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Generalized Vertex Induced Connected Subsets of a Graph

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Abstract: Let k be a positive integer. A graph $G = (V, E)$ is said to be a Π_k - connected graph if for any given subset S of $V(G)$ with $|S| = k$, the subgraph induced by S is connected. In this paper, we explore some properties of Π_k - connectedness and its minimality conditions with respect to other graph theoretic parameters.

Key Words: Graph, subgraph, Π_k - connected graph, minimal Π_k-connected graph.

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§1. Introduction

In this article, we consider finite, undirected, simple and connected graphs $G = (V, E)$ with vertex set V and edge set E. As such $p =| V |$ and $q =| E |$ denote the number of vertices and edges of a graph G, respectively. In general, we use $\langle X\rangle$ to denote the sub graph induced by the set of vertices $X \subseteq V$. $N(v)$ and $N[v]$ denote the open and closed neighborhoods of a vertex v, respectively. A non-trivial graph G is called connected if any two of its vertices are linked by a path in G. A graph G is called n-connected (for $n \in N$) if $|V(G)| > n$ and $G - X$(the graph that results from removing all vertices in X and all edges incident with these vertices) is connected for any vertex set $X \subseteq V(G)$ with $|X| < n$. The greatest integer n such that G is n - connected is called the connectivity $\kappa(G)$ of G. A cut-edge or cut-vertex of G is an edge or a vertex whose deletion increases the number of components. Unless mentioned otherwise for terminology and notation the reader may refer [1] and [5].

The general problem consists of selecting a set of land parcels for conservation to ensure species availability. This problem is also related to site selection, reserve network design and corridor design. Biologists have highlighted the importance of addressing negative ecological impacts of habitats fragmentation when selecting parcels for conservation. Ways to increase the spatial coherence among the set of parcels selected for conservation have been investigated. Conservation planning via Π_k -connected graph model is an important conservation method, in this model they increase the genetic diversity and allow greater mobility for better response to predation and stochastic events such as fire, as well as long term climate change. This motivated us to study Π_k - connectedness in the following manner:

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For any positive integer k. A graph G is said to be a Π_k - connected graph if for any given subset S of $V(G)$ with $|S| = k$, the subgraph induced by S is connected.

For more detail, we refer [2]-[4], [6]– [10] and [13].

§2. Π_k-Connected Graphs

Proposition 2.1 *For any graph G with $p \geq 1$ vertices is a Π_1 - connected graph.*

Proposition 2.2 *For a given $p \geq 3$ vertices, there exist a Π_3 - connected graph of a graph G.*

Proof Removal of t independent edges from a complete graph on p vertices results into a Π_3 - connected graph, where $0 \leq t \leq \frac{p}{2}$ if p is even and $0 \leq t \leq \frac{p-1}{2}$ if p is odd. □

Proposition 2.3 *Let ξ be the number of edges required to make a totally disconnected graph which is a Π_3 - connected graph and hence*

$$\xi(\overline{K}_p) = \begin{cases} \frac{p^2-2p}{2} & \text{if } p \text{ is even} \\ \frac{p^2-2p+1}{2} & \text{if } p \text{ is odd} \end{cases}$$

For a complete bipartite graph $K_{m,n}$, the number of edges to be added to make it a Π_3 - connected graph is given by $\xi(\overline{K}_m) + \xi(\overline{K}_n)$.

Proposition 2.4 *In general the number of edges required to make complete n- partite graph $K_{x_1,x_2,x_3,\ldots,x_n}$ as a Π_3 - connected graph is given by*

$$\xi(\overline{K}_{x_1}) + \xi(\overline{K}_{x_2}) + \cdots + \xi(\overline{K}_{x_n}).$$

Proposition 2.5 *The complete bipartite graph $K_{m,n}$ is a Π_3 - connected graph if $m = 1, 2$ and $n = 1, 2$.*

Theorem 2.1 *For any graph G with $p \geq 3$ vertices is a Π_3 - connected graph if and only if $deg(v_i) \geq p - 2$ for all $v_i \in V(G)$.*

Proof Let G be a Π_3 - connected graph. Suppose on contrary, there exists v_j such that $deg(v_j) \leq p-3$. Let v_1 and v_2 be any two vertices which are not adjacent to v_j, thus the graph induced by v_1, v_2 and v_j is not connected, which is a contradiction. Hence $deg(v_i) \geq p-2$ for all $v_i \in G$.

Conversely, suppose $deg(v_i) \geq p-2$ for all $v_i \in G$, let v_1, v_2, v_3 be any three vertices. Then v_1 is adjacent to at least one of v_2 and v_3. v_2 is adjacent to at least one of v_1 and v_3. Also v_3 is adjacent to at least one of v_1 and v_2. Therefore G is a Π_3 - connected graph. □

Theorem 2.2 *If a graph G is a Π_k- connected graph, then*

$$\delta(G) \geq \begin{cases} t & \textit{if } \ p = t(k-1)+1 \\ t+1 & \textit{if } \ p = t(k-1)+r+1, 1 \leq r \leq k-2 \end{cases}$$

Proof Let G be a Π_k connected graph with $p = t(k-1)+1$ vertices. Suppose on contrary that $\delta(G) = s < t$. Let v be a vertex with $deg(v) = s$. Now we partition the remaining $p-1$ vertices into t vertex disjoint sets such that each set contains a vertex adjacent to v. Since $s < t$, there exists at least one set N which has no vertex adjacent to v. Then $\langle N \cup \{v\} \rangle$ is a disconnected subgraph on k vertices, which is a contradiction.

Now, let $p-1 = t(k-1)+r$. Suppose on contrary that $\delta(G) = s < t+1$, let v be a vertex such that $deg(v) = s$. Now partition the remaining $p-1$ vertices excluding the vertex v in $t+1$ number of vertex disjoint subsets having t subsets with cardinality $k-1$ and one with cardinality r in such a way that each subset contains exactly one vertex adjacent to v. Since $deg(v) < t+1$, there exist at least one subset having no vertex adjacent to v. If the cardinality of such a subset, say T is $k-1$ then $\langle T \cup \{v\} \rangle$ is a disconnected subgraph on k - vertices, again a contradiction. If the cardinality of such a subset, say D is r, then take a vertex which is adjacent to v from a subset A with cardinality $k-1$ to D and any one vertex from D to A, then $\langle A \cup \{v\} \rangle$ is a disconnected subgraph which is induced by k - vertices, a contradiction. Hence

$$\delta(G) \geq \begin{cases} t & \text{if } \ p-1 = t(k-1) \\ t+1 & \text{if } \ p-1 = t(k-1)+r, 1 \leq r \leq k-2 \end{cases}$$ □

Theorem 2.3 *Let G be a Π_k - connectedHence the result follows. Then G is a Π_{k+1} - connected graph with $2 \leq k \leq p-1$.*

Proof On contrary, suppose G is a Π_k - connected graph but not a Π_{k+1} - connected one. Let S be set of $k+1$ vertices on which the graph induced has more than one component. Clearly a subset T consisting of k vertices on which the graph induced is disconnected which is a subgraph of G, which is a contradiction. Hence G is also a Π_{k+1} - connected graph with $2 \leq k \leq p-1$. □

Theorem 2.4 *A graph G is a Π_k-connected graph for all k, $1 \leq k \leq p$ if and only if G is isomorphic to K_p.*

Proof Let G be a Π_k - connected graph for all k, $1 \leq k \leq p$. Clearly G is isomorphic to a complete graph K_p as G is a Π_2 - connected graph.

On the other hand, let G be a complete graph on p vertices. In K_p, every pair of vertices are adjacent. Hence G is a Π_2 - connected graph. As we have proved, every Π_2 - connected graph is Π_t - connected graph for all t, $3 \leq t \leq p$. Therefore G is a Π_k - connected graph with $1 \leq k \leq p$. □

Theorem 2.5 *For any Square tree T^2 of a tree T with diameter $d(T) \geq 5$ is a Π_{p-1} - connected* *graph.*

Proof Let T be any tree with diameter $d(T) \geq 5$. Consider any four non pendent vertices $v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}$ in T such that v_{i+1} is adjacent to v_{i+2}, v_{i+2} is adjacent to v_{i+3}, v_{i+3} is adjacent to v_{i+4}, then in square of T, v_{i+2} is adjacent to v_{i+1}, v_{i+3}, v_{i+4} and v_{i+3} is adjacent to v_{i+1}, v_{i+2}, v_{i+4}. Removal of v_{i+2} and v_{i+3} disconnects the square and since no vertex is a cut vertex, removal of any one vertex does not disconnect T^2. Hence square of any tree T with diameter $d(T) \geq 5$ is a Π_{p-1} - connected graph. □

Proposition 2.6 *For any integer $m \geq 1$, $K_{1,m}$, $K_{1,m}^2$ are Π_2 - connected graphs.*

Proof Let $K_{1,m}$ be any star. Then the square of a star is a complete graph on $m+1$ vertices and hence a Π_2 - connected graph. □

§3. Minimality Conditions on Π_k-Connected Graphs

A Π_k - connected graph G is said to be a vertex minimal Π_k - connected graph if G is not a Π_{k-1} - connected graph. A vertex minimal Π_k - connected graph G is said to be a partially vertex - edge minimal Π_k - connected graph if $G - e$ is not a Π_k - connected graph for some $e \in E(G)$. A vertex minimal Π_k - connected graph G is said to be a totally vertex - edge minimal Π_k - connected graph if $G - e$ is not a Π_k - connected graph for every $e \in E(G)$. For more details, refer [12].

Proposition 3.1 *For any cycle C_p; $p \geq 4$ vertices, the number of edges to be added to make it a Π_3 - connected graph is given by*

$$q \geq \begin{cases} \frac{p^2-4p}{2} & \text{if } p \text{ is even} \\ \frac{p^2-4p+1}{2} & \text{if } p \text{ is odd} \end{cases}$$

where, the equality holds when the resulting graph is a partially vertex - edge minimal Π_3 - connected graph.

Proof Let C_p; $p \geq 4$ vertices be a cycle. Then we have the following cases.

Case 1. Suppose p is even. In any Π_3 - connected graph, the degree each vertex is at least $p-2$. Hence the number of edges in any Π_3 - connected graph is always greater than or equal to $\frac{p(p-2)}{2}$. Therefore the number of edges to be added to C_p is greater than or equal to $\frac{p^2-2p}{2} - p = \frac{p^2-4p}{2}$.

Case 2. Suppose p is odd. In any Π_3 - connected graph on odd number of vertices, the degree of each of the $p-1$ vertices is at least $p-2$ and the degree of one vertex is $p-1$. Hence the number of edges in any Π_3 - connected graph on odd number of vertices is always greater than or equal to $\frac{(p-1)^2}{2}$. Hence the number of edges to be added to C_p is greater than or equal to $\frac{p^2-2p+1}{2} - p = \frac{p^2-4p+1}{2}$. □

Proposition 3.2 *For any path P_p; $p \geq 4$ vertices, the number of edges to be added to make it a Π_3 - connected graph is given by*

$$q \geq \begin{cases} \frac{p^2-4p}{2}+1 & \text{if } p \text{ is even} \\ \frac{p^2-4p+1}{2}+1 & \text{if } p \text{ is odd} \end{cases}$$

where the equality holds when the resulting graph is a partially vertex - edge minimal Π_3 - connected graph.

Proof The proof follows on the same lines as in the above proposition. □

Theorem 3.1 *A connected graph G is a vertex minimal Π_p - connected graph if and only if it has at least one cut vertex.*

Proof Let a connected graph G be a Π_p - connected graph, that is, G is not a Π_{p-1} - connected graph. There exist a vertex v such that the graph induced by $V(G)-v$ is disconnected. Hence v is cut vertex. Conversely, let G be a connected graph with a cut vertex, say v, therefore the subgraph induced by the vertices $V(G)-v$ is disconnected. Hence the graph G is a vertex minimal Π_p - connected graph. □

Theorem 3.2 *For a given $k = 2l+1,\ l \geq 3$, there exists Π_k - connected graph.*

Proof Let $k = 2l+1,\ l \geq 3$ and G be a Π_3 - connected graph on $k-3$ vertices with $V(G) = \{1, 2, 3, \cdots, k-3\}$ and let G' be a graph with $V(G') = \{1, 2, 3, \cdots, k-3, k-2\}$ obtained by adding a vertex $k-2$ and making it adjacent to all the vertices of G. Now take prism of G', label the vertices of second copy of G' in the prism as $\{f(1), f(2), f(3), \cdots, f(k-3), f(k-2)\}$ such that $f(i)$ is the mirror image of i and remove the edge $(k-2, f(k-2))$ from the prism. In the resulting graph H (say), the subgraph induced by any subset $S \subseteq V(H)/\{k-2,\ f(k-2)\}$ containing $k-1$ vertices is connected. The subgraph induced by $V(G) \cup \{k-2,\ f(k-2)\}$ disconnected on $k-3+1+1 = k-1$ vertices and hence every subgraph induced by k vertices is connected. Hence H is Π_k - connected. □

Observation 3.1 The graph obtained in the above theorem is regular when G is a partially vertex - edge minimal Π_3 - connected graph, which is having even order.

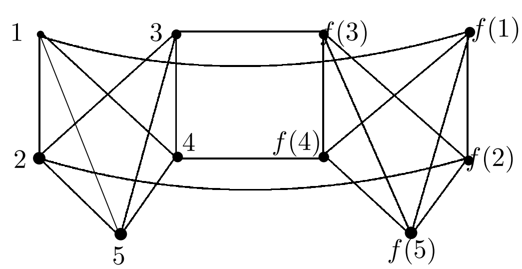

Figure 1. Prism of a Π_3 - connected graph having odd order

For illustration, we construct the above prism of a Π_3 - connected graph having odd order, where prism of a graph G is defined as the cartesian product $G \times K_2$; that is, take two disjoint copies of G and add a matching joining the corresponding vertices in the two copies, [8].

Theorem 3.3 *For any vertex minimal Π_k - connected graph can be embedded in a vertex minimal Π_{k+i} - connected graph, where $i \geq 0$.*

Proof Let G_1 and G_2 be two vertex minimal Π_k and Π_{k+i} - connected graphs. Now we construct a vertex minimal Π_{k+i} - connected graph in which G_1 is an induced vertex minimal Π_k - connected subgraph. Make each vertex of G_1 adjacent to each vertex in G_2 and let the resulting graph be G. Let S be any set of $k+i$ vertices from G. In the following cases we prove the graph induced by S is connected.

Case 1. Suppose $S \cap V(G_1) \cap V(G_2) \neq \varnothing$, then the graph induced by S is connected since each vertex in $S \cap V(G_1)$ is adjacent to every vertex of $S \cap V(G_2)$.

Case 2. Suppose $S \cap V(G_1) \neq \varnothing$ and $S \cap V(G_2) = \varnothing$, then $|S| \geq k$ and $S \subseteq V(G_1)$, the graph induced by S is connected as G_1 is vertex minimal Π_k - connected.

Case 3. Suppose $S \cap V(G_1) = \varnothing$ and $S \cap V(G_2) \neq \varnothing$, then the graph induced by S is connected since S is completely contained in $V(G_2)$ and G_2 is a vertex minimal Π_{k+i} - connected graph. In all the three cases the graph induced by S is connected and G is not Π_{k+i-1} - connected graph since G_2 is a vertex minimal Π_{k+i} - connected graph. Hence the graph G is a vertex minimal Π_{k+i} - connected graph having vertex minimal Π_k - connected graph G_1 as its induced subgraph.

Thus the result follows. □

Theorem 3.4 *A connected graph G is a partially vertex - edge minimal Π_p - connected graph if and only if it has at least one cut edge.*

Proof Let G be a connected graph with a cut edge say $e = uv$. Here u is a cut vertex and also $G - e$ is disconnected, hence G is a partially vertex - edge minimal Π_p - connected graph. Conversely, let G be a partially vertex - edge minimal Π_p - connected graph then $G - e$ is not a Π_p - connected graph. Hence e is a cut edge. □

To prove our next result we make use of the following observations.

Observation 3.2 Removal of $\frac{p}{2}$ independent edges from a complete graph K_p on even number of vertices results into a partially vertex - edge minimal Π_3 - connected graph having $\frac{p(p-2)}{2}$ edges.

Observation 3.3 Removal of $\frac{p-1}{2}$ independent edges from a complete graph on odd number of vertices results into a partially vertex - edge minimal Π_3 - connected graph having $\frac{(p-1)^2}{2}$ edges.

Observation 3.4 Partially vertex - edge minimal Π_3 - connected graph having even order is a regular graph with regularity $p - 2$.

Theorem 3.5 *Let G_1 and G_2 be two partially vertex - edge minimal Π_3 - connected graph. If $V(G_1) = V(G_2)$, then G_1 is isomorphic to G_2.*

Proof Let G_1 and G_2 be partially vertex - edge minimal Π_3 - connected graph having same order. Then, there are following cases:

Case 1. Suppose p is even. As the graphs G_1 and G_2 are partially vertex - edge minimal Π_3 connected graphs, $deg(v) = p - 2$, for all $v \in V(G_1)$ and $deg(v) = p - 2$ for all $v \in V(G_2)$. Clearly G_1 is isomorphic to G_2.

Case 2. Suppose p is odd. As the graphs G_1 and G_2 are partially vertex - edge minimal Π_3 - connected graphs, in the graphs G_1 and G_2 degree of each of $p - 1$ vertices is $p - 2$ and degree of one vertex is $p - 1$. Hence in this case also G_1 is isomorphic to G_2. □

Theorem 3.6 *Any graph G with order p is a totally vertex - edge minimal Π_p - connected graph if and only if G is isomorphic to a tree on p vertices.*

Proof Let G be a graph of order p which is a strongly critical Π_p - connected graph, that is, $G - e$ is not Π_p - connected for all $e \in V(G)$ and G is not a Π_{p-1} - connected graph. The first condition in a totally vertex - edge minimal Π_p - connected graph which implies that every edge in G is a bridge and the second condition in a totally vertex - edge minimal Π_p - connected graph which again implies that every vertex in G is a cut vertex, clearly G is isomorphic to a tree on p vertices.

Conversely, let G is isomorphic to a tree on p vertices. Since every internal vertex is a cut vertex, we have a disconnected induced subgraph on $p - 1$ number of vertices and since every edge is a bridge, $G - e$ is not a Π_p- connected graph for all $e \in V(G)$. Hence G is a totally vertex - edge minimal Π_p - connected graph. □

Theorem 3.7 *Any graph G having even number of vertices is a totally vertex - edge minimal Π_3 - connected graph if and only if $deg(v_i) = p - 2$ for all $v_i \in V(G)$.*

Proof Let G be a totally vertex - edge minimal Π_3 - connected graph on even number of vertices, implies $deg(v) \geq p - 2$ for all $v \in V(G)$ from the Theorem 2.1. Suppose $deg(v) > p - 2$ for some $v \in V(G)$, i.e., $deg(v) = p - 1$. There exist a vertex w adjacent to v such that $deg(w) = p - 1$. The graph $G - vw$ is still a Π_3 - connected graph. Hence G is not a totally vertex - edge minimal Π_3 - connected graph, which is a contradiction. Hence $deg(v) = p - 2$ for all $v \in V(G)$.

Conversely, let G be a graph such that $deg(v) = p - 2$ for all $v \in V(G)$. For every vertex in G there exist an unique non adjacent vertex in G. Hence in $G - e$ there exist two vertices say v and w non adjacent to some vertex say u. The graph induced by these three vertices is disconnected and hence the graph is a totally vertex - edge minimal Π_3 - connected graph. □

Observation 3.5 Any complete graph on $p \geq 3$ vertices is a Π_3 - connected graph but not a partially vertex - edge minimal and totally vertex - edge minimal Π_3 - connected graph.

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References

[1] J. A. Bondy and U. S. R.Murty, *Graph Theory with Applications*, Macmillan, London, 1976.

[2] E.A.Bender, P.J.Cameron, A.M.Odlyzko and L.B.Richmond, Connectedness classes and cycle index, *Probab. Comput.*, 8(1-2)(1999) 31-43.

[3] Carla P. Gomes, Willem-Jan van Hoeve and Ashish Sabharwal, Connections in networks: A hybrid approach, In *CPAIOR*, 5015(2008)303-307.

[4] S.A.Choudum and S.Sivagurunathan, Optimal fault - tolerant networks with a server, *Networks*, 35(2) (2000) 157 - 160.

[5] F.Harary, *Graph theory*, Addison Wesley,Reading Mass., 1969.

[6] F.Harary and J.P.Hayes, Node fault tolerance in graphs, *Networks*, 27 (1996) 19 - 23.

[7] J.P.Hayes, A graph model for fault - tolerant computing systems, *IEEE Trans. on Comp.*, C - 25 (1976) 875 - 884.

[8] G.Sabidussi, Graph derivatives, *Math.Z.*, 76(1961), 385-401.

[9] E.Sampathkumar, *Report on UGC Emeritus Fellowship*, No.F.6-120/2000(SA), 2004.

[10] E.Sampathkumar, Connectivity of a graph-A generalization, *J.Combinatorics, Information and System Sciences*, Vol.9, No. 2, 71-78(1984).

[11] S.Spacapan, Connectivity of Cartesian products of graphs, *Appl. Math. Lett.*, 21(2008), 682 - 685.

[12] H.B.Walikar, B.D.Acharya and E.Sampathkumar, Recent developments in the theory of domination in graphs, *MRI Lecture Notes in Math.*, 1 (1979).

[13] J.C.Williams and Stephanie A. Snyder, Restoring habitat corridors in fragmented landscapes using optimization and percolation models, *Environmental Modeling and Assessment*, 10(3)(2005), 239-250.