*International J.Math. Combin. Vol.2(2015), 133-143*

# Number of Regions in Any Simple Connected Graph

Mushtaq Ahmad Shah

(Department of Mathematics, Vivekananda Global University, Jaipur)

M.H.Gulzar

(Department of Mathematics, University of Kashmir, Srinagar)

Mridula Purohit

(Department of Mathematics, VIT East Jaipur)

E-mail: shahmushtaq81@gmail.com

**Abstract**: A graph $G(v,e)$ is simple if it is without self loops and parallel edges and a graph $G(v,e)$ is connected if every vertex of graph is connected with each other. This paper is dealing with the problem of finding the number of regions in any simple connected graph. In other words this paper generalize the Eulers result on number of regions in planer graphs to all simple non planar graphs according to Euler number of regions in planar graphs is given by $f = e - v + 2$. Now we extend Eulers result to all simple graphs. I will prove that the number of regions in any simple connected graph is equal to

$$f = e - v + 2 + \sum_{j=1}^{r-1} j \sum_{i=2}^{r} C_i, \qquad r \in N$$

The minimum number of regions in any complete graph is

$$f = \frac{1}{4}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right]\left[\frac{n-3}{2}\right] + \frac{n^2-3n+4}{2}$$

Where [ ] represents greatest integer function, and $n$ is the number of vertices of graph.

**Key Words**: Planar graph, simple graph, non-planar graph, complete graph, regions of a graph, crossing number.

**AMS(2010)**: 05C25.

## §1. Introduction

A planar graph is one that can be drawn on a two-dimensional plane such that no two edges cross. A cubic graph is one in which all vertices have degree three. A three connected graph is one that cannot be disconnected by removal of two vertices. A graph is said to be bipartite

[1]Received January 4, 2015, Accepted June 10, 2015.

if the vertices can be colored using exactly two colors such that no two adjacent vertices have the same color. [1] pp 16-20 A plane representation of a graph divides the plane into regions also called windows, faces or meshes. A window is characterized by the set of edges or the set of vertices forming its boundary. Note that windows is not defined in a non planer graph or even in a planer graph not embedded in a plane. Thus a window is a property of the specific plane representation of a graph. The window of a graph may be finite or infinite. The portion of a plane lying outside a graph embedded in a plane is infinite in its extend. Since a planar graph may have different plane representation Euler gives formula for number of windows in a planar graph.[2] pp 88-100.

**Lemma** 1.1([1]) *A graph can be embedded in a surface of a sphere if and only if it can be embedded in a plane.*

**Lemma** 1.2([1]) *A planer graph may be embedded in a plane such that any specified region can be made the infinite region.*

**Lemma** 1.3(Euler theorem, [1], [2]) *A connected planer graph with $n$ vertices and $e$ edges has $e - n + 2$ regions.*

**Lemma** 1.4([2]) *A plane graph is bipartite if and only if each of its faces has an even number of sides.*

**Corollary** 1.5([2]) *In a simple connected planar graph with $f$ regions $n$ vertices and $e$ edges $(e > 2)$ the following inequalities must hold.*

$$e \geq \frac{3}{2}f \quad and \quad e \leq 3n - 6.$$

**Theorem** 1.6([2]) *The spherical embedding of every planar 3-connected graph is unique.*

The crossing number (sometimes denoted as $c(G)$ of a graph $G$ is the smallest number of pair wise crossings of edges among all drawings of $G$ in the plane. In the last decade, there has been significant progress on a true theory of crossing numbers. There are now many theorems on the crossing number of a general graph and the structure of crossing critical graphs, whereas in the past, most results were about the crossing numbers of either individual graphs or the members of special families of graphs. The study of crossing numbers began during the Second World War with Paul Turan. In [4], he tells the story of working in a brickyard and wondering about how to design an efficient rail system from the kilns to the storage yards. For each kiln and each storage yard, there was a track directly connecting them. The problem he Consider was how to lay the rails to reduce the number of crossings, where the cars tended to fall off the tracks, requiring the workers to reload the bricks onto the cars. This is the problem of finding the crossing number of the complete bipartite graph. It is also natural to try to compute the crossing number of the complete graph. To date, there are only conjectures for the crossing numbers of these graphs called Guys conjecture which suggest that crossing number of complete graph $K_n$ is given by $V(K_n) = Z(n)$ [5] [6].

$$Z(n) = \frac{1}{4}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right]\left[\frac{n-3}{2}\right],$$

where [ ] represents greatest integer function which can also be written as

$$Z(n) = \begin{cases} \frac{1}{64}n(n-2)^2(n-4)^2 & \text{... if n is even} \\ \frac{1}{64}(n-1)^2(n-3)^2 & \text{... if n is odd} \end{cases}$$

Guy prove it for $n \leq 10$ in 1972 in 2007 Richter prove it for $n \leq 12$ For any graph $G$, we say that the crossing number $c(G)$ is the minimum number of crossings with which it is possible to draw $G$ in the plane. We note that the edges of $G$ need not be straight line segments, and also that the result is the same whether $G$ is drawn in the plane or on the surface of a sphere. Another invariant of $G$ is the rectilinear crossing number, $c(G)$, which is the minimum number of crossings when $G$ is drawn in the plane in such a way that every edge is a straight line segment. We will find by an example that this is not the same number obtained by drawing $G$ on a sphere with the edges as arcs of great circles. In drawing $G$ in the plane, we may locate its vertices wherever it is most convenient. A plane graph is one which is already drawn in the plane in such a way that no two of its edges intersect. A planar graph is one which can be drawn as a plane graph [9]. In terms of the notation introduced above, a graph $G$ is planar if and only if $c(G) = 0$. The earliest result concerning the drawing of graphs in the plane is due to Fary [7] [10], who showed that any planar graph (without loops or multiple edges) can be drawn in the plane in such a way that every Edge is straight. Thus Farys result may be rephrased: if $c(G) = 0$, then $\bar{c}(G) = 0$. In a drawing, the vertices of the graph are mapped into points of a plane, and the arcs into continuous curves of the plane, no three having a point in common. A minimal drawing does not contain an arc which crosses itself, nor two arcs with more than one point in common. [8][11]In general for a set of $n$ line segments, there can be up to $O(n^2)$ intersection points, since if every segment intersects every other segment, there would be

$$\frac{n(n-1)}{2} = O(n^2)$$

intersection points. To compute them all we require is $O(n^2)$ algorithm.

## §2. Main Result

Before proving the main result we would like to give the detailed purpose of this paper. Euler gives number of regions in planer graphs which is equal to $f = e - v + 2$. But for non planar graphs the number of regions is still unknown. It is obvious that every graph has different representations; there is no particular representation of non planer simple graphs a graph $G(v, e)$ can be represented in different ways. My aim is to find the number of regions in any simple non planer graph in whatever way we draw it. I will prove that number of regions of any simple non planar graph is equal to

$$f = e - v + 2 + \sum_{j=1}^{r-1} j \sum_{i=2}^{r} C_i, \quad r \in N$$

where $\sum C_2$ are the total number of intersection points where two edges have a common point, $\sum C_3$ are the total number of intersection points where three edges have a common point, and so on,$\sum C_r$ are the total number of intersecting points where $r$ edges have a common point.

In whatever way we draw the graph. And the minimum number of regions in a complete graph is equal to

$$f = \frac{1}{4}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right]\left[\frac{n-3}{2}\right] + \frac{n^2-3n+4}{2}, \quad n = \text{number of vertices}$$

This result is depending upon Guys conjecture which is true for all complete graphs $n \leq 12$ therefore my result is true for all complete graphs $n \leq 12$ if conjecture is true for all $n$, then my result is also true for all complete graphs.

**Theorem** 2.1 *The number of regions in any simple graph is given by*

$$f = e - v + 2 + \sum_{j=1}^{r} j \sum_{i=2}^{r} C_i, \quad r \in N$$

*In particular number of regions in any complete graph is given by*

$$f = \frac{1}{4}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right]\left[\frac{n-3}{2}\right] + \frac{n^2-3n+4}{2}$$

*This result of complete graphs is true for all graphs $n \leq 12$ it is true for all $n$ if Guys conjecture is true for all $n$.*

*Proof* Let $G(v,e)$ be a graph contains the finite set of vertices $v$ and finite set of edges $e$. It is obvious that every graph has a planar representation in a certain stage and in that stage according to Euler number of regions are $f = e - v + 2$. Let $n$ edges remaining in the graph by adding a single edge graph becomes non planar that in this stage it has maximum planarity so if we start to add remaining $n$ edges one by one intersecting points occur and number of regions start to increase. Out of remaining $n$ edges let us suppose that there are certain intersecting points where two edges have a common points it is denoted by $C_2$ and total such points can be represented by $\sum C_2$ similarly let us suppose that there are certain intersecting points where three edges have a common points it is denoted by $C_3$ and total such points can be represented by $\sum C_3$ and this process goes on and finally let us suppose that there are certain intersecting points where $r$ edges have a common point it is denoted by $C_r$ and total such points can be represented by $\sum C_r$. It must be kept in mind that graphs cannot be defined uniquely and finitely every graph has different representations. Since my result is true for all representations in whatever way you can represent graph. We first show that if we have finite set of $n$ edges in a plane such that each pair of edges have one common point and no three edges have a common point, number of regions is increased by one by each pair of edges. Let $f_n$ be the number of regions created by finite set of $n$ edges. It is not obvious that every finite set of $n$ edges creates the same number of regions, this follows inductively when we establish a recurrence $f_0$.

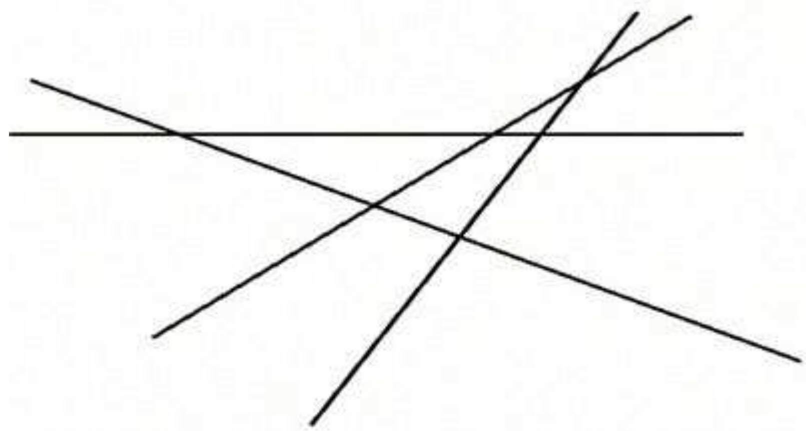

**Fig.1**

We begin with no edges and one region, so $f_0 = 1$. We prove that

$$f_n = f_{n-1} + n$$

if $n \geq 1$. Consider finite set of $n$ edges, with $n \geq 1$ and let $L$ be one of these edges. The other edges form a finite set of $n-1$ edges. We argue that adding $L$ increases the number of regions by $n$. The intersection of $L$ with the other edges partition $L$ into $n$ portions. Each of these portions cuts a region into two. Thus adding $L$ increases the number of regions by $n$ .since this holds for all finite set of edges we have

$$f_n = f_{n-1} + n$$

if $n \geq 1$. This determines a unique sequence starting with $f_0 = 1$, and hence every finite set of edges creates same number of regions. Thus it is clear that if two edges have a common point number of region is increased by one we represent it by $C_2$ and total number of such intersecting point is denoted by $\sum C_2$, similarly if three edges have a common point number of regions is increased by two and we denote it by $C_3$ and total number of such intersection points is denoted by $\sum C_3$ and number of regions are $2\sum C_3$ this process goes on and finally if $r$ lines have a common point number of regions is increased by $r$ and it is denoted by $C_r$ and total number of such intersection points are denoted by $\sum C_r$ and number of regions increased by $(r-1)\sum C_r$ it must be noted that every graph has different representations any number of intersection points can occur. Thus we conclude that number of regions in any simple graph is given by

$$\begin{aligned} f &= e - v + 2 + \text{sum of all intersecting points where two edges have common point} \\ &\quad +2(\text{sum of all intersecting points where three edges have common point}) \\ &\quad +3(\text{sum of all intersecting points where four edges have common point}) \\ &\quad + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\quad +(r-1)(\text{sum of all intersecting points where r edges have common point}), \end{aligned}$$

written to be

$$f = e - v + 2 + \sum C_2 + 2\sum C_3 + 3\sum C_4 + ... + (r-1)\sum C_r,$$

which can be expressed as

$$f = e - v + 2 + \sum_{j=1}^{r-1} j \sum_{i=2}^{r} C_i, \quad r \in N$$

It should be noted that Figures 2-4 below illustrate above result.

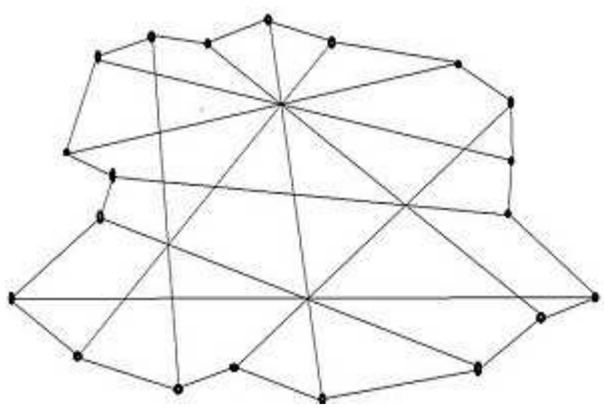

**Figure** 2

Figure 2 has 20 vertices and 30 edges, there are 9 intersection points where two edges have common point, 2 intersection points where three edges have common point,1 intersection points where four edges have common point, 1 intersection points where five edges have common point, and number of regions is 32 we now verify it by above formula.

$$\begin{aligned} f &= e - v + 2 + \sum_{j=1}^{r} j \sum_{i=2}^{r} C_i \\ &= e - v + 2 + \sum C_2 + 2\sum C_3 + 3\sum C_4 + 4\sum C_5. \end{aligned}$$

Substitute above values we get that

$$f = 30 - 20 + 2 + 9 + 4 + 3 + 4 = 32,$$

which verifies that above result.

Figure 3 below has 14 vertices 24 edges 15 intersecting points where two edges have common point. 2 intersection points where three edges have common point,1 intersection points where four edges have common point, and number of regions is 34 we now verify it by above formula.

$$\begin{aligned} f &= e - v + 2 + \sum_{j=1}^{r} j \sum_{i=2}^{r} C_i \\ &= e - v + 2 + \sum C_2 + 2\sum C_3 + 3\sum C_4 = 24 - 14 + 2 + 15 + 4 + 3 = 34, \end{aligned}$$

which again verifies that above result.

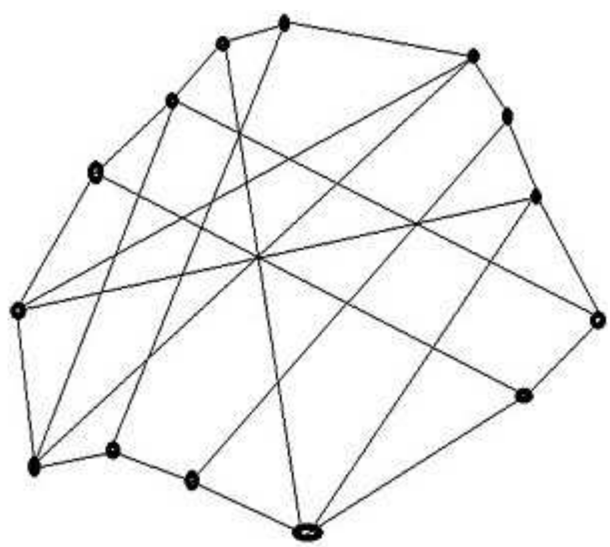

**Figure** 3

Figure 4 below has 6 vertices 11 edges 2 intersecting points where two edges have common point. 1 intersection points where three edges have common point, and number of regions is 11 we now verify it by above formula.

$$\begin{aligned} f &= e - v + 2 + \sum_{j=1}^{r} j \sum_{i=2}^{r} C_i \\ &= e - v + 2 + \sum C_2 + 2\sum C_3 = 11 - 6 + 2 + 2 + 2 = 11, \end{aligned}$$

verifies that above result again.

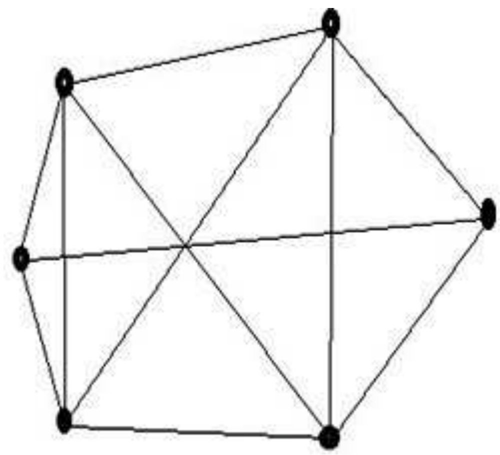

**Figure** 4

Now if the graph is complete with $n$ vertices then the number of edges in it is $\frac{n(n-1)}{2}$ and minimum number of crossing points are given by Guys conjecture that is

$$Z(n) = \frac{1}{4}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right]\left[\frac{n-3}{2}\right]$$

which is true for all $n \leq 12$ thus above result is true for all $n \leq 12$, if Guys conjecture is true, then my result is true for all n. We know that every complete graph has a planar representation in a certain stage. When we start to draw any complete graph we add edge one by one and a stage comes when graph has maximum planarity in that stage number of regions according to Euler is $f = e - v + 2$, when we start to add more edges one by one number of crossing numbers occur but according to definition of crossing numbers two edges have a common point and no three edges have a common point it has been shown that if two edges have a common point number of regions is increased by $\sum C_2$ , thus the number of regions is given by

$$f = e - v + 2 + \sum C_2,$$

where

$$\sum C_2 = \frac{1}{4}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right]\left[\frac{n-3}{2}\right]$$

is the minimum number of crossing points ( Guys conjecture), $e$ the number of edges and $v$ number of vertices.

Let us suppose that graph has $n$ vertices and number of edges is $\frac{n(n-1)}{2}$ substitute these values above we get minimum number of regions in a complete graph is given by

$$\begin{aligned} f &= e - v + 2 + \sum C_2 \\ &= \frac{n(n-1)}{2} - n + 2 + \frac{1}{4}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right]\left[\frac{n-3}{2}\right] \end{aligned}$$

$$= \frac{1}{4}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right]\left[\frac{n-3}{2}\right] + \frac{n^2-3n+4}{2}.$$

That proves the result. □

Figures 5-6 below illustrates this result. The Figure 5 below is the complete graph of six vertices and number of regions are as

$$\begin{aligned} f &= \frac{1}{4}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right]\left[\frac{n-3}{2}\right] + \frac{n^2-3n+4}{2} \\ &= \frac{1}{4}\left[\frac{6}{2}\right]\left[\frac{6-1}{2}\right]\left[\frac{6-2}{2}\right]\left[\frac{6-3}{2}\right] + \frac{6^2-3(6)+4}{2} \\ &= \frac{1}{4} \times 3 \times 2 \times 2 \times 1 + \frac{36-18+4}{2} = 14 \end{aligned}$$

This shows that the above result is true.

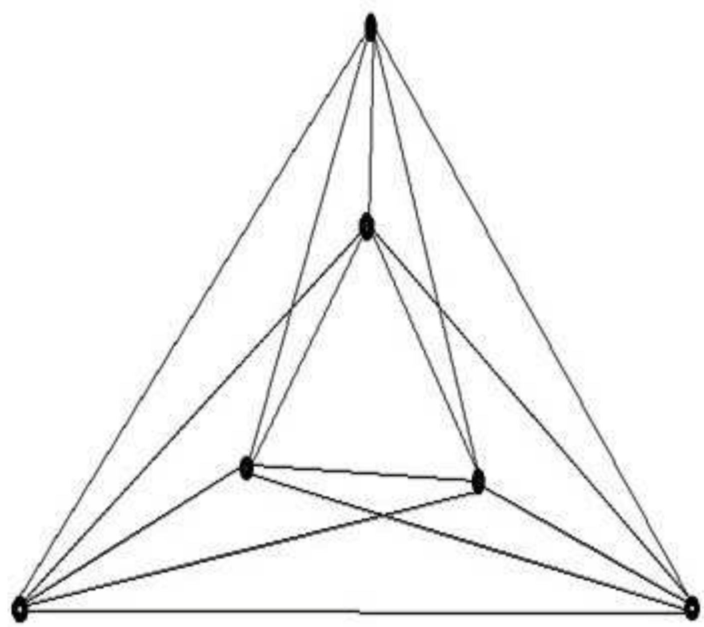

**Figure** 5

Figure 6 below is the complete graph of 5 vertices and number of regions are as

$$\begin{aligned} f &= \frac{1}{4}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right]\left[\frac{n-3}{2}\right] + \frac{n^2-3n+4}{2} \\ &= \frac{1}{4}\left[\frac{5}{2}\right]\left[\frac{5-1}{2}\right]\left[\frac{5-2}{2}\right]\left[\frac{5-3}{2}\right] + \frac{5^2-3(5)+4}{2} \\ &= \frac{1}{4} \times 2 \times 2 \times 1 \times 1 + \frac{25-15+4}{2} = 8 \end{aligned}$$

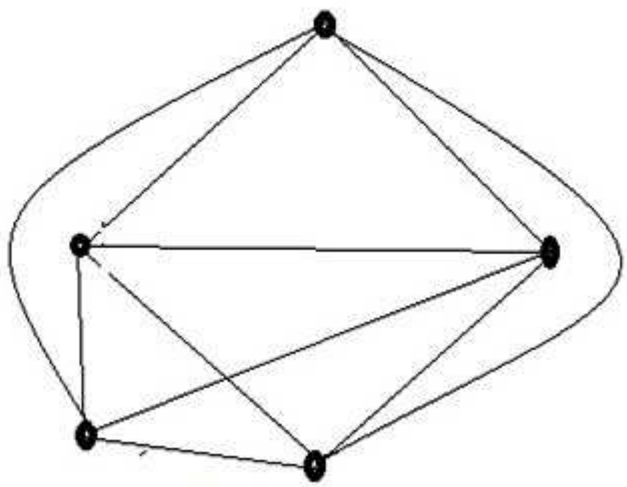

**Figure** 6

**Example** 1 Find the number of regions of a complete graph of 8 vertices with minimum crossings. Find number of regions?

*Solution* Apply the above result we get

$$\begin{aligned} f &= \frac{1}{4}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right]\left[\frac{n-3}{2}\right]+\frac{n^2-3n+4}{2} \\ &= \frac{1}{4}\left[\frac{8}{2}\right]\left[\frac{8-1}{2}\right]\left[\frac{8-2}{2}\right]\left[\frac{8-3}{2}\right]+\frac{8^2-3\times 8+4}{2} \\ &= \frac{1}{4}\times 4\times 3\times 3\times 2+\frac{64-24+4}{2}=40 \end{aligned}$$

**Example** 2 A graph has 10 vertices and 24 edges, there are three points where two edges have a common point, and there is one point where three edges have a common point find the number of regions of a graph?

*Solution* By applying above formula we get

$$\begin{aligned} f &= e-v+2+\sum_{j=1}^{r} j\sum_{i=2}^{r} C_i \\ &= e-v+2+\sum C_2+2\sum C_3 = 24-10+2+3+2=21 \end{aligned}$$

Thus number of regions is 21.

**Acknowledgement**

We are highly thankful to Vivekananda Global University, Jaipur for facilitating us to complete our work smoothly.

**References**

[1] Taro Herju, *Lecture Notes on Graph Theory*, Cambridge University Press, 2011, pages 16-20 and 61-65.

[2] NARSINGH DEO , *Graph Theory with Applications*, Prentice - Hall of India Private limited New Delhi, 11000/2005, pp 88-100 and 96-100.

[3] A.Hertel, *Hamiltonian Cycle in Sparse Graphs*, Masters thesis, University of Toronto, 2004.

[4] P.Turan, A note of welcome, *J. Graph Theory*, 1(1977), 15.

[5] Erdos and R.Guy, crossing number problem, *American Math Month*, 80(1973), 52-57.

[6] R.Guy, A combinatorial problem, *Bull. Malayan Math. Soc.*, 7 (1960),68-72.

[7] I.Fary, On straight line representation of planar graphs, *Acta Univ. Szeged*, 11 (1948), 229-233.

[8] F.Harry and A.Hill, on the number of crossings in a complete graphs, *Proc. Edinburgh Math. Soc.*, 13 (1962) 333-338.

[9] J.W.Moon, On the distribution of crossing in random complete graphs, *SAIM. J. Appl .Math.*, 13(1965) 506-510.

[10] L.Saaty, The minimum number of intersection in complete graph, *Proceedings of the National Academy of Science*, USA, 1964 September, Vol 52(3), 688-690.

[11] R.B.Richter and G.Salazak, Crossing number, A complete chapter, www.math.uwaterloo.ca/brichter/pubs/June2008.pdf.