

Smarandache-R-Module and Mcrita Context

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Abstract: In this paper we introduced Smarandache-2-algebraic structure of R -module namely Smarandache- R -module. A Smarandache-2-algebraic structure on a set N means a weak algebraic structure A_0 on N such that there exist a proper subset M of N , which is embedded with a stronger algebraic structure A_1 , stronger algebraic structure means satisfying more axioms, by proper subset one understands a subset from the empty set, from the unit element if any, from the whole set. We define Smarandache- R -module and obtain some of its characterization through S -algebra and Morita context. For basic concept we refer to Raul Padilla.

Key Words: R -module, Smarandache- R -module, S -algebras, Morita context and Cauchy modules.

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§1. Preliminaries

Definition 1.1 Let S be any field. An S -algebra A is an (R, R) -bimodule together with module morphisms $\mu : A \otimes_R A \rightarrow A$ and $\eta : R \rightarrow A$ called multiplication and unit linear maps respectively such that

$$A \otimes_R A \otimes_R A \xrightarrow[\mu]{\mu \otimes 1_A} A \otimes_R A \xrightarrow{\mu} A \text{ with } \mu \circ (\mu \otimes 1_A) = \mu \circ (1_A \otimes \mu) \text{ and}$$

$$R \xrightarrow[\eta]{\eta \otimes 1_A} A \otimes_R A \xrightarrow{\mu} A \text{ with } \mu \circ (\eta \otimes 1_A) = \mu \circ (1_A \otimes \eta).$$

Definition 1.2 Let A and B be S -algebras. Then $f : A \rightarrow B$ is an S -algebra homomorphism if $\mu_B \circ (f \otimes f) = f \circ \mu_A$ and $f \circ \eta_A = \eta_B$.

Definition 1.3 Let S be a commutative field with 1_R and A an S -algebra M is said to be a left A -module if for a natural map $\pi : A \otimes_R M \rightarrow M$, we have $\pi \circ (1_A \otimes \pi) = \pi \circ (\mu \otimes 1_M)$.

Definition 1.4 Let S be a commutative field. An S -coalgebra is an (R, R) -bimodule C with R -linear maps $\Delta : C \rightarrow C \otimes_R C$ and $\varepsilon : C \rightarrow R$, called comultiplication and counit respectively such

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that $C \xrightarrow{\Delta} C \otimes_R C \xrightarrow[\Delta \otimes 1_C]{1_C \otimes \Delta} C \otimes_R C \otimes_R C$ with $(1_C \otimes \Delta) \circ \Delta = (\Delta \otimes 1_C) \circ \Delta$ and $C \xrightarrow{\Delta} C \otimes_R C \xrightarrow[\varepsilon \otimes 1_C]{1_C \otimes \varepsilon} R$ with $(1_C \otimes \varepsilon) \circ \Delta = 1_C = (\varepsilon \otimes 1_C) \circ \Delta$.

Definition 1.5 Let C and D be S -coalgebras. A coalgebra morphism $f : C \rightarrow D$ is a module morphism if it satisfies $\Delta_D \circ f = (f \otimes f) \circ \Delta_C$ and $\varepsilon_D \circ f = \varepsilon_C$.

Definition 1.6 Let A be an S -algebra and C an S -coalgebra. Then the convolution product is defined by $f * g = \mu \circ (f \otimes g) \circ \Delta$ with $1_{\text{Hom}_R(C, A)} = \eta \circ (1_R)$ for all $f, g \in \text{Hom}_R(C, A)$.

Definition 1.6 For a commutative field S , an S -bialgebra B is an R -module which is an algebra (B, μ, η) and a coalgebra (B, Δ, ε) such that Δ and ε are algebra morphisms or equivalently μ and η are coalgebra morphisms.

Definition 1.7 Let R, S be fields and M an (R, S) -bimodule. Then, $M^* = \text{Hom}_R(M, R)$ is an (S, R) -bimodule and for every left R -module L , there is a canonical module morphism $\alpha_L^M : M^* \otimes_R L \rightarrow \text{Hom}_R(M, L)$ defined by $\alpha_L^M(m^* \otimes l)(m) = m^*(m)l$ for all $m \in M, m^* \in M^*$ and $l \in L$. If α_L^M is an isomorphism for each left R -module L , then ${}_R M_S$ is called a Cauchy module.

Definition 1.8 Let R, S be fields with multiplicative identities 1 , an (S, R) -bimodule M and N , an (R, S) -bimodule. Then the six-tuple datum $K = [R, S, M, N, \langle, \rangle_R, \langle, \rangle_S]$ is said to be a Morita context if the maps $\langle, \rangle_R : N \otimes_S M \rightarrow R$ and $\langle, \rangle_S : M \otimes_R N \rightarrow S$ are binmodule morphisms satisfying the following associativity conditions:

$$m' \langle n, m \rangle_R = \langle m', n \rangle_S m \text{ and } \langle n, m \rangle_R n' = n \langle m, n' \rangle_S$$

\langle, \rangle_R and \langle, \rangle_S are called the Morita maps.

§2. Smarandache- R -Modules

Definition 2.1 A Smarandache- R -module is defined to be such an R -module that there exists a proper subset A of R which is an S -Algebra with respect to the same induced operations of R .

§3. Results

Theorem 3.1 Let R be a R -module. There exists a proper subset A of R which is an S -coalgebra iff A^* is an S -algebra.

Proof Let us assume A^* is an S -algebra. For proving that A is an S -coalgebra we check the counit conditions as follows:

$$\varepsilon : A \cong A \otimes_R S \xrightarrow{1_A \otimes \mu} A \otimes_R A^* \xrightarrow{\psi_A} R.$$

Next, we check the counit condition as follows:

$$\begin{aligned}\Delta : A &\cong A \otimes_R S \otimes_R S \xrightarrow{1_A \otimes \eta \text{End}_S(A)} A \otimes_R (A^* \otimes_R A) \otimes_R A^* \\ &\xrightarrow{1_A \otimes A \otimes A^*} (A \otimes_R A) \otimes_R (A \otimes_R A)^* \xrightarrow{1_A \otimes R \otimes A} (A \otimes_R A) \otimes_R (A \otimes_R A)^* \\ &\xrightarrow{1_A \otimes A} (A \otimes_R A) \otimes_R A^* \xrightarrow{\cong} (A \otimes_R A) \otimes_R A \xrightarrow{\cong} A \otimes_R A \xrightarrow{\cong} A.\end{aligned}$$

Thus, A is an S -coalgebra.

Conversely, Let us assume A is an S -coalgebra. Now to prove that A^* is an S -algebra, we check the unit conditions as follows

$$\eta : R \xrightarrow{\eta \text{End}_S(A)} A \otimes_R A^* \rightarrow 1_A \otimes A \xrightarrow{\cong} A.$$

We check the multiplication conditions as follows A is a Cauchy module. Notice that

$$\begin{aligned}A \otimes_R A &\rightarrow R, \\ A &\cong A \otimes_R A \otimes_R R \xrightarrow{1_A \otimes \eta \text{End}_S(A)} A \otimes_R A \otimes_R A^* \rightarrow R \otimes_R A^* \xrightarrow{\cong} A^*, \\ \mu : A \otimes_R A &\xrightarrow{\varepsilon \otimes 1_A} A^* \otimes_R A \xrightarrow{\cong} R \otimes_R A^* \xrightarrow{\cong} A^*.\end{aligned}$$

Thus, A^* is an S -algebra. By definition, R is a smarandache R -module. \square

Theorem 3.2 *Let R be an R -module. Then there exists a proper subset $\text{End}_S(M)^*$ of R which is an S -algebra.*

Proof Let us assume that R be an R -module. For proving that $\text{End}_S(M)$ is an S -coalgebra which satisfies multiplication and unit conditions $\mu : \text{End}_S(M) \otimes_R \text{End}_S(M) \rightarrow \text{End}_S(M)$ and $\eta : R \rightarrow \text{End}_S(M)$, we check the comultiplication condition as follows:

$$\Delta : \text{End}_S(M) \cong \text{End}_S(M) \otimes_R \xrightarrow{1_{\text{End}(M)} \otimes \eta} \text{End}_S(M) \otimes_R \text{End}_S(M).$$

Next, we check the counit conditions as follows:

$$\begin{aligned}\varepsilon : \text{End}_S(M) &\cong \text{End}_S(M) \otimes_R R \xrightarrow{1_{\text{End}(M)} \otimes \eta} \text{End}_S(M) \otimes_R \text{End}_S(M) \\ &\xrightarrow{\cong \otimes \cong} \text{Hom}_R(M, M) \otimes_R \text{Hom}_R(M, M) \\ &\xrightarrow{\cong \otimes \cong} (M' \otimes_R M) \otimes_R (M' \otimes_R M) \xrightarrow{\psi M \otimes \psi M} R \otimes_R R \xrightarrow{\cong} R.\end{aligned}$$

Thus $\text{End}_S(M)$ is an S -coalgebra. By Theorem 3.1, $\text{End}_S(M)^*$ is an S -algebra. Hence, R is a Smarandache R -module. \square

Theorem 3.3 *Let R be an R -module. Then there exists a proper subset $M \otimes_R M^*$ of R which is an S -algebra.*

Proof For proving that $M \otimes_R M^*$ is an S -algebra, we check the multiplication and unit

conditions as follows:

$$\begin{aligned} \mu : (M \otimes_R M^*) \otimes (M \otimes_R M^*) &\xrightarrow{\cong} M \otimes_R (M^* \otimes_R M) \otimes_R M^* \\ &\xrightarrow{1_M \psi_M \otimes 1_M} M \otimes_R R \otimes_R M^* \\ &\xrightarrow{1_M \otimes \psi_M} M \otimes_R M^*. \end{aligned}$$

As M is a Cauchy module, we have

$$\eta : R \rightarrow \text{End}_S(M) \xrightarrow{\cong} M \otimes_R M^*,$$

which implies that $M \otimes_R M^*$ is an S -algebra. Hence, R is a Smarandache R -module. \square

Theorem 3.4 *Let R be an R -module. Then there exists a proper subset the datum $[R, M, N, \langle, \rangle_R]$ a morita context $(M \otimes_R N)^*$ of R which is an S -algebra.*

Proof Let us assume that R be an R -module. For proving that $M \otimes_R N$ is an S -algebra, we have

$$\begin{aligned} \mu : (M \otimes_R N) \otimes_R (M \otimes_R N) &\rightarrow M \otimes_R (N \otimes_R M) \otimes_R N \\ &\xrightarrow{1_M \otimes \langle, \rangle \otimes 1_N} M \otimes_R R \otimes_R N \xrightarrow{\cong} M \otimes_R N, \end{aligned}$$

which shows that the multiplication condition is satisfied.

Also, since M and N are Cauchy R -modules, there exist maps

$$\eta \text{End}_R(M) : R \rightarrow M^* \otimes_R M \quad \text{and} \quad \eta \text{End}_S(N) : R \rightarrow N^* \otimes_R N$$

that can be used to prove the unit condition as follows:

$$\begin{aligned} \eta : R \cong R \otimes_R R &\xrightarrow{\eta \text{End}_S(M) \otimes \eta \text{End}_S(N)} (M^* \otimes_R M) \otimes_R (N^* \otimes_R N) \\ &\xrightarrow{\cong \otimes 1_{M \otimes N}} (M^* \otimes_R N^*) \otimes_R (M \otimes_R N) \\ &\xrightarrow{\cong \otimes 1_{M \otimes N}} (M \otimes_R N)^* \otimes_R (M \otimes_R N) \\ &\xrightarrow{\cong \otimes 1_{M \otimes N}} R^* \otimes_R (M \otimes_R N) \\ &\xrightarrow{\cong} R \otimes_R (M \otimes_R N) \xrightarrow{\cong} (M \otimes_R N), \end{aligned}$$

which implies that $M \otimes_R N$ is an S -algebra. By definition, R is a Smarandache R -module.

Theorem 3.5 *Let R be an R -module. Then there exists a proper subset the datum $[R, M, N, \langle, \rangle_R]$ a morita context $M \otimes_R N$ of R which is an S -coalgebra.*

Proof Let us assume that R be an R -module. For proving that $(M \otimes_R N)$ is an S -coalgebra,

we have

$$\begin{aligned}
\Delta : M \otimes_R N &= (M \otimes_R N) \otimes_R (R \otimes_R R) \\
&\xrightarrow{1_{M \otimes_R N} \otimes \eta \text{End}_S(M) \otimes \eta \text{End}_S(N)} (M^* \otimes_R M) \otimes_R (N^* \otimes_R N) \\
&\xrightarrow{1_{M \otimes_R N} \otimes \cong} (M \otimes_R N) \otimes_R (M \otimes_R N) \otimes_R (M^* \otimes_R N^*) \\
&\xrightarrow{1_{M \otimes_R N} \otimes \cong} (M \otimes_R N) \otimes_R (M \otimes_R N) \otimes_R (M \otimes_R N)^* \\
&\xrightarrow{1_{M \otimes_R N} \otimes \langle \cdot, \cdot \rangle_R^*} (M \otimes_R N) \otimes_R (M \otimes_R N) \otimes_R R \\
&\xrightarrow{\cong} (M \otimes_R N) \otimes_R (M \otimes_R N).
\end{aligned}$$

Also, we have the counit condition as follows:

$$\begin{aligned}
\varepsilon : M \otimes_R N &\cong (M \otimes_R N) \otimes_R R \xrightarrow{1_{M \otimes_R N} \otimes \eta \text{End}_S(M)} (M \otimes_R N) \otimes_R (M^* \otimes_R M) \\
&\xrightarrow{\langle \cdot, \cdot \rangle_R \otimes 1_{M^* \otimes M}} R \otimes_R M^* \otimes_R M \xrightarrow{\cong} M^* \otimes_R M \xrightarrow{\psi_M} R,
\end{aligned}$$

which implies that $\Rightarrow M \otimes_R N$ is an S -coalgebra. Hence, R is a Smarandache R -module. \square

Theorem 3.6 *Let R be an R -module. Then there exists a proper subset the datum $[R, M, N, \langle \cdot, \cdot \rangle_R]$ a Morita context iff $M \otimes_R N$ is an S -bialgebra.*

Proof First, if $M \otimes_R N$ is an S -bialgebra by Theorem 3.5, we know that $M \otimes_R N$ is an S -algebra and $M \otimes_R N$ is an S -coalgebra. Hence by definition, R is a Smarandache R -module.

If $M \otimes_R N$ is an S -bialgebra, we have the map

$$\varepsilon = \langle \cdot, \cdot \rangle_R : M \otimes_R N \rightarrow R.$$

Associativity of the map $\varepsilon = \langle \cdot, \cdot \rangle_R$ holds because the diagram

$$\begin{array}{ccc}
(M \otimes_R N) \otimes_R M & \xrightarrow{\cong} & M \otimes_R (N \otimes_R M) \\
\varepsilon \otimes 1_M \searrow & & \swarrow 1_M \otimes \varepsilon \\
& M &
\end{array}$$

is commutative. Hence the datum $[R, M, N, \langle \cdot, \cdot \rangle_R]$ is a Morita context. \square

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