $\|\mathbf{a}\|_G$ and is defined by

$$\|\mathbf{a}\|_G = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle_G} = |x_1| \tag{9}$$

and the norm of every special vector $\mathbf{a} = (0, x_2) \in G^2$ on the Galilean Plane is denoted by $\|\mathbf{a}\|_\delta$ and is defined by

$$\|\mathbf{a}\|_\delta = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle_\delta} = |x_2|. \tag{10}$$

The distance between points $A(x_1, x_2)$ and $B(y_1, y_2)$ on the Galilean Plane is denoted by d_{AB} and is defined by

$$d_{AB} = \sqrt{\langle \mathbf{AB}, \mathbf{AB} \rangle_G} = y_1 - x_1, \tag{11}$$

where $y_1 > x_1$ (see, Figure 3).

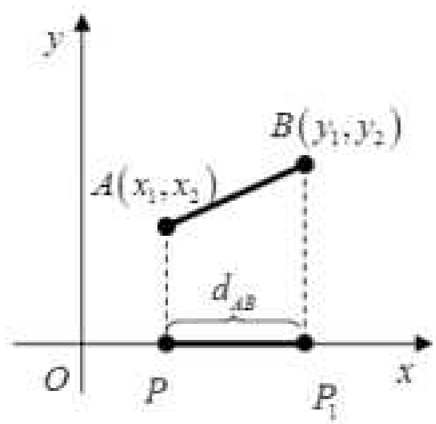

Figure 3 The distance between two points in G^2

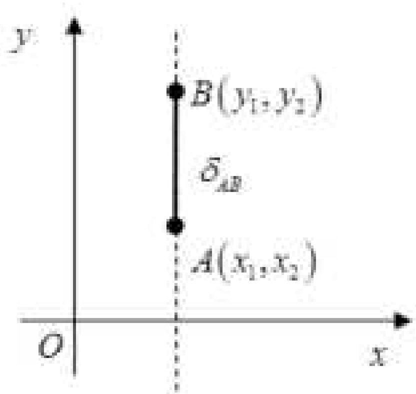

Figure 4 The special distance between two points

That is, d_{AB} is equal to $\|\mathbf{PP}_1\|$ in the sense of Euclidean Geometry. If the distance between

$A(x_1, x_2)$ and $B(y_1, y_2)$ is equal to zero $(x_1 = y_1)$, then special distance of the points $A(x_1, x_2)$ and $B(y_1, y_2)$ is denoted by δ_{AB} and is defined by

$$\delta_{AB} = \sqrt{\langle \mathbf{AB}, \mathbf{AB} \rangle_\delta} = y_2 - x_2 \tag{12}$$

here $y_2 > x_2$ (see, Figure 4). The set of points $M(x, y)$ whose distances from a fixed point $Q(a, b)$ have constant absolute value r is called a *Galilean circle*, and is denoted by S. Thus, the circle S in the Galilean Plane is defined by

$$(x - a)^2 = r^2 \tag{13}$$

or

$$x^2 + 2px + q = 0 \tag{14}$$

where $p = -a$, $q = a^2 - r^2$. Also in the Galilean Plane, lines are parallel to y-axis are separable from class of lines and these lines are called *special lines* and others *are called ordinary lines.* Therefore, the circle S in the Galilean Plane consists of two special lines whose distance from Q is r (see, Figure 5).

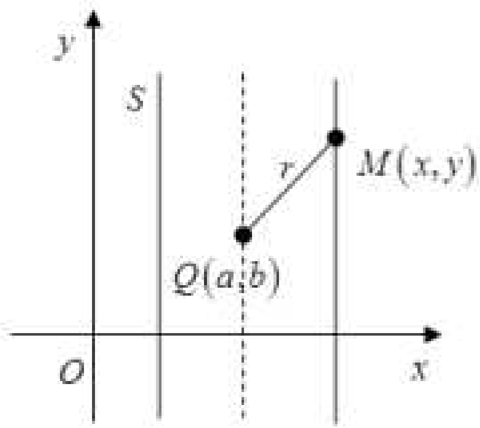

Figure 5 The circle in G^2

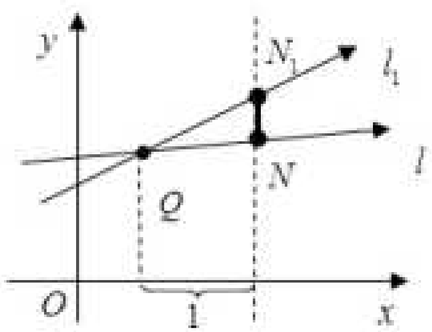

Figure 6 The angle between two intersecting lines

However, the angle between two ordinary lines $y = kx + s$ and $y = k_1x + s_1$ intersecting at a point $Q = (x_0, y_0)$ (see, Figure 6) is defined by

$$\delta_{ll_1} = k_1 - k. \tag{15}$$

But the *right angle* is defined by angle between ordinary line and special line in the Galilean Plane. So, the special lines are perpendicular to ordinary lines and also special vectors are perpendicular to ordinary vectors. Consequently, let S be a unit circle with centered at O and $M\,(x, y)$ be a point on S. Assume that l denotes line OM and α denotes δ_{Ol} (see, Figure 7).

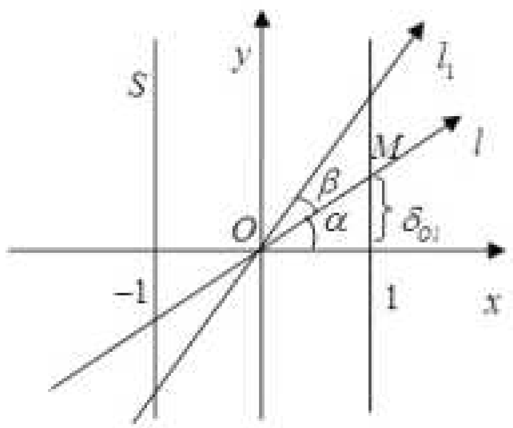

Figure 7 The trigonometry in G^2

Then, we have

$$cosg\alpha = 1 \tag{16}$$

and

$$sing\alpha = \alpha. \tag{17}$$

Also, suppose that l_1 be another ordinary line and $\delta_{ll_1} = \beta$. Then we get

$$\cos g\,(\alpha + \beta) = 1 \tag{18}$$

and

$$\sin g\,(\alpha + \beta) = \sin g\alpha \cos g\beta + \cos g\alpha \sin g\beta. \tag{19}$$

We can define a circle by another definition in Euclidean Geometry that the set of points M from which a given ordinary segment AB (i.e., a segment on an ordinary line) is seen at a constant directed angle α. If we use this definition in the Galilean Plane, we have equation

$$ax^2 + 2b_1x + 2b_2y + c = 0 \tag{20}$$

which are (Euclidean) parabolas and this set is called a *Galilean cycle* and denoted by Z. Here each of lines which are parallel to y-axis, is a *diameter* of cycle Z and it is denoted by d (see, Figure 8).

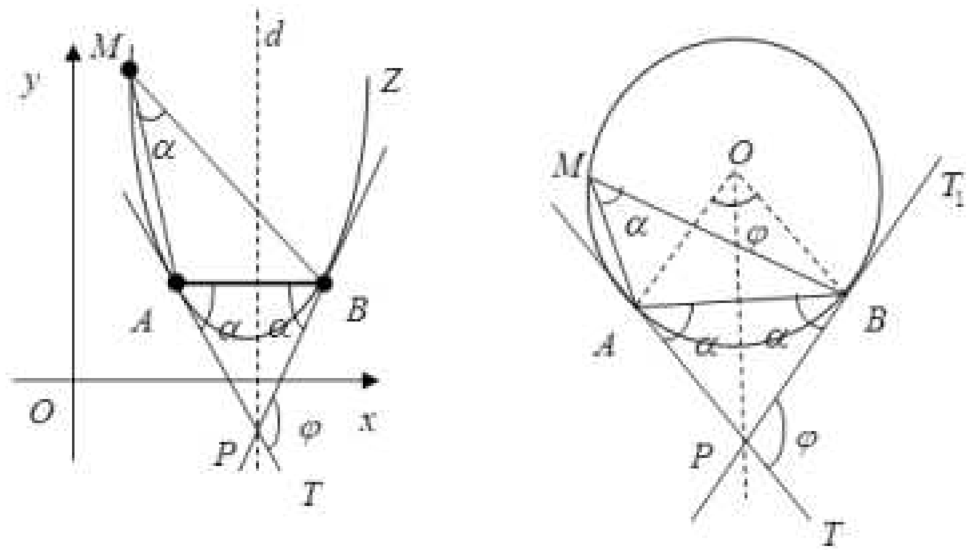

Figure 8 The cycle in G^2 and circle in E^2

Also, the length of an arc AB of a curve Γ is equal to the length $s = d_{AB}$ of the cord AB (see, Figure 9).

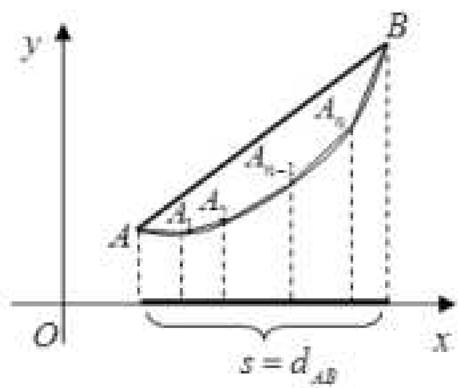

Figure 9 The length of an arc in G^2

Thus, the radius of cycle Z is defined by

$$r = \frac{1}{2a}. \tag{21}$$

Furthermore, the curvature ρ of Γ at A is defined as the rate change of the tangent at A, that is, the curvature of Γ at A is

$$\rho = \lim_{\triangle s \to 0} \frac{\triangle \varphi}{\triangle s} \tag{22}$$

where $\triangle \varphi = TAT_0'$ is angle between the two neighboring tangents, $\triangle s = arcAM$ is the scalar arc element of Γ such that M is a point of curve Γ (see, Figure 10).

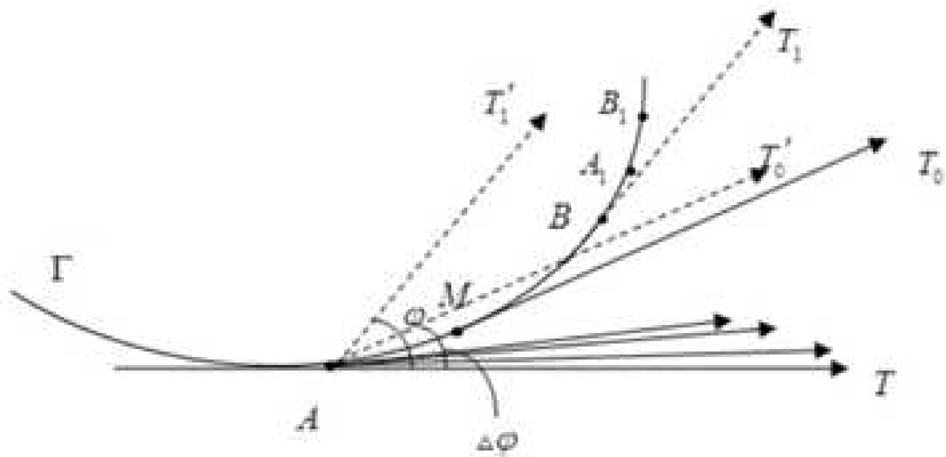

Figure 10 The curvature in G^2

Therefore, the radius of curvature of Γ at A, denoted r, is

$$r = \frac{1}{\rho}. \tag{23}$$

Now, let consider all cycles passing through the points of Γ and having the same tangent $AT = l$ at A as Γ. From these cycles, we select that for any other cycle Z_0 which is closest distance MM' between points on Γ and Z sufficiently close to A which project to the same point N on l is larger than the distance between the corresponding points M and M_0' of Γ and Z_0. This cycle Z_0 is called the *osculating cycle of curve* Γ at A (see, Figure 11), [1].

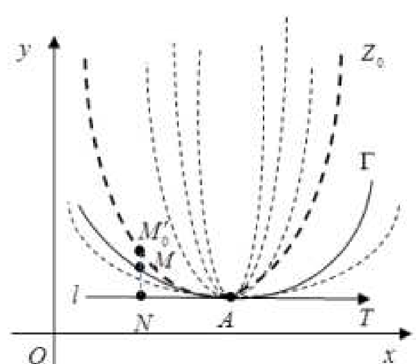

Figure 11 The cycle in G^2

§3. One Parameter Planar Galilean Motion

Let G and G' $\left(G = G' = G^2\right)$ be moving and fixed Galilean planes and $\{O; \mathbf{g}_1, \mathbf{g}_2\}$ and $\{O'; \mathbf{g}_1', \mathbf{g}_2'\}$ be their coordinate systems, respectively. The motion defined by the transformation

$$\mathbf{x} = \mathbf{x}' - \mathbf{u} \tag{24}$$

is called as *one parameter planar Galilean motion* and denoted by $B = G/G'$ where

$$\mathbf{OO'} = \mathbf{u} = u_1\mathbf{g}_1 + u_2\mathbf{g}_2 \tag{25}$$

for $u_1, u_2 \in \mathbb{R}$, $\mathbf{x}, \mathbf{x'}$ are the coordinate vectors with respect to the moving and fixed rectangular coordinate systems of a point $X = (x_1, x_2) \in G$ respectively. Also, these vectors $\mathbf{x}, \mathbf{x'}$ and $\mathbf{u}$ and shear rotation angle φ between $\mathbf{g}_1$ and $\mathbf{g}'_1$ are continuously differentiable functions of a time parameter t (see, Figure 12).

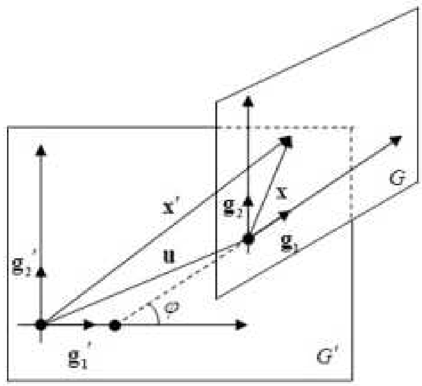

Figure 12 The motion $B = G/G'$

We can write

$$\begin{aligned} \mathbf{g}_1 &= \mathbf{g}'_1 + \varphi \mathbf{g}'_2 \\ \mathbf{g}_2 &= \mathbf{g}'_2 \end{aligned} \tag{26}$$

for the shear rotation angle $\varphi = \varphi(t)$ between $\mathbf{g}_1$ and $\mathbf{g}'_1$. In this study, we suppose that

$$\dot{\varphi}(t) = \frac{d\varphi}{dt} \neq 0 \tag{27}$$

where "." denotes the derivation with respect to "t". By differentiating the equations (25) and (26), the derivative formulae of the motion $B = G/G'$ are

$$\begin{aligned} \dot{\mathbf{g_1}} &= \dot{\varphi}\mathbf{g}_2 \\ \dot{\mathbf{g_2}} &= \mathbf{0} \\ \dot{\mathbf{u}} &= u_1\mathbf{g}_1 + (u_2 + u_1\dot{\varphi})\,\mathbf{g}_2. \end{aligned} \tag{28}$$

The velocity of the point X with respect to G is defined as the relative velocity $\mathbf{V}_r$ and is founded by

$$\mathbf{V}_r \dot{x_1}\mathbf{g}_1 + \dot{x_2}\mathbf{g}_2. \tag{29}$$

Furthermore, velocity of the point $X \in G$ according to G' is known as the absolute velocity, and is found as

$$\mathbf{V_a} = -\dot{u_1}\mathbf{g}_1 + (-\dot{u_2} - u_1\dot{\varphi} + x_1\dot{\varphi})\,\mathbf{g}_2 + \mathbf{V}_r. \tag{30}$$

Thus, we get the sliding velocity

$$\mathbf{V}_f = -\dot{u_1}\mathbf{g}_1 + (-\dot{u_2} - u_1\dot{\varphi} + x_1\dot{\varphi})\,\mathbf{g}_2. \tag{31}$$

In the general one parameter motions, the points whose sliding velocity is zero, i.e.,$\mathbf{V}_f = \mathbf{0}$ are called the *pole point* or *instantaneous shear rotation pole point* and in the Galilean Plane G, the pole point $P = (p_1, p_2) \in G$ of the motion $B = G/G'$ is defined by

$$p ... \begin{cases} p_1 = u_1 + \frac{\dot{u_2}(t)}{\dot{\varphi}(t)} \\ p_2 = p_2\left(t\left(\lambda\right)\right) \end{cases} \tag{32}$$

for $\lambda \in \mathbb{R}$ (see, Figure 13) [17].

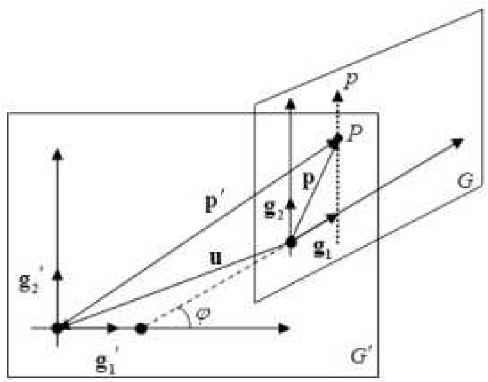

Figure 13 The pole line in Galilean plane

Corollary 3.1([17]) *During one parameter planar motion $B = G/G'$ invariants points in both planes at any instant t have been on a special line in the plane G. That is, there only exists pole line in the Galilean Plane G at any instant t. For all $t \in I$, this pole lines are parallel to y-axis and these pole lines form bundles of parallel lines. Using equations* (31)*and* (32)*, for sliding velocity, we can write*

$$\mathbf{V}_f = \{0\mathbf{g}_1 + (x_1 - p_1)\,\mathbf{g}_2\}\,\dot{\varphi}. \tag{33}$$

Corollary 3.2([17]) *During one parameter planar motion $B = G/G'$, the pole ray $\mathbf{PX} = (x_1 - p_1)\,\mathbf{g}_1 + (x_2 - p_2)\,\mathbf{g}_2$ and $\mathbf{V}_f = \{0\mathbf{g}_1 + (x_1 - p_1)\,\mathbf{g}_2\}\,\dot{\varphi}$ are perpendicular vectors, i.e., $\left\langle \mathbf{PX}, \mathbf{V}_f \right\rangle_G = 0$. Thus, under the motion $B = G/G'$, the focus of the points $X \in G$ is an orbit curve that is normal pass through the shear rotation pole P.*

Corollary 3.3([17]) *Under the motion $B = G/G'$, the norm of the sliding velocity $\mathbf{V}_f$ is*

$$\|\mathbf{V}_f\|_\delta = \|\mathbf{PX}\|_G \left|\dot{\varphi}\right|. \tag{34}$$

That is, during the motion $B = G/G'$, all of the orbits of the points $X \in G$ are such curves *whose normal lines pass thoroughly the pole point P. At any instant t, the motion $B = G/G'$ is a Galilean instantaneous shear rotation with the angular velocity $\dot{\varphi}$ about the pole point P.*

Since there exist pole points in every moment t, during the one-parameter plane motion $B = G/G'$ any pole point P is situated various positions on the plane G and G'. The position of the pole point P on the moving plane G is usually a curve and this curve is called *moving pole curve* and is denoted by (P). Also the position of this pole point P on the fixed plane G' is usually a curve and this curve is called *fixed pole curve* and is denoted by (P').

§4. The Moving Coordinate System on the Galilean Planes

In this section, we study on three Galilean planes, and investigate relative, sliding and absolute velocity, a point of X on a plane according to the other two plane and relations between the pole points. Let A and G be moving and G' be fixed Galilean plane and $\{B, \mathbf{a}_1, \mathbf{a}_2\}$, $\{O; \mathbf{g}_1, \mathbf{g}_2\}$ and $\{O'; \mathbf{g}'_1, \mathbf{g}'_2\}$ their coordinate systems, respectively (see, Figure 14).

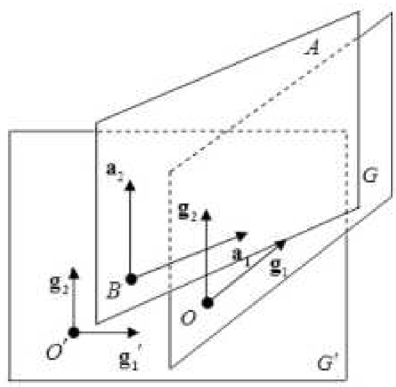

Figure 14 The two moving and one fixed coordinate system

Assume that φ and ψ are rotation angles of one parameter planar motions A/G and A/G', respectively. Let us consider a point X with the coordinates of (x_1, x_2) in moving plane A. Since

$$\mathbf{BX} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 \tag{35}$$

$$\mathbf{O}B = \mathbf{b} = b_1\mathbf{a}_1 + b_2\mathbf{a}_2 \tag{36}$$

$$\mathbf{O'B} = \mathbf{b}' = b'_1\mathbf{a}_1 + b'_2\mathbf{a}_2 \tag{37}$$

are vectors on the moving system of A, we have

$$\mathbf{x} = \mathbf{OX} = \mathbf{OB} + \mathbf{BX} = \mathbf{b} + x_1\mathbf{a}_1 + x_2\mathbf{a}_2 \tag{38}$$

$$\mathbf{x}' = \mathbf{O'X} = \mathbf{O'B} + \mathbf{BX} = \mathbf{b}' + x_1\mathbf{a}_1 + x_2\mathbf{a}_2 \tag{39}$$

where vector $\mathbf{x}$ and $\mathbf{x}'$ denote the point X with respect to the coordinate systems of G and G', respectively. Let's find the velocities of one parameter motion with the help of the differentiation the equations (38) and (39). Assume that "$d...$" denotes the differential with respect to G and "$d'^{...}$" denotes the differential with respect to G'.

The derivative equations in motion $B = A/G$, are

$$d\mathbf{a}_1 = d\varphi \mathbf{a}_2 \tag{40}$$

$$d\mathbf{a}_2 = 0 \tag{41}$$

$$db = db_1 \mathbf{a}_1 + (b_1 d\varphi + db_2)\,\mathbf{a}_2 \tag{42}$$

and the derivative equations in motion $B = A/G'$ taking $d'b = d'b'$, are

$$d'\mathbf{a}_1 = d'\psi \mathbf{a}_2 \tag{43}$$

$$d'\mathbf{a}_2 = 0 \tag{44}$$

$$d'b = db'_1 \mathbf{a}_1 + (b'_1 d\psi + db'_2)\,\mathbf{a}_2. \tag{45}$$

So differential of X with respect to G is

$$d\mathbf{x} = (\sigma_1 + dx_1)\,\mathbf{a}_1 + (\sigma_2 + \tau x_1 + dx_2)\,\mathbf{a}_2 \tag{46}$$

where $\sigma_1 = db_1, \sigma_2 = db_2 + b_1 d\varphi, \tau = d\varphi$. Therefore the relative velocity vector of X with respect to G is

$$\mathbf{V}_r = \frac{d\mathbf{x}}{dt} \tag{47}$$

and also differential of X with respect to G' is

$$d'\mathbf{x} = (\sigma'_1 + dx_1)\,\mathbf{a}_1 + (\sigma'_2 + \tau' x_1 + dx_2)\,\mathbf{a}_2 \tag{48}$$

where $\sigma'_1 = db'_1, \sigma'_2 = db'_2 + b'_1 d\psi, \tau' = d\psi$. Thus, the absolute velocity vector of X with respect to G' is

$$\mathbf{V}_a = \frac{d'\mathbf{x}}{dt}. \tag{49}$$

Here $\sigma_1, \sigma_2, \sigma'_1, \sigma'_2, \tau$ and τ' are the *Pfaffian forms* of one parameter motion with respect to t. If $V_r = 0$ and $V_a = 0$ then the point X is fixed in the planes G and G', respectively. Thus, the conditions that the point is fixed in planes G and G' become

$$dx_1 = -\sigma_1, dx_2 = -\sigma_1 - \tau x_1 \tag{50}$$

and

$$dx_1 = -\sigma'_1, dx_2 = -\sigma'_2 - \tau' x_1, \tag{51}$$

respectively. Substituting equation (50) into equation (48) and considering that the sliding velocity of the point X is $\mathbf{V}_f = \frac{d_f \mathbf{x}}{dt}$, we have

$$\mathbf{d}_f \mathbf{x} = \{\sigma'_1 - \sigma_1\}\,\mathbf{a}_1 + \{(\sigma'_2 - \sigma_2) + (\tau' - \tau)\}\,\mathbf{a}_2. \tag{52}$$

Therefore from (46), (48) and (52) we may give the following theorem.

Theorem 4.1 *If X is a fixed point on G, then we have*

$$\mathbf{d}'\mathbf{x} = \mathbf{d}_f\mathbf{x} + \mathbf{dx}, \tag{53}$$

that is, $\mathbf{V}_a = \mathbf{V}_f + \mathbf{V}_r$. *Thus, velocities law is preserved.*

Remark 4.2 In the motion A/G, the absolute velocity $\widetilde{\mathbf{V}}a$ corresponds the differential of

$$d\mathbf{x} = \sigma_1\mathbf{a}_1 + \{\sigma_2 + \tau x_1\}\,\mathbf{a}_2 + dx_1\mathbf{a}_1 + dx_2\mathbf{a}_2 \tag{54}$$

according to plane G of the point X, and the relative velocity $\widetilde{\mathbf{V}}_r$ which is the velocity of X according to the plane A, is equal to the differential of

$$dx_1\mathbf{a}_1 + dx_2\mathbf{a}_2 \tag{55}$$

with respect to A of the point X. Thus the sliding velocity $\widetilde{\mathbf{V}}_f$ with respect to motion A/G is the differential of

$$\sigma_1\mathbf{a}_1 + \{\sigma_2 + \tau x_1\}\,\mathbf{a}_2 \tag{56}$$

according to G of the point X. Similarly, in the motion A/G', the absolute velocity $\widetilde{\mathbf{V}}'_a$ is equal to the differential of

$$d'\mathbf{x} = \sigma'_1\mathbf{a}_1 + \{\sigma'_2 + \tau' x_1\}\,\mathbf{a}_2 + dx_1\mathbf{a}_1 + dx_2\mathbf{a}_2 \tag{57}$$

with respect to G' of the point X, and the relative velocity $\widetilde{\mathbf{V}}'_r$ is the differential of

$$dx_1\mathbf{a}_1 + dx_2\mathbf{a}_2 \tag{58}$$

with respect to A of the point X. So the sliding velocity $\widetilde{\mathbf{V}}'_f$ corresponds the differential of

$$\sigma'_1\mathbf{a}_1 + \{\sigma'_2 + \tau' x_1\}\,\mathbf{a}_2 \tag{59}$$

with respect to G' of the point X.Since the motion G/G' is characterized by the inverse motion of A/G and the motion A/G', we have the sliding motion $\mathbf{d}_f\mathbf{x}$ when we subtract from the sliding velocity of the motion A/G' to the sliding velocity of the motion A/G. So we can write

$$\mathbf{V}_f = \widetilde{\mathbf{V}}'_f - \widetilde{\mathbf{V}}_f. \tag{60}$$

To avoid the cases of pure translation we assume that $\dot{\varphi} \neq 0, \dot{\psi} \neq 0$.

In one parameter planar Galilean motions the pole point is characterised by vanishing sliding velocity, i.e., $d_f x = 0$. So, the pole point P of the one parameter planar motion G/G' is obtained as

$$p_1 = -\frac{\sigma'_2 - \sigma_2}{\tau' - \tau} \tag{61}$$

$$p_2 = p_2(\lambda), \lambda \in \mathbb{R} \tag{62}$$

where $\mathbf{BP} = p_1\mathbf{a}_1 + p_2\mathbf{a}_2$. Note that here we find

$$\sigma_1' - \sigma_1 = 0. \tag{63}$$

§5. The Shear Rotation Poles for Moving Galilean Planes with Respect to the Other

Let us have three planes such as A, G, G' moving with respect to together and also occur in two one parameter planar Galilean motion with respect to each other. In the determined time t, pairs of plane $(A, G), (A, G')$ and (G, G') have a determined shear rotation pole line, and instantaneous shear rotation motions arise with angular velocity about the pole line. Accordingly, three planes moving with respect to together constitute a three-member kinematic chain.

Motion A/G of plane A with respect to plane G is formulated by equation system (41). Here $d\varphi = \tau$ is infinitesimal shear rotation angle, that is, $\frac{\tau}{dt}$ is an angular velocity. Differential of point X with respect to plane A

$$d\mathbf{BX} = dx_1\mathbf{a}_1 + dx_2\mathbf{a}_2. \tag{64}$$

The differential corresponds to relative velocity with respect to plane A. If point X is fixed, then we can write $d\mathbf{BX} = \mathbf{0}$. In the equation (46), the differential of point X is given with respect to plane G. From here, sliding velocity of point X with respect to motion A/G is

$$\sigma_1\mathbf{a}_1 + (\sigma_2 + \tau x_1)\mathbf{a}_2. \tag{65}$$

However the shear rotation pole of motion is characterized by vanishing the sliding velocity. So, for the shear rotation pole line q of motion A/G, we have

$$q ... \left\{ q_1 = -\frac{\sigma_2}{\tau}, q_2 = q_2(\xi) \;\; \xi \in \mathbb{R}. \right. \tag{66}$$

Similarly for the shear rotation pole line q' of motion A/G', we get

$$q' ... \left\{ q_1' = -\frac{\sigma_2'}{\tau'}, q_2' = q_2'(\mu), \;\; \mu \in \mathbb{R}. \right. \tag{67}$$

And also, the angular velocity of motion G/G' is

$$\frac{d(\psi - \varphi)}{dt} = \frac{\tau' - \tau}{dt} \tag{68}$$

and for the shear rotation pole line p, from equations (61) and (62) we can rewrite

$$p ... \left\{ \begin{array}{l} p_1 = -\frac{\sigma_2' - \sigma_2}{\tau' - \tau} \\ p_2 = p_2(\lambda), \lambda \in \mathbb{R} \end{array} \right. . \tag{69}$$

So, we give following theorem.

Theorem 5.1 *If three Galilean planes form one parameter planar Galilean motions pairwisely, there exist three shear rotation pole lines at every moment t, and each of these three lines is parallel to the others.*

Corollary 5.2 *Generally, if there are n−Galilean planes which form one parameter planar Galilean motions pairwisely, then we tell of n−member kinematic chain. If the each motions is connected time (real) parameter t , there exit* $\begin{pmatrix} n \\ 2 \end{pmatrix}$ *relative shear rotation pole lines at every moment t and every each line is parallel to each others.*

Theorem 5.3 *The rate of the distance of three shear rotation poles is as the rate of their angular velocities.*

Proof Since $q_1 = -\frac{\sigma_2}{\tau}, q_1' = -\frac{\sigma_2'}{\tau'}$ and $p_1 = -\frac{\sigma_2' - \sigma_2}{\tau' - \tau}$, it is hold

$$(q_1 - q_1) : (p_1 - q_1') : (q_1 - p_1) = (\tau' - \tau) : \tau : -\tau'. \tag{70}$$

Thus, we can write

$$\left\|\overrightarrow{QQ'}\right\|_G : \left\|\overrightarrow{Q'P}\right\|_G : \left\|\overrightarrow{PQ}\right\|_G = (\tau' - \tau) : \tau : -\tau'. \tag{71}$$

§6. Euler-Savary's Formula for One Parameter Motions in the Galilean Plane

We studied one parameter Galilean motion adding $\{B, \mathbf{a}_1, \mathbf{a}_2\}$ moving system to the motion of G with respect to G'. Now, in this section, we choose a special relative system $\{B, \mathbf{a}_1, \mathbf{a}_2\}$ satisfying the following conditions:

(i) The initial point B of the system is a pole point P on the pole line that coordinates are p_1 and p_2.

(ii) The axes $\{B; \mathbf{a}_1\}$ coincides with the common tangent of the pole curves (P) and (P').

If we consider the condition i), then from equations (61) and (62) we have

$$p_1 = p_2 = 0. \tag{72}$$

Thus, we can write

$$\sigma_1' = \sigma_1, \sigma_2' = \sigma_2. \tag{73}$$

Therefore we get

$$\mathbf{db} = \mathbf{dp} = \sigma_1 \mathbf{a}_1 + \sigma_2 \mathbf{a}_2 = \mathbf{d'p} = \mathbf{db}'. \tag{74}$$

Considering the condition ii), then we have $\sigma_2' = \sigma_2 = 0$. So if we choose $\sigma_1' = \sigma_1 = \sigma$, then the derivative equations for the canonical relative system $\{P, \mathbf{a}_1, \mathbf{a}_2\}$ are

$$\begin{aligned} d\mathbf{a}_1 &= \tau\mathbf{a}_2, \quad d'\mathbf{a}_1 = \tau'\mathbf{a}_2 \\ d\mathbf{a}_2 &= \mathbf{0}, \qquad d'\mathbf{a}_2 = \mathbf{0} \\ d\mathbf{p} &= \sigma\mathbf{a}_1, \quad d'\mathbf{p} = \sigma\mathbf{a}_1. \end{aligned} \tag{75}$$

Moreover, τ is the cotangent angle, that is, two neighboring tangets angle of curve (P), and τ' is also the cotangent angle of curve (P') where, $\sigma = ds$ is the scalar arc element of the pole curves (P) and (P'). And so $\tau : \sigma$ is the curvature of the moving pole curve (P). Similarly, $\tau' : \sigma$ is the curvature of the fixed pole curve (P'). Hence from (23) the radius of curvature of the pole curves (P) and (P') are

$$r = \frac{\sigma}{\tau} \tag{76}$$

and

$$r' = \frac{\sigma}{\tau'} \tag{77}$$

respectively. Moving plane G rotates the infinitesimal instantaneous angle of the $d\phi = \tau' - \tau$ around the shear rotation pole P within the time scale dt with respect to fixed plane G'. Therefore the angular velocity of shear rotational motion of G with respect to G' is

$$\frac{\tau' - \tau}{ds} = \frac{d\phi}{ds} = \dot{\phi}. \tag{78}$$

Hence we get

$$\frac{\tau' - \tau}{ds} = \frac{d\phi}{ds} = \frac{1}{r'} - \frac{1}{r} \tag{79}$$

from equations (76), (77) and (78). We accept that for the direction of unit tangent vector $\mathbf{a}_1$, pole curves (P) and (P') are drawn to the positive x-axis direction that is, $\frac{ds}{dt} > 0$, and so we have $r > 0$. Similarly we can write $r' > 0$.

Now we will investigate case of the point X' which is on the diameter d of osculating cycle of trajectory curve which is drawn in the fixed plane G' by a point X of moving plane G in the movement G/G'. In the canonical relative system, let coordinates of points X at plane G and X' at plane G' be the (x_1, x_2) and (x_1', x_2'), respectively. In the movement $\mathbb{D}/\mathbb{D}'$, there is a point X' which is on center of curvature of osculating cycle of trajectory curve of X are situated together with the instantaneous rotation pole P in every moment t such that

$$\mathbf{PX} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2$$

and

$$\mathbf{PX}' = x_1'\mathbf{a}_1 + x_2'\mathbf{a}_2$$

have same direction which passes the pole point P. So we can write

$$x_1 : x_2 = x_1' : x_2'$$

or

$$x_1 x_2' - x_1' x_2 = 0. \tag{80}$$

Considering the condition ii) we obtain the condition that the point X to be the fixed in the moving plane G is

$$dx_1 = -\sigma, dx_2 = -\tau x_1 \tag{81}$$

and the point X' to be the fixed in the moving plane G' is

$$dx_1' = -\sigma, dx_2' = -\tau' x_1'. \tag{82}$$

Differentiating the equation(80) and from the conditions (81) and (82) we have

$$(x_2 - x_2')\,\sigma + x_1 x_1'\,(\tau - \tau') = 0. \tag{83}$$

If the polar coordinates are passed, then we get

$$x_1 = a\cos g\alpha = a,\ x_2 = a\sin g\alpha = a\alpha \tag{84}$$

$$x_1' = a'\cos g\alpha = a',\ x_2' = a'\sin g\alpha = a'\alpha. \tag{85}$$

Thus, we can write

$$(a\sin g\alpha - a'\sin g\alpha)\,\sigma + aa'\,(\tau - \tau') = 0. \tag{86}$$

So from last equation and equation (79) we have

$$\left(\frac{1}{a'} - \frac{1}{a}\right)\sin g\alpha = \frac{1}{r'} - \frac{1}{r} = \frac{d\phi}{ds}. \tag{87}$$

Here, r and r' are the radii of curvature of the pole curves P and P', respectively. ds represents the scalar arc element and $d\phi$ represents the infinitesimal Galilean angle of the motion of the pole curves. The equation (87) is called the *Euler-Savary formula* for one-parameter motion in Galilean plane G.

Consequently, the following theorem can be given.

Theorem 6.1 *Let G and G' be the moving and fixed Galilean planes, respectively. A point $X \in G$, draws a trajectory whose a point at the normal axis of curvature is X' on the plane G' in one-parameter planar motion G/G'. In the inverse motion of G/G', a point X' assumed on G' draws a trajectory whose a point at the normal axis of curvature is X on the plane G. The relation between the points X and X' which is given by the Euler-Savary formula given in the equation (87).*

References

[1] Yaglom I.M., *A simple non-Euclidean Geometry and its Physical Basis*, Springer-Verlag, New York, 1979.

[2] O. Röschel, *Die Geometrie des Galileischen Raumes*, Habilitationsschrift, Institut für

Math. und Angew. Geometrie, Leoben, 1984.

[3] Pavković B.J. and Kamenarović, I., The equiform differential geometry of curves in the Galilean space, *Glasnik Matematićki*, 22(42), 449-457, 1987.

[4] Helzer G., Special relativity with acceleration, Mathematical Association of America, *The American Mathematical Monthly*, Vol.107, No. 3, 219-237, 2000.

[5] Alexander J.C., Maddocks J.H., On the maneuvering of vehicles, *SIAM J. Appl. Math.*, 48(1): 38-52,1988.

[6] Buckley R., Whitfield E.V., The Euler-Savary formula, *The Mathematical Gazette*, 33(306): 297-299, 1949.

[7] Dooner D.B., Griffis M.W., On the spatial Euler-Savary equations for envelopes, *J. Mech. Design*, 129(8): 865-875, 2007.

[8] Ito N., Takahashi K.,Extension of the Euler-Savary equation to hypoid gears, *JSME Int.Journal. Ser C. Mech Systems*, 42(1): 218-224, 1999.

[9] Pennock G.R., Raje N.N., Curvature theory for the double flier eight-bar linkage, *Mech. Theory*, 39: 665-679, 2004.

[10] Blaschke W., and Müller H.R., *Ebene Kinematik*, Verlag Oldenbourg, München, 1959.

[11] A.A.Ergin, On the one-parameter Lorentzian motion, *Comm. Fac. Sci. Univ. Ankara*, Series A 40, 59–66, 1991.

[12] A.A.Ergin, Three Lorentzian planes moving with respect to one another and pole points, *Comm. Fac. Sci.Univ. Ankara*, Series A 41,79-84,1992.

[13] I.Aytun, *Euler-Savary Formula for One-Parameter Lorentzian Plane Motion and its Lorentzian Geometrical Interpretation*, M.Sc. Thesis, Celal Bayar University, 2002.

[14] T.Ikawa, Euler-Savary's formula on Minkowski geometry, *Balkan Journal of Geometry and Its Applications*, 8 (2), 31-36, 2003.

[15] Otto Röschel, Zur Kinematik der isotropen Ebene., *Journal of Geometry*,**21**,146–156,1983.

[16] Otto Röschel, Zur Kinematik der isotropen Ebene. II., *Journal of Geometry*, **24**, 112–122,1985.

[17] Akar M., Yuce S. and Kuruoglu N, One parameter planar motion in the Galilean plane, *International Electronic Journal of Geometry*, 6(1), 79-88, 2013.