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A Characterization of Directed Paths

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Abstract: In this note, the non-trivial connected digraphs D with vertex set $V(D) = \{v_1, v_2, \ldots, v_n\}$ satisfying $\sum\limits_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) = n - 2$ are characterized, where $d^-(v_i)$ and $d^+(v_i)$ be the in-degree and out-degree of vertices of D, respectively.

Key Words: Directed path, directed cycle, directed tree, tournament.

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§1. Introduction

Notations and definitions not introduced here can be found in [1]. For a simple graph G with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$, V.R.Kulli[2] gave the following characterization. A graph G is a non-empty path if and only if it is connected graph with $n \geq 2$ vertices and $\sum\limits_{i=1}^{n} d_i^2 - 4n + 6 = 0$, where d_i is the degree of vertices of G. In this note, we extend the characterization of paths to directed paths, which is needed to characterize the maximal outer planarity property of some digraph operator(digraph valued function).

We need some concepts and notations on directed graphs. A *directed graph*(or just *digraph*) D consists of a finite non-empty set $V(D)$ of elements called *vertices* and a finite set $A(D)$ of ordered pair of distinct vertices called *arcs*. Here, $V(D)$ is the *vertex set* and $A(D)$ is the *arc set* of D. A *directed path* from v_1 to v_n is a collection of distinct vertices $v_1, v_2, v_3, \ldots, v_n$ together with the arcs $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$ considered in the following order: $v_1, v_1v_2, v_2, v_2v_3, \ldots, v_{n-1}v_n, v_n$. A directed path is said to be *non-empty* if it has at least one arc. An *arborescence* is a directed graph in which, for a vertex u called the *root*(i.e., a vertex of in-degree zero) and any other vertex v, there is exactly one directed path from u to v. A *directed cycle* is obtained from a nontrivial directed path on adding an arc from the terminal vertex to the initial vertex of the directed path. A *directed tree* is a directed graph which would be a tree if the directions on the arcs are ignored. The *out-degree* of a vertex v, written $d^+(v)$, is the number of arcs going out

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from v and the *in-degree* of a vertex v, written $d^-(v)$, is the number of arcs coming into v.

The *total degree* of a vertex v, written $td(v)$, is the number of arcs incident with v. We immediately have $td(v) = d^-(v) + d^+(v)$. A *tournament* is a nontrivial complete asymmetric digraph.

§2. Characterization

Theorem 2.1 *A connected digraph D with vertex set $V(D) = \{v_1, v_2, \cdots, v_n\}$, $n \geq 2$ is a non-empty directed path if and only if*

$$\sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) = n - 2. \tag{1}$$

Proof Let D be a directed path with n vertices $v_1, v_2, \cdots, v_n$. Then it is easy to verify that the sum of product of in-degree and out-degree of its vertices is $(n-2)$.

To prove the sufficiency part, we are given that D is connected with n vertices $v_1, v_2, \ldots, v_n$ and equation (1) is satisfied. If $n = 2$, then the only connected digraph is a tournament with two vertices(or a directed path with two vertices) and (1) is trivially verified.

Now, suppose that D is connected with $n \geq 3$ vertices. We consider the following two cases:

(i) The total degree of every vertex of D is at most two;

(ii) There exists at least one vertex of D whose total degree is at least three.

In the former case, since D is connected, it is either a directed path or a directed tree or a directed cycle.

Suppose that D is a directed tree with $n \geq 3$ vertices. Then there exists exactly two vertices of total degree one, and $(n-2)$ vertices of total degree two. Thus, $\sum\limits_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) = \phi < n-2$ violating the condition (1), where ϕ is the number of vertices of D whose in-degree and out-degree are both one. Hence D cannot be a directed tree. On the other hand, if D is a directed cycle with $n \geq 3$ vertices, then $\sum\limits_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) = n > n-2$, again violating the condition (1). Hence D cannot be a directed cycle also. In the latter case, we prove as follows.

Case 1. Suppose that a connected digraph D with $n \geq 3$ vertices has exactly one vertex of total degree three, and remaining vertices of total degree at most two. We consider the following two subcases of Case 1.

Subcase 1. If D is a directed tree, then clearly it has three vertices of total degree one, and $(n-4)$ vertices of total degree two. Thus, $\sum\limits_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) \leq \phi' < n-2$, where ϕ' is the number of vertices of D whose in-degree and out-degree are both at least one.

Subcase 2. If D is cyclic, then it has a vertex of total degree one, and $(n-2)$ vertices of total degree two. Thus, $\sum_{i=1}^{n} d^{-}(v_i) \cdot d^{+}(v_i) = n > n-2$.

Case 2. Finally, consider any connected digraph with n vertices having more than one vertex of total degree at least three. Clearly, such a digraph can be obtained by adding new arcs joining pairs of non-adjacent vertices of a digraph described in Case 1. The addition of new arcs increases the total degree of some vertices and there by the above inequality is preserved in this case also. Therefore in all cases, we arrive at a contradiction if we assume that D has some vertices of total degree at least three. Hence we conclude that D is a non-empty directed path. This completes the proof. $\square$

Remark 2.1 It is known that a directed path is a special case of an arborescence. Hence equation (1) is satisfied for an arborescence whose root vertex has out-degree exactly one. For an example, see Fig.1, Fig.2. It is easy to verify that equation (1) is satisfied for an arborescence showed in Fig.1, but not in Fig.2.

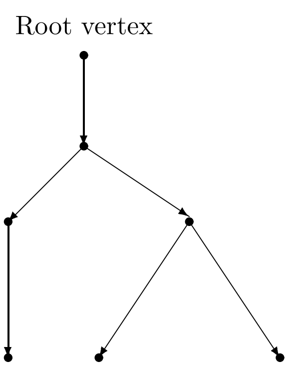

Fig.1

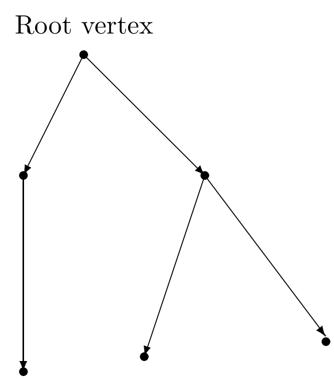

Fig.2

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References

[1] Jorgen Bang-Jensen, Gregory Gutin, *Digraphs Theory, Algorithms and applications*, Springer-Verlag London Limited(2009).

[2] V.R.Kulli, A Characterization of Paths, *The Mathematical Education*, 1975, pp 1-2.