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Laplacian Energy of Binary Labeled Graph

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Abstract: Let G be a binary labeled graph and $A_l(G) = (l_{ij})$ be its label adjacency matrix. For a vertex v_i, we define label degree as $L_i = \sum\limits_{j=1}^{n} l_{ij}$. In this paper, we define label Laplacian energy $LE_l(G)$. It depends on the underlying graph G and labels of the vertices. We compute label Laplacian spectrum of families of graph. We also obtain some bounds for label Laplacian energy.

Key Words: Label Laplacian matrix, label Laplacian eigenvalues, label Laplacian energy.

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§1. Introduction

For an n-vertex graph G with adjacency matrix A whose eigenvalues are $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$, the energy of the graph G is defined as $E(G) = \sum\limits_{i=1}^{n} |\lambda_i|$. The concept of Energy of graph was introduced by Ivan Gutman, in connection with the π-molecular energy. The matrix $L(G) = D(G) - A(G)$ is the Laplacian matrix of (n, m) graph G. If $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n = 0$ are the eigenvalues of $L(G)$, then the Laplacian energy of G is defined as

$$LE(G) = \sum_{i=1}^{n} |\mu_i - \tfrac{2m}{n}|$$

However, in the last few years, research on graph energy has much intensified, resulting in a very large number of publications which can be found in the literature [4, 5, 6, 7, 9, 16]. In spectral graph theory , the eigenvalues of several matrices like adjacency matrix, Laplacian matrix [8], distance matrix [10] etc. are studied extensively for more than 50 years. Recently minimum covering matrix, color matrix, maximum degree etc are introduced and studied in [1,2,3].

Motivated by this, P.G. Bhat and S. D'Souza have introduced a new matrix $A_l(G)$ called label matrix [14] of a binary labeled graph $G = (V, X)$, whose elements are defined as follows:

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$$l_{ij} = \begin{cases} a, & \text{if } v_iv_j \in X(G) \text{ and } l(v_i) = l(v_j) = 0, \\ b, & \text{if } v_iv_j \in X(G) \text{ and } l(v_i) = l(v_j) = 1, \\ c, & \text{if } v_iv_j \in X(G) \text{ and } l(v_i) = 0, l(v_j) = 1 \text{ or vice-versa}, \\ 0, & \text{otherwise.} \end{cases}$$

where a, b, and c are distinct non zero real numbers. The eigenvalues λ_1, λ_2, $\ldots$, λ_n of $A_l(G)$ are said to be label eigenvalues of the graph G and form its label spectrum. The label eigenvalues satisfy the following simple relations:

$$\sum_{i=1}^{n} \lambda_i = 0 \quad and \quad \sum_{i=1}^{n} \lambda_i^2 = 2Q \tag{1.1}$$

where

$$Q = n_1a^2 + n_2b^2 + n_3c^2 \tag{1.2}$$

Where n_1, n_2 and n_3 denotes number of edges with $(0,0)$, $(1,1)$ and $(0,1)$ as end vertex labels respectively.

The *label degree* of the vertex v_i, denoted by L_i, is given by $L_i = \sum\limits_{i=1}^{n} l_{ij}$. A Graph G is said to be *k-label regular* if $L_i = k$ for all i. The *label Laplacian matrix* of a binary labeled graph G is defined as

$$L_l(G) = Diag(L_i) - A_l(G)$$

where $Diag(L_i)$ denotes the diagonal matrix of the label degrees. Since $L_l(G)$ is real symmetric, all its eigenvalues μ_i, $i = 1, 2, \ldots, n$, are real and can be labeled as $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n$. These form the *label Laplacian spectrum* of G. Several results on Laplacian of Graph G are reported in the Literature ([6, 11, 12, 13, 16]).

This paper is organized as follows. In section 2, we establish relationship between λ_i and μ_i and some general results on Laplacian label eigenvalues μ_i. In section 3, lower bound and upper bounds for $LE_l(G)$ are obtained. In the last section label Laplacian spectrum is derived for family of graphs.

§2. Label Laplacian Energy

The following Lemma 2.1 shows the similarities between the spectra of label matrix and label Laplacian matrix. For a labeled graph, let $P_A(x)$ and $P_L(x)$ denote the label and label Laplacian characteristic polynomials respectively.

Lemma 2.1 *If* $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *is the label spectrum of k-label regular graph G, then* $\{k - \lambda_n, k - \lambda_{n-1}, \ldots, k - \lambda_1\}$ *is the label Laplacian spectrum of G.*

Proof The label Laplacian characteristic polynomial for k-label regular graph G is given by

$$P_L(x) = \det(L_l(G) - xI) = (-1)^n \det(A_l(G) - (k - x)I) = (-1)^n P_A(k - x) \tag{2.1}$$

Thus, if $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ is the label spectrum of k-label regular graph G, then from equation (2.1), it follows that $k - \lambda_n \geqslant k - \lambda_{n-1} \geqslant \ldots \geqslant k - \lambda_1$ is the label Laplacian spectrum of G.□

We first introduce the auxiliary eigenvalues γ_i, defined as

$$\gamma_i = \mu_i - \tfrac{1}{n} \sum_{j=1}^{n} L_j$$

Lemma 2.2 *If $\mu_1, \mu_2, \cdots, \mu_n$ are the label Laplacian eigenvalues of $L_l(G)$, then*

$$\sum_{i=1}^{n} \mu_i^2 = 2Q + \sum_{i=1}^{n} L_i^2.$$

Proof We have

$$\begin{aligned} \sum_{i=1}^{n} \mu_i^2 &= trace \ \ (L_l(G))^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} l_{ij} l_{ji} = 2\sum_{i<j} l_{ij}^2 + \sum_{i=1}^{n} l_{ii}^2 \\ &= 2[n_1(a)^2 + n_2(b)^2 + n_3(c)^2] + \sum_{i=1}^{n} L_i^2 = 2Q + \sum_{i=1}^{n} L_i^2 \end{aligned}$$ □

Lemma 2.3 *Let G be a binary labeled graph of order n. Then $\sum_{i=1}^{n} \gamma_i = 0$ and $\sum_{i=1}^{n} \gamma_i^2 = 2R$, where*

$$R = Q + \tfrac{1}{2} \sum_{i=1}^{n} \left(L_i - \tfrac{1}{n} \sum_{j=1}^{n} L_j \right)^2$$

and Q is given by equation (1.2).

Proof Note that

$$\sum_{i=1}^{n} \mu_i = tr(L_l(G)) = \sum_{i=1}^{n} L_i \ \text{ and } \ \sum_{i=1}^{n} \mu_i^2 = \sum_{i=1}^{n} L_i^2 + 2Q$$

From which we have,

$$\sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n} \left(\mu_i - \tfrac{1}{n} \sum_{j=1}^{n} L_j \right) = \sum_{i=1}^{n} \mu_i - \sum_{j=1}^{n} L_j = 0$$

and

$$\begin{aligned}
\sum_{i=1}^{n}\gamma_i^2 &= \sum_{i=1}^{n}\left(\mu_i-\frac{1}{n}\sum_{j=1}^{n}L_j\right)^2\\
&= \sum_{i=1}^{n}\mu_i^2-\frac{2}{n}\sum_{i=1}^{n}L_j\sum_{i=1}^{n}\mu_i+\left(\frac{1}{n}\sum_{i=1}^{n}L_j\right)^2\\
&= \sum_{i=1}^{n}L_i^2+2Q-\frac{2}{n}\left(\sum_{i=1}^{n}L_j\right)^2+\left(\frac{1}{n}\sum_{i=1}^{n}L_j\right)^2\\
&= 2Q+\sum_{i=1}^{n}\left(L_i-\frac{1}{n}\sum_{i=1}^{n}L_j\right)^2=2R \qquad \square
\end{aligned}$$

Definition 2.1 *Let G be a binary labeled graph of order n. Then the label Laplacian energy of G, denoted by $LE_l(G)$, is defined as $\sum\limits_{i=1}^{n}|\gamma_i|$, i.e.*

$$LE_l(G)=\sum_{i=1}^{n}|\mu_i-\tfrac{1}{n}\sum_{j=1}^{n}L_j|$$

In 2006, I.Gutman and B.Zhou defined Laplacian energy $LE(G)$ of a graph G. More on Laplacian energy reader can refer ([8], [15], [17], [18]). In Chemistry, there are situations where chemists use labeled graphs, such as vertices represent two distinct chemical species and the edges represent a particular reaction between two corresponding species. We mention that this paper deals only the mathematical aspects of label Laplacian energy of a graph and it is a new concept in the literature.

Lemma 2.4 *If G is k- label regular, then $LE_l(G)=E_l(G)$.*

If G is k- label regular, then $k=L_i=\frac{1}{n}\sum\limits_{j=1}^{n}L_j$ for $i=1,2,\cdots,n$. Using Lemma 2.1,

$$\gamma_i=\mu_i-\tfrac{1}{n}\sum_{j=1}^{n}L_j=(k-\lambda_{n+1-i})-k=-\lambda_{n+1-i}$$

for $i=1,2,\cdots,n$. Hence, the lemma follows from the definitions of the label energy and label Laplacian energy.

§2. Bounds for the Label Laplacian Energy

Lemma 3.1([17]) *Let $a_1,a_2,\ldots,a_n$ be non-negative numbers. Then*

$$n\left[\frac{1}{n}\sum_{i=1}^{n}a_i-\left(\prod_{i=1}^{n}a_i\right)^{\frac{1}{n}}\right]\leqslant n\sum_{i=1}^{n}a_i-\left(\sum_{i=1}^{n}\sqrt{a_i}\right)^2\leqslant n(n-1)\left[\frac{1}{n}\sum_{i=1}^{n}a_i-\left(\prod_{i=1}^{n}a_i\right)^{\frac{1}{n}}\right].$$

Theorem 3.1 *Let G be a binary labeled graph with n vertices and m edges. Then*

$$\sqrt{2R+n(n-1)\Delta^{\frac{2}{n}}} \leqslant LE_l(G) \leqslant \sqrt{2(n-1)R+n\Delta^{\frac{2}{n}}},$$

where $\Delta = \left|\det\left(L_l(G) - \frac{1}{n}\sum\limits_{j=1}^{n} L_j I\right)\right|$.

Proof Note that

$$\sum_{i=1}^{n}|\gamma_i| = LE_l(G) \quad \text{and} \quad \sum_{i=1}^{n}\gamma_i^2 = 2R,$$

where $R = \left[n_1(a)^2 + n_2(b)^2 + n_3(c)^2\right] + \dfrac{1}{2}\sum\limits_{i=1}^{n}\left(L_i - \frac{1}{n}\sum\limits_{j=1}^{n} L_j\right)^2$.

Using Lemma 3.1, it can be easily checked that Theorem 3.1 is true if $\Delta = 0$. Now we assume that $\Delta \neq 0$. By setting $a_i = \gamma_i^2, \quad i = 1, 2, \ldots, n$, and

$$K = n\left[\frac{1}{n}\sum_{i=1}^{n}\gamma_i^2 - \left(\prod_{i=1}^{n}\gamma_i^2\right)^{\frac{1}{n}}\right] \geqslant 0.$$

From Lemma 3.1, we have

$$K \leqslant n\sum_{i=1}^{n}\gamma_i^2 - \left(\sum_{i=1}^{n}|\gamma_i|\right)^2 \leqslant (n-1)K,$$

which can be further expressed as

$$K \leqslant 2nR - (LE_l(G))^2 \leqslant (n-1)K$$

$$2nR - (n-1)K \leqslant (LE_l(G))^2 \leqslant 2nR - K, \tag{3.1}$$

where

$$K = n\left[\frac{1}{n}\gamma_i^2 - \left(\prod_{i=1}^{n}\gamma_i^2\right)^{\frac{1}{n}}\right] = n\left[\frac{1}{n}2R - \Delta^{\frac{2}{n}}\right] = 2R - n\Delta^{\frac{2}{n}}$$

Substituting in equation (3.1), we obtain

$$\sqrt{2R+n(n-1)\Delta^{\frac{2}{n}}} \leqslant LE_l(G) \leqslant \sqrt{2R(n-1)+n\Delta^{\frac{2}{n}}}. \qquad \square$$

Theorem 3.2 *Let G be a binary labeled graph of order $n \geqslant 2$. Then*

$$2\sqrt{R} \leqslant LE_l(G) \leqslant \sqrt{2nR}.$$

Proof Consider the term

$$\begin{aligned} S &= \sum_{i=1}^{n}\sum_{j=1}^{n}(|\gamma_i|-|\gamma_j|)^2 = 2n\sum_{i=1}^{n}|\gamma_i|^2 - 2\left(\sum_{i=1}^{n}|\gamma_i|\right)\left(\sum_{j=1}^{n}|\gamma_j|\right) \\ &= 2n.2R - 2(LE_l(G))^2 = 4nR - 2(LE_l(G))^2 \end{aligned}$$

Note that $S \geqslant 0$, i.e., $4nR - 2(LE_l(G))^2 \geqslant 0$, which implies $LE_l(G) \leqslant \sqrt{2nR}$. We have $\left(\sum\limits_{i=1}^{n}\gamma_i\right)^2 = 0$ and the fact that $R \geqslant 0$,

$$2R = \sum_{i=1}^{n}\gamma_i^2 = \left(\sum_{i=1}^{n}\gamma_i\right)^2 - 2\sum_{1\leqslant i<j\leqslant n}\gamma_i\gamma_j \leqslant 2\left|\sum_{1\leqslant i<j\leqslant n}\gamma_i\gamma_j\right| \leqslant 2\sum_{1\leqslant i<j\leqslant n}|\gamma_i||\gamma_j| \qquad (3.2)$$

Thus,

$$\begin{aligned} LE_l(G)^2 &= \left(\sum_{i=1}^{n}|\gamma_i|\right)^2 = \sum_{i=1}^{n}|\gamma_i|^2 + 2\sum_{1\leqslant i<j\leqslant n}|\gamma_i||\gamma_j| \\ &\geqslant 2R + 2R = 4R \end{aligned}$$

from Lemma 2.3 and equation (3.1). Hence, $LE_l(G) \geqslant 2\sqrt{R}$. □

Theorem 3.3 *Let G be a labelled graph of order n. Then*

$$LE_l(G) \leqslant \frac{1}{n}\sum_{i=1}^{n}L_i + \sqrt{(n-1)\left[2R - \left(\frac{1}{n}\sum_{i=1}^{n}L_i\right)^2\right]}.$$

Proof We have

$$\gamma_n = 0 - \frac{1}{n}\sum_{i=1}^{n}L_i = \frac{1}{n}\sum_{i=1}^{n}L_i.$$

Consider the non-negative term

$$\begin{aligned} S &= \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(|\gamma_i|-|\gamma_j|)^2 \\ &= 2(n-1)\sum_{i=1}^{n}\gamma_i^2 - 2\left(\sum_{i=1}^{n}|\gamma_i|\right)\left(\sum_{j=1}^{n}|\gamma_j|\right) \\ &= 2(n-1)\left[2R - \left(\frac{1}{n}\sum_{i=1}^{n}L_i\right)^2\right] - 2\left(LE_l(G) - \frac{1}{n}\sum_{i=1}^{n}L_i\right)^2 \geqslant 0. \end{aligned}$$

Hence,

$$LE_l(G) \leqslant \frac{1}{n}\sum_{i=1}^{n} L_i + \sqrt{(n-1)\left[2R - \left(\frac{1}{n}\sum_{i=1}^{n} L_i\right)^2\right]}. \qquad \square$$

§4. Label Laplacian Spectrum of Some Graphs

Theorem 4.1 *For $n \geqslant 2$, the label Laplacian spectrum of complete graph K_n is*

$$\left\{ \begin{array}{cccc} 0 & ma+(n-m)c & (n-m)b+mc & nc \\ 1 & m-1 & n-m-1 & 1 \end{array} \right\}$$

where m vertices are labeled zero, $n-m$ vertices are labeled one and $0 \leqslant m \leqslant n$.

Proof Let $v_1, v_2, \cdots, v_m$ vertices of K_n be labeled zero and $v_{m+1}, v_{m+2}, \cdots, v_n$ be labeled 1. Then the label degree of vertex v_i is $L(v_i) = (m-1)a + (n-m)c$ for $i = 1, 2, \ldots, m$, $L(v_i) = (n-m-1)b + mc$ for $i = m+1, m+2, \ldots, n$,

$$L_l(K_n) = \left[\begin{array}{c|c} [ma+(n-m)c]I_m - aJ_{m\times m} & -cJ_{m\times(n-m)} \\ \hline -cJ_{(n-m)\times m} & [(n-m)b+mc]I_{n-m} - bJ_{(n-m)\times(n-m)} \end{array}\right]$$

Consider

$$\det(\mu I - L_l(K_n)) = \left|\begin{array}{c|c} [\mu - \{ma+(n-m)c\}]I_m + aJ_{m\times m} & cJ_{m\times(n-m)} \\ \hline cJ_{(n-m)\times m} & [\mu - \{(n-m)b+mc\}]I_{n-m} + bJ_{(n-m)\times(n-m)} \end{array}\right|.$$

Step 1 Replacing column C_1 by $C_1' = C_1 + C_2 + \ldots + C_n$, we obtain determinant $\mu \det(B)$.

Step 2 In determinant B, replace the row R_i by $R_i' = R_i - R_{i-1}$ for $i = 2, 3, \ldots, m, m+2, m+3, \ldots, n$, we obtain

$$\det(B) = (\mu - \{(m-1)a + (n-m)c\})^{m-1} (\mu - \{(n-m-1)b + mc\})^{n-m-1} \det(C).$$

Step 3 By changing C_i by $C_i' = C_i + C_{i+1} + \ldots + C_n$ for $i = m+1$ *to* n in determinant C, we get a new determinant D of order $m+1$, i.e.

$$\det(D) = \begin{vmatrix} 1 & a & a & \ldots & a & (n-m)c \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 1 & c & c & \ldots & c & \mu - mc \end{vmatrix}$$

Step 4 By expanding determinant D over the first column , we obtain $\det(D) = \mu - mc + (-1)^{m+2}(-1)^{m+1}(n-m)c = \mu - nc$.

Step 5 By back substitution,

$$\det(\mu I - L_l(G)) = \mu\,(\mu - [ma + (n-m)c])^{m-1}\,(\mu - [(n-m)b + mc])^{n-m-1}\,(\mu - nc).$$

Hence, label Laplacian spectrum of K_n is,

$$\begin{Bmatrix} 0 & ma+(n-m)c & (n-m)b+mc & nc \\ 1 & m-1 & n-m-1 & 1 \end{Bmatrix}$$

□

Corollary 4.1 *For $n \geqslant 2$, the label Laplacian spectrum of $K_n - \{(0,0)\}$ is*

$$\begin{Bmatrix} 0 & (m-2)a+(n-m)c & ma+(n-m)c & (n-m)b+mc & nc \\ 1 & 1 & m-2 & n-m-1 & 1 \end{Bmatrix}$$

where m vertices are labeled zero, $n-m$ vertices are labeled one and $0 \leqslant m \leqslant n$.

Corollary 4.2 *For $n \geqslant 2$, the label Laplacian spectrum of $K_n - \{(1,1)\}$ is*

$$\begin{Bmatrix} 0 & ma+(n-m)c & (n-m-2)b+mc & (n-m)b+mc & nc \\ 1 & m-1 & 1 & n-m-2 & 1 \end{Bmatrix}$$

where m vertices are labeled zero, $n-m$ vertices are labeled one and $0 \leqslant m \leqslant n$.

Theorem 4.2 *The label Laplacian spectrum of star graph S_n is*

$$\begin{Bmatrix} 0 & a & c & \frac{\alpha+\beta}{2} & \frac{\alpha-\beta}{2} \\ 1 & m-2 & n-m-1 & 1 & 1 \end{Bmatrix}$$

where m denotes the number of vertices including the central vertex labeled zero, remaining vertices labeled one, $m \leq n$, $\alpha = ma + (n-m+1)c$ and $\beta = \sqrt{[ma + (n-m+1)c]^2 - 4acn}$.

Proof Let $v_1, v_2, \cdots, v_m$ be labeled as zero and remaining vertices be labeled as one,where v_1 is the central vertex. Then, $L(v_1) = a(m-1) + c(n-m)$ and

$$L(v_i) = \begin{cases} a, & \text{for} \quad i = 2, 3, \cdots, m \\ c, & \text{for} \quad i = m+1, m+2, \cdots, n \end{cases}$$

$$L_l(S_n) = \begin{bmatrix} a(m-1)+c(n-m) & -a & -a & \cdots & -a & -c & -c & \cdots & -c \\ -a & a & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ -a & 0 & a & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -a & 0 & 0 & \cdots & 0 & 0 & 0 & \dots & 0 \\ -a & 0 & 0 & \cdots & a & 0 & 0 & \cdots & 0 \\ -c & 0 & 0 & \cdots & 0 & c & 0 & \cdots & 0 \\ -c & 0 & 0 & \cdots & 0 & 0 & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -c & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & c \end{bmatrix}$$

where rows and columns are denoted by $v_1, v_2, \cdots, v_m, v_{m+1}, v_{m+2}, \cdots, v_n$ for the matrix $L_l(S_n)$. Consider $det(\mu I - L_l(G))$.

Step 1 Replace the column C_1 by $C_1' = C_1+C_2+\cdots+C_n$. Then we see that $det(\mu I - L_l(G))$ is of the form $\mu \det(B)$.

Step 2 In $det(B)$, replace R_i by $R_i' = R_i - R_{i-1}$, $i = 3, 4, \cdots, m, m+2, \cdots, n$. Simplifying we get $\det(B) = (\mu - a)^{m-2}(\mu - c)^{n-m-1}\det(C)$.

Step 3 In $\det(C)$, replace C_i by $C_i' = C_i + C_{i+1} + \cdots + C_n$ for $i = m, m+1, \cdots, n$. Then it reduces to the order $m+1$.

Step 4 In $\det(C)$, Replacing C_i by $C_i' = C_i + C_{i+1} + \cdots + C_m$ for $i = 2, 3, \cdots, (m-1)$, we get

$$\det(C) = \begin{vmatrix} 1 & (m-1)a & (m-2)a & \dots & 2a & a & (n-m)c \\ 1 & (\mu - a) & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & (\mu - c) \end{vmatrix}$$

Expanding over the last column, we get $\det(C) = \mu^2 - [am + (n-m+1)c]\mu + acn$. By back substitution, we obtain

$$\det(\mu I - L_l(G)) = \mu(\mu - a)^{m-2}(\mu - c)^{n-m-1}\left(\mu^2 - [am + (n-m+1)c]\mu + acn\right).$$

Hence, label Laplacian spectrum of S_n is given by

$$\begin{Bmatrix} 0 & a & c & \frac{\alpha+\beta}{2} & \frac{\alpha-\beta}{2} \\ 1 & m-2 & n-m-1 & 1 & 1 \end{Bmatrix}$$

where $\alpha = ma + (n-m+1)c$ and $\beta = \sqrt{[ma+(n-m+1)c]^2 - 4acn}$. □

Corollary 4.3 *The label Laplacian spectrum of star graph S_n is*

$$\begin{Bmatrix} 0 & b & c & \frac{\delta+\gamma}{2} & \frac{\delta-\gamma}{2} \\ 1 & m-2 & n-m-1 & 1 & 1 \end{Bmatrix}$$

where m denotes the number of vertices including the central vertex labeled zero, remaining vertices labeled one, $m \leq n$, $\delta = mb + (n-m+1)c$ and $\gamma = \sqrt{[mb+(n-m+1)c] - 4bcn}$.

Proof Let v_1 be the central vertex . Let $v_1, v_2, \cdots, v_m$ be labeled as one and remaining vertices be labeled as zero. Then , $L(v_1) = b(m-1) + c(n-m)$ and

$$L(v_i) = \begin{cases} b, & \text{for} \quad i = 2, 3, \cdots, m \\ c, & \text{for} \quad i = m+1, m+2, \cdots, n \end{cases}$$

The remaining proof of this corollary is similar to Theorem 4.2. □

Corollary 4.4 *If the vertices of cycle C_{2n} are labeled 0 and 1 alternately, then $LE_l(C_{2n}) = cLE(C_{2n}) = cE(C_{2n})$.*

Corollary 4.5 *If the vertices of path P_n are labeled 0 and 1 alternately, then $LE_l(P_n) = cLE(P_n)$.*

Lemma 4.1([10]) *Let M, N, P, Q be matrices, M invertible and*

$$S = \begin{pmatrix} M & N \\ P & Q \end{pmatrix}$$

Then $\det(S) = \det(M)\det(Q - PM^{-1}N)$. *Furthermore, if M, P are commute, then* $\det(S) = \det(MQ - PN)$.

Theorem 4.3 *The label Laplacian spectrum of complete bipartite graph $K(r,s)$ with $m_1 \leqslant r, m_2 \leqslant s$, the number of zeros in the vertex set of order r, s respectively, is given by*

$$\begin{Bmatrix} 0 & (am_2 + (s-m_2)c) & (cm_2 + (s-m_2)b) & (am_1 + (r-m_1)c) & (cm_1 + (r-m_1)b) \\ 1 & m_1 - 1 & r - m_1 - 1 & m_2 - 1 & s - m_2 - 1 \end{Bmatrix}$$

and the roots of

$$\begin{aligned}&[\mu^3-\mu^2\{(a+c)(m_1+m_2)+(b+c)((r+s)-(m_1+m_2))\}\\&+\mu\{ac(m_1^2+m_2^2)+(c^2+ab)(m_2(s-m_2)+m_1(r-m_1))\\&+(ab+bc+ca)(m_1(s-m_2)+m_1(r-m_1))+bc((r-m_1)\\&+(s-m_2))^2+(b^2+c^2)(r-m_1)(s-m_2)\}-\{(a+c)bc(r-m_1)(s-m_2)(m_1+m_2)\\&+abc\,(m_1(s-m_2)(s+m_1-m_2)+m_2(r-m_1)(r+m_2-m_1))\\&+bc^2(r-m_1)(s-m_2)(r+s-m_1-m_2)\\&+ac(a+c)m_1^2m_2+c(ac+c^2+ab)m_1m_2(s-m_2)+ac(b+c)m_1m_2(r-m_1)\}]=0.\end{aligned}$$

Proof Let the labels of $r+s$ vertices of $K(r,s)$ be $\underbrace{000\cdots0}_{m_1}\underbrace{111\cdots1}_{r-m_1}$ and $\underbrace{000\cdots0}_{m_2}\underbrace{111\cdots1}_{s-m_2}$.

$$L_l(K(r,s))=\left[\begin{array}{c|c}A_{r\times r}&-B_{r\times s}\\\hline -B^T_{s\times r}&C_{s\times s}\end{array}\right]_{(r+s)\mathrm{x}(r+s)}\quad\text{with } B=\left[\begin{array}{c|c}aJ_{m_1\times m_2}&cJ_{m_1\times s-m_2}\\\hline cJ_{r-m_1\times m_2}&bJ_{r-m_1\times s-m_2}\end{array}\right]_{r\mathrm{x}s}$$

Characteristic polynomial of $L_l(K(r,s))$ is

$$\phi(L_l(K(r,s)),\mu)=\left|\begin{array}{c|c}(\mu I-A)_{r\times r}&B_{r\times s}\\\hline B^T_{s\times r}&(\mu I-C)_{s\times s}\end{array}\right|=|\mu I-A||(\mu I-C)-B^TA^{-1}B|\qquad(4.1)$$

by Lemma 4.1. Let us denote the label degree of m_1, $r-m_1$, m_2 and $s-m_2$ vertices as $W=cm_2+(s-m_2)b$, $X=cm_2+(s-m_2)b$, $Y=am_1+(r-m_1)c$ and $Z=cm_1+(r-m_1)b$ respectively. Then $A=Diag[\mu-W,\mu-W,\ldots,\mu-W,\mu-W,\mu-X,\mu-X,\ldots,\mu-X]$, $C=Diag[\mu-Y,\mu-Y,\ldots,\mu-Y,\mu-Y,\mu-Z,\mu-Z,\ldots,\mu-Z]$. Note that

$$B^TA^{-1}B=\left[\begin{array}{c|c}\left\{\frac{m_1a^2}{\mu-W}+\frac{c^2(r-m_1)}{\mu-X}\right\}J_{m_2\times m_2}&\left\{\frac{acm_1}{\mu-W}+\frac{bc(r-m_1)}{\mu-X}\right\}J_{m_2\times s-m_2}\\\hline\left\{\frac{acm_1}{\mu-W}+\frac{bc(r-m_1)}{\mu-X}\right\}J_{s-m_2\times m_2}&\left\{\frac{c^2m_1}{\mu-W}+\frac{b^2(r-m_1)}{\mu-X}\right\}J_{s-m_2\times s-m_2}\end{array}\right]$$

By applying elementary transformations, $\det(C-B^TA^{-1}B)$ reduces to order m_2+1. Hence,

$$|C-B^TA^{-1}B|=(\mu-Y)^{m_2-1}(\mu-Z)^{s-m_2-1}\det(E),\qquad(4.2)$$

where

$$\det(E)=\begin{vmatrix}(\mu-Y)-m_2G&-(m_2-1)G&(m_2-2)G&\ldots&-G&-(s-m_2)H\\0&1&0&\ldots&0&0\\0&0&1&\ldots&0&0\\\vdots&&&\ldots&&\vdots\\-m_2H&-(m_2-1)H&-(m_2-2)H&\ldots&-H&(\mu-Z)-(s-m_2)K\end{vmatrix}$$

and $G = \frac{m_1 a^2}{\mu - W} + \frac{c^2(r-m_1)}{\mu - X}$, $H = \frac{acm_1}{\mu - W} + \frac{bc(r-m_1)}{\mu - X}$ and $K = \frac{c^2 m_1}{\mu - W} + \frac{b^2(r-m_1)}{\mu - X}$.

Now expression (4.1) becomes

$$\begin{aligned}&\phi(L_l(K(r,s),\mu) = (\mu - W)^{m_1-1}(\mu - X)^{r-m_1-1}(\mu - Y)^{m_2-1}(\mu - Z)^{s-m_2-1}\\&\times[(\mu - W)(\mu - X)(\mu - Y)(\mu - Z) - c^2 m_1(s-m_2)(\mu - X)(\mu - Y)\\&-b^2(r-m_1)(s-m_2)(\mu - W)(\mu - Y) - a^2 m_1 m_2(\mu - Z)(\mu - X)\\&-c^2(r-m_1)m_2(\mu - Z)(\mu - W) + (r-m_1)m_1 m_2(s-m_2)(c^4 + a^2b^2 - 2abc^2)].\end{aligned}$$

On further simplification, we obtain

$$\begin{aligned}&\phi(L_l(K(r,s),\mu) = \mu(\mu - W)^{m_1-1}(\mu - X)^{r-m_1-1}(\mu - Y)^{m_2-1}(\mu - Z)^{s-m_2-1}\\&[\mu^3 - \mu^2(X+Y+Z+W) + \mu(WX + XY + YZ + ZW + WY + XZ)\\&-(XYZ + XYW + XWZ + WYZ) - c^2 m_1(s-m_2)(\mu - (X+Y))\\&-b^2(r-m_1)(s-m_2)(\mu - (Y+W)) - a^2 m_1 m_2(\mu - (X+Z))\\&-c^2(r-m_1)m_2(\mu - (W+Z))].\end{aligned}$$

Substituting W,X,Y and Z and reducing the terms we get

$$\left\{\begin{matrix}0 & (am_2 + (s-m_2)c) & (cm_2 + (s-m_2)b) & (am_1 + (r-m_1)c) & (cm_1 + (r-m_1)b)\\ 1 & m_1 - 1 & r - m_1 - 1 & m_2 - 1 & s - m_2 - 1\end{matrix}\right\}$$

and the roots of

$$\begin{aligned}&[\mu^3 - \mu^2\{(a+c)(m_1+m_2) + (b+c)((r+s) - (m_1+m_2))\}\\&+\mu\{ac(m_1^2 + m_2^2) + (c^2 + ab)(m_2(s-m_2) + m_1(r-m_1))\\&+(ab + bc + ca)(m_1(s-m_2) + m_1(r-m_1)) + bc((r-m_1)\\&+(s-m_2))^2 + (b^2 + c^2)(r-m_1)(s-m_2)\} - \{(a+c)bc(r-m_1)(s-m_2)(m_1+m_2)\\&+abc\,(m_1(s-m_2)(s+m_1-m_2) + m_2(r-m_1)(r+m_2-m_1))\\&+bc^2(r-m_1)(s-m_2)(r+s-m_1-m_2)\\&+ac(a+c)m_1^2 m_2 + c(ac + c^2 + ab)m_1 m_2(s-m_2) + ac(b+c)m_1 m_2(r-m_1)\}] = 0.\end{aligned}$$

□

Theorem 4.4 *Let $S(m,n)$ be a double star graph with central vertices labeled zero and the pendent vertices labeled one. Then the characteristic polynomial of label Laplacian matrix of $S(m,n)$ is*

$$\mu(\mu - c)^{m+n-4}\left(\mu^3 - [(m+n)c + 2a]\mu^2 + [ac(m+n) + c^2 mn + 2ac]\mu - [ac^2(m+n)]\right).$$

Proof Let v_m and v_{m+1} be the central vertices of $S(m,n)$ with zero labels. Remaining $m+n-2$ vertices be given label one. Characteristic polynomial of $L_l(S(m,n))$ is

$$|\mu I - L_l(S(m,n))| = \left|\begin{array}{cc|cc} (\mu-c)I_{m-1} & cJ_{m-1\times 1} & O_{m-1\times 1} & O_{m-1\times n-1} \\ cJ_{1\times m-1} & (\mu-(m-1)c-a)I_1 & aI_1 & O_{1\times n-1} \\ \hline O_{1\times m-1} & aI_1 & (\mu-(n-1)c-a)I_1 & cJ_{1\times n-1} \\ O_{n-1\times m-1} & O_{m-1\times 1} & cJ_{n-1\times 1} & (\mu-c)I_{n-1} \end{array}\right|$$

Using elementary transformations, we get

$$|\mu I - L_l(S(m,n))| = \mu(\mu-c)^{m+n-4}\begin{vmatrix} (\mu-mc-a) & a & 0 \\ a-c & (\mu-(n-1)c-a) & (n-1)c \\ -c & c & (\mu-c) \end{vmatrix}$$

Hence, the characteristic polynomial of $S(m,n)$ is

$$\begin{aligned}\phi(L_l(S(m,n)),\mu) &= \mu(\mu-c)^{m+n-4}\left(\mu^3-[(m+n)c+2a]\mu^2+[ac(m+n)\right.\\ &\quad \left.+c^2mn+2ac]\mu-[ac^2(m+n)]\right).\end{aligned}$$

□

Corollary 4.6 *Let $S(m,n)$ be a double star graph with central vertices labeled one and the pendent vertices labeled zero. Then the characteristic polynomial of label Laplacian matrix of $S(m,n)$ is*

$$\mu(\mu-c)^{m+n-4}\left(\mu^3-[(m+n)c+2b]\mu^2+[bc(m+n)+c^2mn+2bc]\mu-[bc^2(m+n)]\right).$$

Definition 4.1 *The crown graph S_n^0 for an integer $n \geqslant 3$ is the graph with vertex set $\{u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n\}$ and edge set $\{u_iv_i : 1 \leqslant i,j \leqslant n, i \neq j\}$.*

Lemma 4.2([10]) *Let*

$$A = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$$

be a 2×2 block symmetric matrix. Then the eigenvalues of A are the eigenvalues of the matrices A_0+A_1 and A_0-A_1.

Theorem 4.5 *The label Laplacian spectrum of crown graph S_n^0 of order $2n$ is*

$$\left\{\begin{array}{ccccccc} 0 & ma+(n-m)c & (n-m)b+mc & nc & (m-2)a+c(n-m) & \frac{\xi+\eta}{2} & \frac{\xi-\eta}{2} \\ 1 & m-1 & n-m-1 & 1 & m-1 & 1 & 1 \end{array}\right\}$$

where $X = 2a(m-1)+2b(n-m-1)+cn$, $Y = 4ab(m-1)(n-m-1)+2bc(n-m-1)(n-m)+2acm(m-1)$, $\xi = X$, $\eta = \sqrt{X^2-4Y}$ and m denotes the number of vertices labelled zero in each vertex set of the crown graph.

Proof Let the labels of n vertices of S_n^0 be $\underbrace{000\ldots0}_{m}\underbrace{111\cdots1}_{n-m}$ in each partite set. Then

$$L_l(S_n^0) = \left[\begin{array}{c|c} A & -B \\ \hline -B & A \end{array}\right]$$

where

$$A = Diag\begin{bmatrix} v_1 & v_2 & \cdots & v_m & v_{m+1} & v_{m+2} & \cdots & v_n \\ W & W & \cdots & W & Z & Z & \cdots & Z \end{bmatrix},$$

W and Z are the label degrees of the m zero label vertices and $n-m$ one label vertices respectively given by $W = a(m-1) + c(n-m)$ and $Z = b(n-m-1) + cm$. Note that

$$B = \left[\begin{array}{c|c} a(J-I)_{m\times m} & cJ_{m\times n-m} \\ \hline cJ_{n-m\times m} & a(J-I)_{n-m\times n-m} \end{array}\right]$$

From Lemma 4.2, the label Laplacian spectrum of $L_l(S_n^0)$ is the union of spectrum of $A+B$ and $A-B$. Observe that $A+B = L_l(K_n)$. Hence, by Theorem 4.2, we obtain

$$Spec_l(A+B) = \left\{\begin{array}{cccc} 0 & ma+(n-m)c & (n-m)b+mc & nc \\ 1 & m-1 & n-m-1 & 1 \end{array}\right\} \tag{4.3}$$

Also, $A-B = A + A_l(K_n)$. Consider $\det(\mu I - (A + A_l(K_n)))$.

Step 1 Replacing R_i by $R_i' = R_i - R_{i-1}$, for $i = 2, 3, \cdots, m, m+2, m+3, \cdots, n$, we obtain

$$\det(\mu I - (A + A_l(K_n))) = (\mu - (a(m-2) + c(n-m)))^{m-1}(\mu - (b(n-m-2) + cm)))^{n-m-1}\det(E).$$

Step 2 Replacing C_i by $C_i' = C_i + C_{i+1} + \ldots + C_m$, $i = 1, 2, \ldots, m-1$ and replacing C_j by $C_j = C_j + C_{j+1} + \ldots + C_n$, $j = 1, 2, \ldots, n-1$, the $\det(E)$ reduces to a determinant of order $m+1$.

$$\det(E) = \begin{vmatrix} \sigma & -a(m-1) & -a(m-2) & \ldots & -a & -c(n-m) \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ -cm & -c(m-1) & -c(m-2) & \ldots & -c & \varsigma \end{vmatrix},$$

where $\sigma = \mu - [2a(m-1) + c(n-m)]$ and $\varsigma = \mu - [2b(n-m-1) + cm]$.

Step 3 Expanding over the first column

$$\begin{aligned} \det(E) &= \mu^2 - \mu[2a(m-1) + 2b(n-m-1) + cm] + 4ab(m-1)(n-m-1) \\ &\quad +2bc(n-m-1)(n-m) + 2cam(m-1). \end{aligned}$$

Step 4 Substituting $\det(E)$ in Step 1,

$$\det(\mu I-(A+A_l(K_n)))=(\mu-(a(m-2)+c(n-m)))^{m-1}$$
$$\times(\mu-(b(n-m-2)+cm)))^{n-m-1}\{\mu^2-\mu[2a(m-1)+2b(n-m-1)+cm]$$
$$+4ab(m-1)(n-m-1)+2bc(n-m-1)(n-m)+2cam(m-1)\}.$$

Hence, label Laplacian spectrum of $A-B$ is

$$Spec_l(A-B)=\begin{pmatrix} a(m-2)+c(n-m) & m-1 \\ b(n-m-2)+cm & n-m-1 \\ \frac{X+\sqrt{X^2-4Y}}{2} & 1 \\ \frac{X-\sqrt{X^2-4Y}}{2} & 1 \end{pmatrix} \tag{4.4}$$

where

$$X = 2a(m-1)+2b(n-m-1)+cn,$$
$$Y = 4ab(m-1)(n-m-1)+2bc(n-m-1)(n-m)+2acm(m-1).$$

The union of expressions (4.3) and (4.4) is the label Laplacian spectrum of S_n^0. □

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