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One Modulo N Gracefullness Of Arbitrary Supersubdivisions of Graphs

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Abstract: A function f is called a graceful labelling of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, 2, \ldots, q\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. A graph G is said to be one modulo N graceful (where N is a positive integer) if there is a function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \ldots, N(q-1), N(q-1)+1\}$in such a way that (i) ϕ is $1-1$ (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$where $\phi^*(uv)$=$|\phi(u) - \phi(v)|$. In this paper we prove that the arbitrary supersubdivisions of paths, disconnected paths, cycles and stars are one modulo N graceful for all positive integers N.

Key Words: Modulo graceful graph, Smarandache modulo graceful graph, supersubdivisions of graphs, paths, disconnected paths, cycles and stars.

AMS(2010): 05C78

§1. Introduction

S.W.Golomb introduced graceful labelling ([1]). The odd gracefulness was introduced by R.B.Gnanajothi in [2]. C.Sekar introduced one modulo three graceful labelling ([8]) recently. V.Ramachandran and C.Sekar ([6]) introduced the concept of one modulo N graceful where N is any positive integer.In the case $N = 2$, the labelling is odd graceful and in the case $N = 1$ the labelling is graceful.We prove that the the arbitrary supersubdivisions of paths, disconnected paths, cycles and stars are one modulo N graceful for all positive integers N.

§2. Main Results

Definition 2.1 *A graph G is said to be one Smarandache modulo N graceful on subgraph $H < G$ with q edges (where N is a positive integer) if there is a function ϕ from the vertex set*

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of G to $\{0, 1, N, (N+1), 2N, (2N+1), \cdots, N(q-1), N(q-1)+1\}$ in such a way that (i) ϕ is $1-1$ (ii) ϕ induces a bijection ϕ^* from the edge set of H to $\{1, N+1, 2N+1, \cdots, N(q-1)+1\}$, and $E(G) \setminus E(h)$ to $\{1, 2, \cdots, |E(G)|-q\}$, where $\phi^*(uv)=|\phi(u)-\phi(v)|$. Particularly, if $H = G$ such a graph is said to be one modulo N graceful graph.

Definition 2.2([9]) *In the complete bipartite graph $K_{2,m}$ we call the part consisting of two vertices, the 2-vertices part of $K_{2,m}$ and the part consisting of m vertices the m-vertices part of $K_{2,m}$.Let G be a graph with p vertices and q edges. A graph H is said to be a supersubdivision of G if H is obtained by replacing every edge e_i of G by the complete bipartite graph $K_{2,m}$ for some positive integer m in such a way that the ends of e_i are merged with the two vertices part of $K_{2,m}$ after removing the edge e_i from G.H is denoted by $SS(G)$.*

Definition 2.3([9]) *A supersubdivision H of a graph G is said to be an arbitrary supersubdivision of the graph G if every edge of G is replaced by an arbitrary $K_{2,m}$ (m may vary for each edge arbitrarily). H is denoted by $ASS(G)$.*

Definition 2.4 *A graph G is said to be connected if any two vertices of G are joined by a path. Otherwise it is called disconnected graph.*

Definition 2.5 *A star S_n with n spokes is given by (V, E) where $V(S_n) = \{v_0, v_1, \ldots, v_n\}$ and $E(S_n) = \{v_0 v_i / i = 1, 2 \ldots, n\}$. v_0 is called the centre of the star.*

Definition 2.6 *A cycle C_n with n points is a graph given by (V, E) where $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$.*

Theorem 2.7 *Arbitrary supersubdivisions of paths are one modulo N graceful for every positive integer N.*

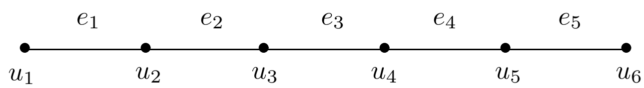
Proof Let P_n be a path with successive vertices $u_1, u_2, u_3, \cdots, u_n$ and let e_i $(1 \le i \le n-1)$ denote the edge u_iu_{i+1} of P_n. Let H be an arbitrary supersubdivision of the path P_n where each edge e_i of P_n is replaced by a complete bipartite graph K_{2,m_i} where m_i is any positive integer,such as those shown in Fig.1 for P_6. We observe that H has $M = 2(m_1+m_2+\cdots+m_{n-1})$ edges.

Define $\phi(u_i) = N(i-1), i = 1, 2, 3, \cdots, n$. For $k = 1, 2, 3, \cdots, m_i$, let

$$\phi(u_{i,i+1}^{(k)}) = \begin{cases} N(M-2k+1)+1 & \text{if } i = 1, \\ N(M-2k+i) - 2N(m_1+m_2+\cdots+m_{i-1})+1 & \text{if } i = 2, 3, \cdots n-1. \end{cases}$$

It is clear from the above labelling that the m_i+2 vertices of K_{2,m_i} have distinct labels and the $2m_i$ edges of K_{2,m_i} also have distinct labels for $1 \le i \le n-1$. Therefore, the vertices of each K_{2,m_i}, $1 \le i \le n-1$ in the arbitrary supersubdivision H of P_n have distinct labels and also the edges of each K_{2,m_i}, $1 \le i \le n-1$ in the arbitrary supersubdivision graph H of P_n have distinct labels. Also the function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \cdots, N(q-1), N(q-1)+1\}$ is in such a way that (i) ϕ is $1-1$, and (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \cdots, N(q-1)+1\}$, where $\phi^*(uv) = |\phi(u)-\phi(v)|$.

Hence H is one modulo N graceful.



Path P_6

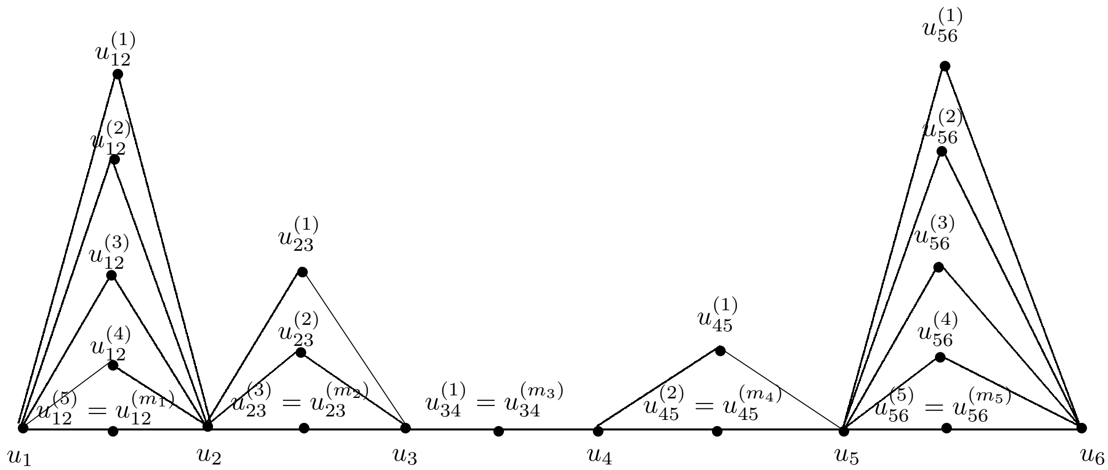


Fig.1 An arbitrary supersubdivision of P_6

Clearly, ϕ defines a one modulo N graceful labelling of arbitrary supersubdivision of the path P_n. □

Example 2.8 An odd graceful labelling of $ASS(P_5)$ is shown in Fig.2.

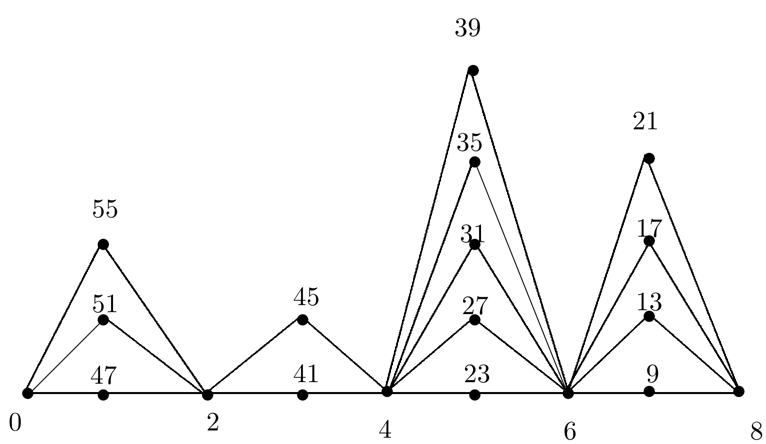


Fig.2

Example 2.9 A graceful labelling of $ASS(P_6)$ is shown in Fig.3.

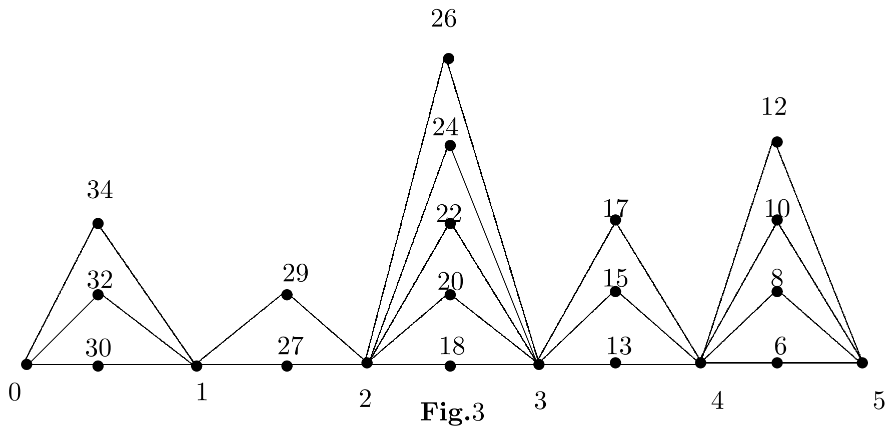


Fig.3

Example 2.10 A one modulo 7 graceful labelling of $ASS(P_6)$ is shown in Fig.4.

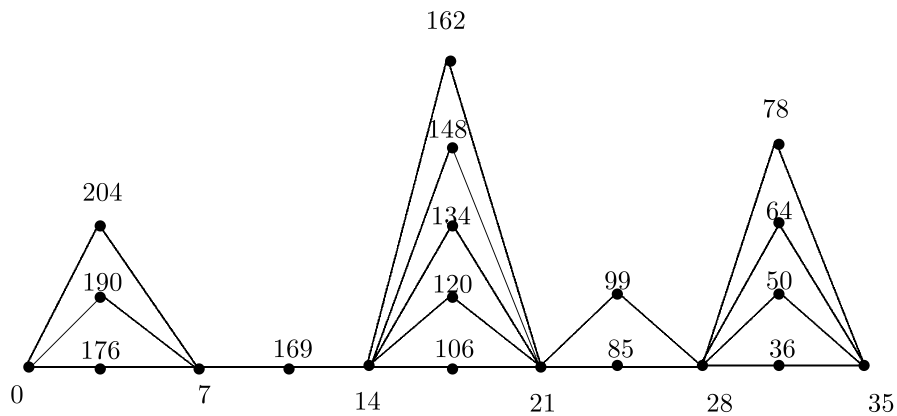


Fig.4

Theorem 2.11 *Arbitrary supersubdivision of disconnecte paths $P_n \cup P_r$ are one modulo N graceful provided the arbitrary supersubdivision is obtained by replacing each edge of G by $K_{2,m}$ with $m \geqslant 2$.*

Proof Let P_n be a path with successive vertices $v_1, v_2, \cdots, v_n$ and let e_i $(1 \leq i \leq n-1)$ denote the edge v_iv_{i+1} of P_n. Let P_r be a path with successive vertices $v_{n+1}, v_{n+2}, \cdots, v_{n+r}$ and let $e_i(n+1 \leq i \leq n+r-1)$ denote the edge v_iv_{i+1}.

Let H be an arbitrary supersubdivision of the disconnected graph $P_n \cup P_r$ where each edge e_i of $P_n \cup P_r$ is replaced by a complete bipartite graph K_{2,m_i} with $m_i \geqslant 2$ for $1 \leq i \leq n-1$ and $n+1 \leq i \leq n+r-1$. We observe that H has $M = 2(m_1+m_2+\cdots+m_{n-1}+m_{n+1}+\cdots+m_{n+r-1})$ edges.

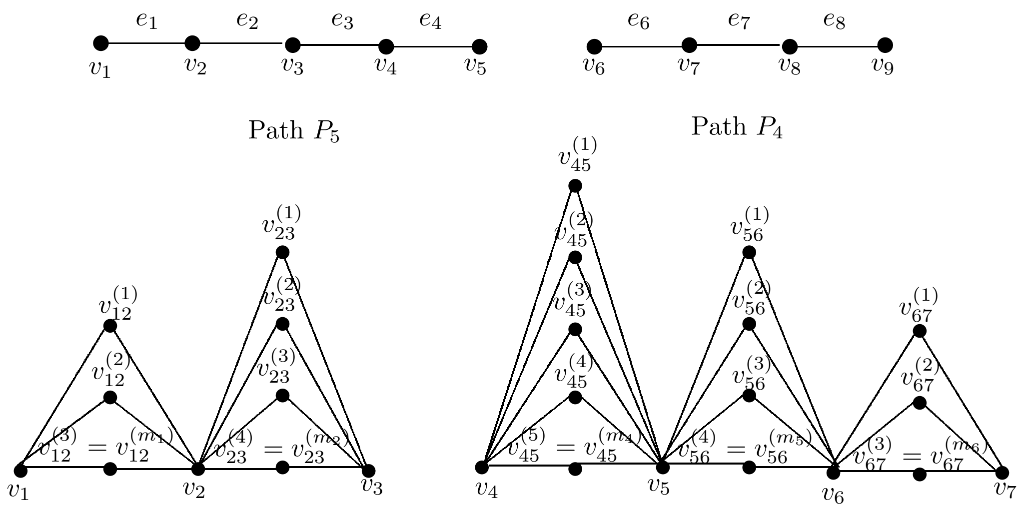


Fig.5 An arbitrary supersubdivision of $P_3 \cup P_4$

Define $\phi(v_i) = N(i-1), i = 1, 2, 3, \cdots, n$, $\phi(v_i) = N(i), i = n+1, n+2, n+3, \cdots, n+r$. For $k = 1, 2, 3, \ldots, m_i$, let

$$\phi(v_{i,i+1}^{(k)}) = \begin{cases} N(M-2k+1)+1 & \text{if } i = 1, \\ N(M-2+i)+1-2N(m_1+m_2+\cdots+m_{i-1}+k-1) & \text{if } i = 2, 3, \cdots n-1, \\ N(M-1+i)+1-2N(m_1+m_2+\cdots+m_{n-1}+k-1) & \text{if } i = n+1, \\ N(M-1+i)+1-2N[(m_1+m_2+\cdots+m_{n-1})+ & \\ (m_{n+1}+\cdots+m_{i-1})+k-1] & \text{if } i = n+2, n+3, \cdots n+r-1. \end{cases}$$

It is clear from the above labelling that the m_i+2 vertices of K_{2,m_i} have distinct labels and the $2m_i$ edges of K_{2,m_i} also have distinct labels for $1 \leq i \leq n-1$ and $n+1 \leq i \leq n+r-1$.Therefore the vertices of each K_{2,m_i}, $1 \leq i \leq n-1$ and $n+1 \leq i \leq n+r-1$ in the arbitrary supersubdivision H of $P_n \cup P_r$ have distinct labels and also the edges of each K_{2,m_i}, $1 \leq i \leq n-1$ and $n+1 \leq i \leq n+r-1$ in the arbitrary supersubdivision graph H of $P_n \cup P_r$ have distinct labels. Also the function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \ldots, N(q-1), N(q-1)+1\}$ is in such a way that (i) ϕ is $1-1$, and (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \cdots, N(q-1)+1\}$, where $\phi^*(uv)$=$|\phi(u) - \phi(v)|$. Hence H is one modulo N graceful.

Clearly, ϕ defines a one modulo N graceful labelling of arbitrary supersubdivisions of disconnected paths $P_n \cup P_r$. □

Example 2.12 An odd graceful labelling of $ASS(P_6 \cup P_3)$ is shown in Fig.6.

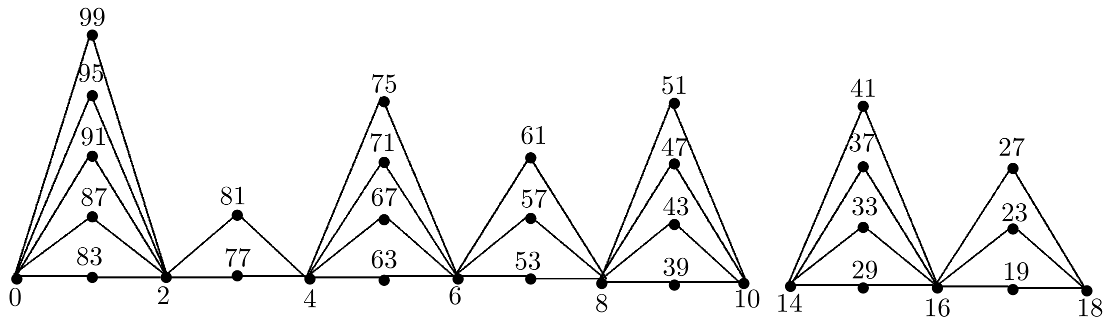


Fig.6

Example 2.13 A graceful labelling of $ASS(P_3 \cup P_4)$ is shown in Fig.7.

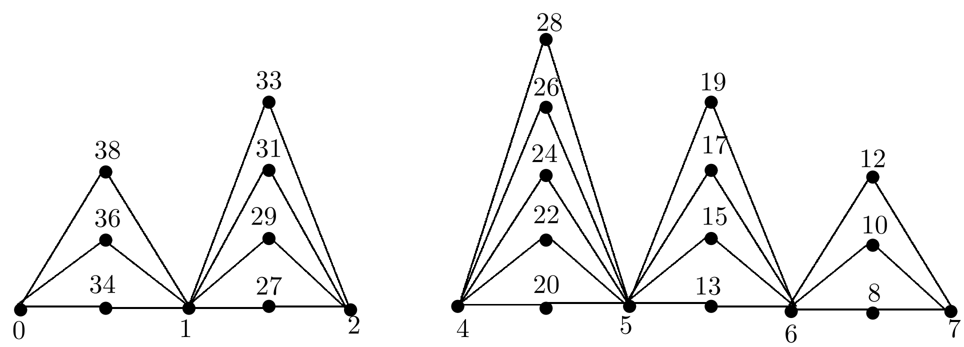


Fig.7

Example 2.14 A one modulo 4 graceful labelling of $ASS(P_4 \cup P_3)$ is shown in Fig.8.

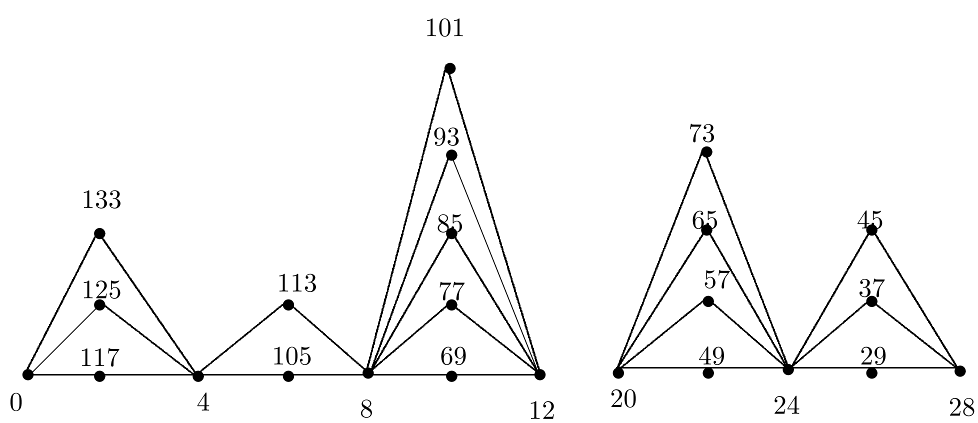


Fig.8

Theorem 2.15 *For any any $n \geq 3$, there exists an arbitrary supersubdivision of C_n which is* *one modulo N graceful for every positive integer N.*

Proof Let C_n be a cycle with consecutive vertices $v_1, v_2, v_3, \cdots, v_n$. Let G be a supersubdivision of a cycle C_n where each edge e_i of C_n is replaced by a complete bipartite graph K_{2,m_i} where m_i is any positive integer for $1 \le i \le n-1$ and $m_n = (n-1)$. It is clear that G has $M = 2(m_1 + m_2 + \cdots + m_n)$ edges. Here the edge $v_{n-1}v_1$ is replaced by $K_{2,n-1}$ for the construction of arbitrary supersubdivision of C_n.

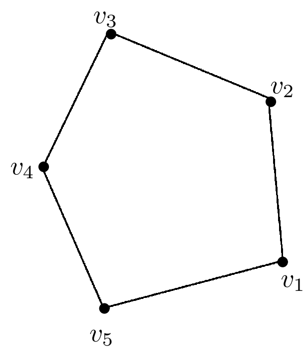


Fig.9 Cycle C_n

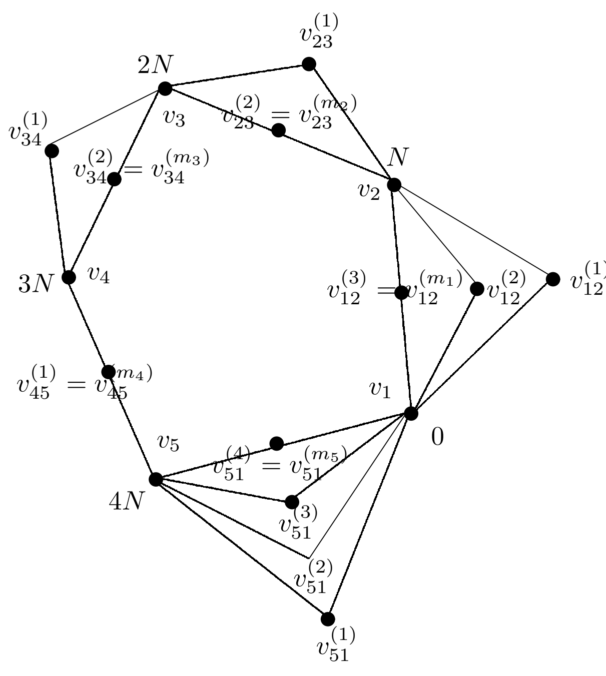


Fig.10 An arbitrary Supersubdivision of C_5

Define $\phi(v_i) = N(i-1), i = 1, 2, 3, \cdots, n$. For $k = 1, 2, 3, \ldots, m_i$, let

$$\phi(v_{i,i+1}^{(k)}) = \begin{cases} N(M-2k+1)+1 & \text{if } i = 1, \\ N(M-2k+i)+1-2N(m_1+m_2+\cdots+m_{i-1}) & \text{if } i = 2, 3, \cdots n-1. \end{cases}$$

and $\phi(v_{n,1}^{(k)}) = N(n-k+m_n-1)+1$.

It is clear from the above labelling that the function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \cdots, N(q-1), N(q-1)+1\}$ is in such a way that (i) ϕ is $1-1$ (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \cdots, N(q-1)+1\}$where $\phi^*(uv)$=$|\phi(u)-\phi(v)|$. Hence, H is one modulo N graceful. Clearly, ϕ defines a modulo N graceful labelling of arbitrary supersubdivision of cycle C_n. □

Example 2.16 An odd graceful labelling of $ASS(C_5)$ is shown in Fig.11.

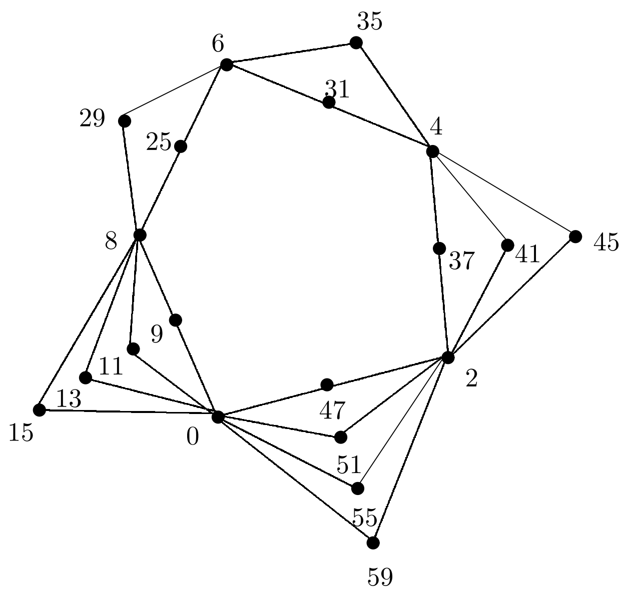


Fig.11

Example 2.17 A graceful labelling of $ASS(C_5)$ is shown in Fig.12.

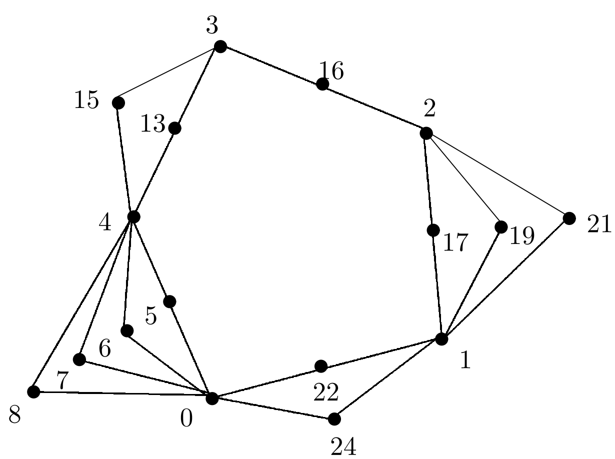


Fig.12

Example 2.18 A one modulo 3 graceful labelling of $ASS(C_4)$ is shown in Fig.13.

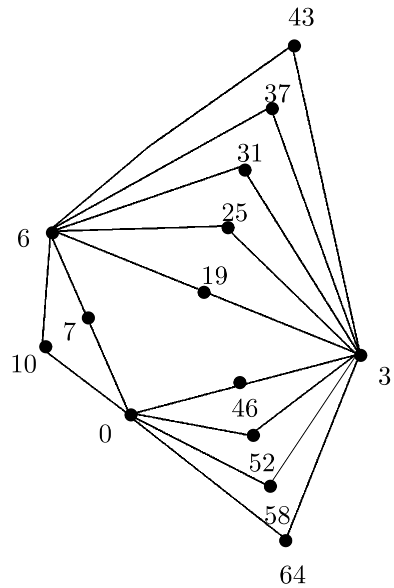


Fig.13

Theorem 2.19 *Arbitrary supersubdivision of any star is one modulo N graceful for every positive integer N.*

Proof The proof is divided into 2 cases.

Case 1 $N = 1$

It has been proved in [4] that arbitrary supersubdivision of any star is graceful.

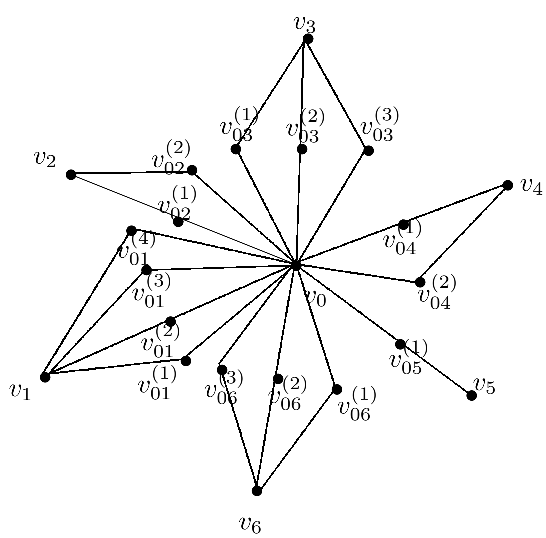


Fig.14 An arbitrary supersubdivision of S_6

Case 2 $N > 1$.

Let S_n be a star with vertices $v_0, v_1, v_2, \cdots, v_n$ and let e_i denote the edge v_0v_i of S_n for $1 \leq$

$i \leq n$. Let H be an arbitrary supersubdivision of S_n. That is for $1 \leq i \leq n$ each edge e_i of S_n is replaced by a complete bipartite graph K_{2,m_i} with m_i is any positive integer for $1 \leq i \leq n-1$ and $m_n = (n-1)$. It is clear that H has $M = 2(m_1+m_2+\cdots+m_n)$ edges. The vertex set and edge set of H are given by $V(H) = \{v_0, v_1, v_2\cdots, v_n, v_{01}^{(1)}, v_{01}^{(2)}\cdots, v_{01}^{(m_1)}, v_{02}^{(1)}, v_{02}^{(2)}, \cdots, v_{02}^{(m_2)}, \cdots, v_{0n}^{(1)}, v_{0n}^{(2)}, \cdots, v_{0n}^{(m_n)}\}$.

Define $\phi : V(H) \rightarrow \{0, 1, 2, \cdots 2\sum_{i=1}^{n} m_i\}$ as follows:

let $\phi(v_0) = 0$. For $k = 1, 2, 3, \ldots, m_i$, let

$$\phi(v_{0i}^{(k)}) = \begin{cases} N(M-k)+1 & \text{if } i = 1, \\ N(M-k)+1-N(m_1+m_2+\cdots+m_{i-1}) & \text{if} i = 2, 3, \cdots n. \end{cases}$$

$$\phi(v_i) = \begin{cases} N(M-m_1) & \text{if } i = 1, \\ NM - N(2m_1+2m_2+\cdots+2m_{i-1}+m_i) & \text{if } i = 2, 3, \cdots n. \end{cases}$$

It is clear from the above labelling that the function ϕ from the vertex set of G to $\{0, 1, N, (N+1), 2N, (2N+1), \cdots, N(q-1), N(q-1)+1\}$ is in such a way that (i) ϕ is $1-1$ (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$where $\phi^*(uv)$=$|\phi(u) - \phi(v)|$. Hence H is one modulo N graceful.

Clearly, ϕ defines a one modulo N graceful labelling of arbitrary supersubdivision of star S_n. □

Example 2.20 A one modulo 5 graceful labelling of $ASS(S_4)$ is shown in Fig.14.

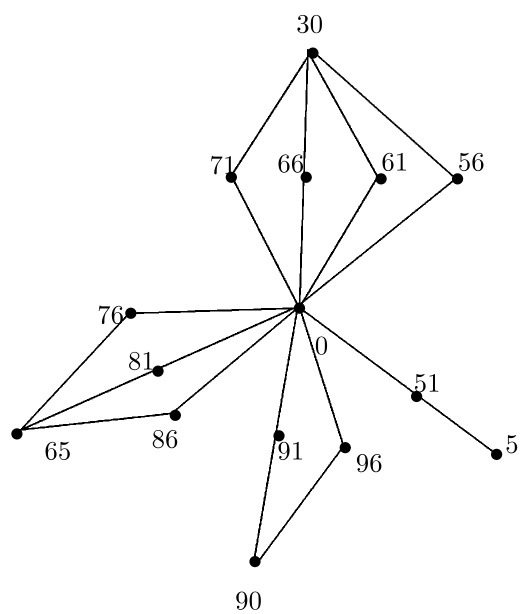


Fig.14

Example 2.21 An odd graceful labelling of $ASS(S_6)$ is shown in Fig.15.

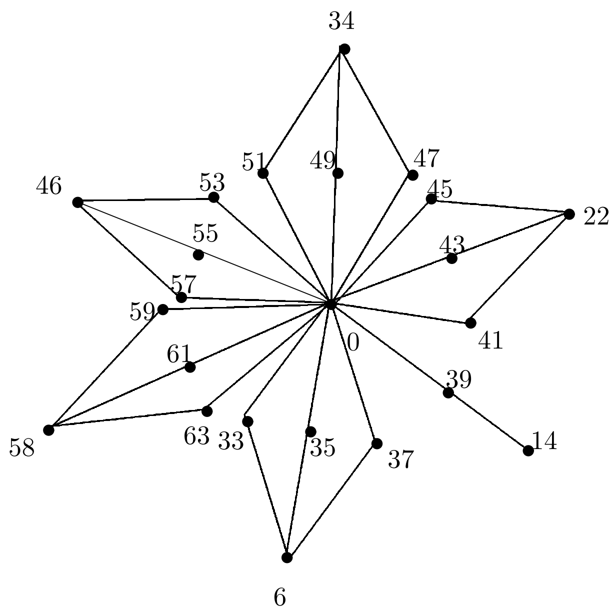


Fig.15

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