

On Pathos Adjacency Cut Vertex Jump Graph of a Tree

Nagesh.H.M

(Department of Science and Humanities, PES Institute of Technology, Bangalore - 560 100, India)

R.Chandrasekhar

(Department of Mathematics, Atria Institute of Technology, Bangalore - 560 024, India)

E-mail: nageshnm@pes.edu, dr.chandri@gmail.com

Abstract: In this paper the concept of pathos adjacency cut vertex jump graph $PJC(T)$ of a tree T is introduced. We also present a characterization of graphs whose pathos adjacency cut vertex jump graphs are planar, outerplanar, minimally non-outerplanar, Eulerian and Hamiltonian.

Key Words: Jump graph $J(G)$, pathos, Smarandache pathos-cut jump graph, crossing number $cr(G)$, outerplanar, minimally non-outerplanar, inner vertex number $i(G)$.

AMS(2010): 05C78

§1. Introduction

For standard terminology and notation in graph theory, not specifically defined in this paper, the reader is referred to Harary [2]. The concept of *pathos* of a graph G was introduced by Harary [3], as a collection of minimum number of edge disjoint open paths whose union is G . The *path number* of a graph G is the number of paths in any pathos. The path number of a tree T is equal to k , where $2k$ is the number of odd degree vertices of T . A *pathos vertex* is a vertex corresponding to a path P in any pathos of T .

The *line graph* of a graph G , written $L(G)$, is the graph whose vertices are the edges of G , with two vertices of $L(G)$ adjacent whenever the corresponding edges of G are adjacent.

The *jump graph* of a graph G ([1]), written $J(G)$, is the graph whose vertices are the edges of G , with two vertices of $J(G)$ adjacent whenever the corresponding edges of G are not adjacent. Clearly, the jump graph $J(G)$ is the complement of the line graph $L(G)$ of G .

The *pathos jump graph* of a tree T [5], written $J_P(T)$, is the graph whose vertices are the edges and paths of pathos of T , with two vertices of $J_P(T)$ adjacent whenever the corresponding edges of T are not adjacent and the edges that lie on the corresponding path P_i of pathos of T .

The *cut vertex jump graph* of a graph G ([6]), written $JC(G)$, is the graph whose vertices are the edges and cut vertices of G , with two vertices of $JC(G)$ adjacent whenever the corresponding edges of G are not adjacent and the edges incident to the cut vertex of G .

¹Received December 11, 2013, Accepted June 5, 2014.

The *edge degree* of an edge pq of a tree T is the sum of the degrees of p and q . A graph G is *planar* if it can be drawn on the plane in such a way that no two of its edges intersect. If all the vertices of a planar graph G lie in the exterior region, then G is said to be an outerplanar.

An outerplanar graph G is *maximal outerplanar* if no edge can be added without losing its outer planarity. For a planar graph G , the *inner vertex number* $i(G)$ is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of G in the plane. A graph G is said to be *minimally non-outerplanar* if the inner vertex number $i(G) = 1$ ([4]).

The least number of edge-crossings of a graph G , among all planar embeddings of G , is called the *crossing number* of G and is denoted by $cr(G)$.

A *wheel graph* W_n is a graph obtained by taking the join of a cycle and a single vertex. The *Dutch windmill graph* $D_3^{(m)}$, also called a *friendship graph*, is the graph obtained by taking m copies of the cycle graph C_3 with a vertex in common, and therefore corresponds to the usual *windmill graph* $W_n^{(m)}$. It is therefore natural to extend the definition to $D_n^{(m)}$, consisting of m copies of C_n .

A *Smarandache pathos-cut jump graph* of a tree T on subtree $T_1 < T$, written $SPJC(T_1)$, is the graph whose vertices are the edges, paths of pathos and cut vertices of T_1 and vertices $V(T) - V(T_1)$, with two vertices of $SPJC(T_1)$ adjacent whenever the corresponding edges of T_1 are not adjacent, edges that lie on the corresponding path P_i of pathos, the edges incident to the cut vertex of T_1 and edges in $E(T) \setminus E(T_1)$. Particularly, if $T_1 = T$, such a graph is called *pathos adjacency cut vertex jump graph* and denoted by $PJC(T)$. Two distinct pathos vertices P_m and P_n are adjacent in $PJC(T)$ whenever the corresponding paths of pathos $P_m(v_i, v_j)$ and $P_n(v_k, v_l)$ have a common vertex, say v_c in T .

Since the pattern of pathos for a tree is not unique, the corresponding pathos adjacency cut vertex jump graph is also not unique.

In the following, Fig.1 shows a tree T and Fig.2 is its corresponding $PJC(T)$.

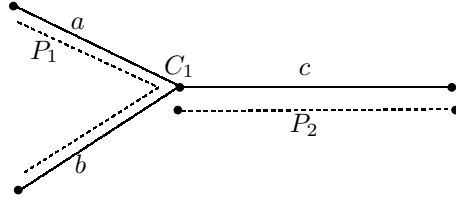


Fig.1 Tree T

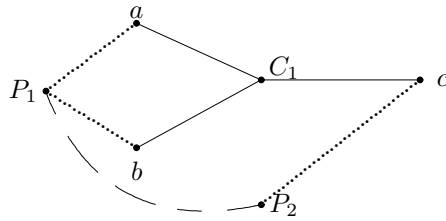


Fig.2 $PJC(T)$

The following existing result is required to prove further results.

Theorem A([2]) *A connected graph G is Eulerian if and only if each vertex in G has even degree.*

Some preliminary results which satisfies for any $PJC(T)$ are listed following.

Remark 1 For any tree T with $n \geq 3$ vertices, $J(T) \subseteq J_P(T)$ and $J(T) \subseteq JC(T) \subseteq PJC(T)$. Here $\subseteq$ is the subgraph notation.

Remark 2 If the edge degree of an edge pq in a tree T is even(odd) and p and q are the cut vertices, then the degree of the corresponding vertex pq in $PJC(T)$ is even.

Remark 3 If the edge degree of a pendant edge pq in T is even(odd), then the degree of the corresponding vertex pq in $PJC(T)$ is even.

Remark 4 If T is a tree with p vertices and q edges, then the number of edges in $J(T)$ is

$$\frac{q(q+1) - \sum_{i=1}^p d_i^2}{2},$$

where d_i is the degree of vertices of T .

Remark 5 Let T be a tree(except star graph). Then the number of edges whose end vertices are the pathos vertices in $PJC(T)$ is $(k-1)$, where k is the path number of T .

Remark 6 If T is a star graph $K_{1,n}$ on $n \geq 3$ vertices, then the number of edges whose end vertices are the pathos vertices in $PJC(T)$ is $\frac{k(k-1)}{2}$, where k is the path number of T . For example, the edge P_1P_2 in Fig.2.

§2. Calculations

In this section, we determine the number of vertices and edges in $PJC(T)$.

Lemma 2.1 *Let T be a tree(except star graph) on p vertices and q edges such that d_i and C_j are the degrees of vertices and cut vertices C of T , respectively. Then $PJC(T)$ has $(q+k+C)$ vertices and*

$$\frac{q(q+1) - \sum_{i=1}^p d_i^2}{2} + \sum_{j=1}^C C_j + q + (k-1)$$

edges, where k is the path number of T .

Proof Let T be a tree(except star graph) on p vertices and q edges. The number vertices of $PJC(T)$ equals the sum of edges, paths of pathos and cut vertices C of T . Hence $PJC(T)$ has $(q+k+C)$ vertices. The number of edges of $PJC(T)$ equals the sum of edges in $J(T)$, degree

of cut vertices, edges that lie on the corresponding path P_i of pathos of T and the number of edges whose end vertices are the pathos vertices. By Remark 4 and 5, the number of edges in $PJC(T)$ is given by

$$\frac{q(q+1) - \sum_{i=1}^p d_i^2}{2} + \sum_{j=1}^C C_j + q + (k-1). \quad \square$$

Lemma 2.2 *If T is a star graph $K_{1,n}$ on $n \geq 3$ vertices and m edges, then $PJC(T)$ has $(m+k+1)$ vertices and $\frac{4m+k(k-1)}{2}$ edges, where k is the path number of T .*

Proof Let T be a star graph $K_{1,n}$ on $n \geq 3$ vertices and m edges. By definition, $PJC(T)$ has $(m+k+1)$ vertices. Also, for a star graph, the number of edges of $PJC(T)$ equals the sum of edges in $J(T)$, i.e., zero, twice the number of edges of T and the number of edges whose end vertices are the pathos vertices. By Remark 6, the number of edges in $PJC(T)$ is given by

$$2m + \frac{k(k-1)}{2} \Rightarrow \frac{4m+k(k-1)}{2}. \quad \square$$

§3. Main Results

Theorem 3.1 *The pathos adjacency cut vertex jump graph $PJC(T)$ of a tree T is planar if and only if the following conditions hold:*

- (i) T is a path P_n on $n=3$ and 4 vertices;
- (ii) T is a star graph $K_{1,n}$, on $n=3,4,5$ and 6 vertices.

Proof (i) Suppose $PJC(T)$ is planar. Assume that T is a path P_n on $n \geq 5$ vertices. Let T be a path P_5 and let the edge set $E(P_5) = \{e_1, e_2, e_3, e_4\}$. Then the jump graph $J(T)$ is the path $P_4 = \{e_3, e_1, e_4, e_2\}$. Since the path number of T is exactly one, $J_P(T)$ is $W_n - e$, where W_n is the join of a cycle with the vertices corresponding to edges of T and a single vertex corresponding to pathos vertex P , and e is an edge between any two vertices corresponding to arcs of T in W_n . Let $\{C_1, C_2, C_3\}$ be the cut vertex set of T . Then the edges joining to $J(T)$ from the corresponding cut vertices gives $PJC(T)$ such that the crossing number of $PJC(T)$ is one, i.e., $cr(PJC(T)) = 1$, a contradiction.

For sufficiency, we consider the following two cases.

Case 1 If T is a path P_3 , then $PJC(T)$ is cycle C_4 , which is planar.

Case 2 Let T be a path P_4 and let $E(P_4) = \{e_1, e_2, e_3\}$. Also, the path number of T is exactly one, i.e., P . Then $J_P(T)$ is $K_{1,3} + e$, where P is the vertex of degree three, and e is an edge between any two vertices corresponding to edges of T in $K_{1,3}$. Let $\{C_1, C_2\}$ be the cut vertex set of T . Then the edges joining to $J(T)$ from the corresponding cut vertices gives $PJC(T) = W_n - \{a, b\}$, where W_n is join of a cycle with the vertices corresponding to edges and cut vertices of T and a single vertex corresponding to pathos vertex P , and $\{a, b\}$ are the edges between pathos vertex P and cut vertices C_1 and C_2 of W_n . Clearly, $cr(PJC(T)) = 0$. Hence $PJC(T)$ is planar.

(ii) Suppose that $PJC(T)$ is planar. Let T be a star graph $K_{1,n}$ on $n \geq 7$ vertices. If T is $K_{1,7}$, then $J(T)$ is a null graph of order seven. Since each edge in T lies on exactly one cut vertex C , $JC(T)$ is a star graph $K_{1,7}$. Furthermore, the path number of T is exactly four. Hence $PJC(T)$ is $D_4^{(4)} - v$, where v is a vertex at distance one from the common vertex in $D_4^{(4)}$. Finally, on embedding $PJC(T)$ in any plane for the adjacency of pathos vertices corresponding to paths of pathos in T , by Remark 6, $cr(PJC(T)) = 1$, a contradiction.

Conversely, suppose that T is a star graph $K_{1,n}$ on $n=3,4,5$ and 6 vertices. For $n=3,4,5$ and 6 vertices, $J(T)$ is a null graph of order n . Since each edge in T lies on exactly one cut vertex C , $JC(T)$ is a star graph of order $n+1$. The path number of T is at most 3. Now, for $n=4$, $PJC(T)$ is the join of two copies of cycle C_4 with a common vertex and for $n=6$, $PJC(T)$ is the join of three copies of cycle C_4 with a common vertex. Next, for $n=3$, $PJC(T)$ is $D_4^{(2)} - v$, and $n=5$, $PJC(T)$ is $D_4^{(3)} - v$, respectively, where v is the vertex at distance one from the common vertex. Finally, on embedding $PJC(T)$ in any plane for the adjacency of pathos vertices corresponding to paths of pathos in T , by Remark 6, $cr(PJC(T)) = 0$. Hence $PJC(T)$ is planar. $\square$

Theorem 3.2 *The pathos adjacency cut vertex jump graph $PJC(T)$ of a tree T is an outerplanar if and only if T is a path P_3 .*

Proof Suppose that $PJC(T)$ is an outerplanar. By Theorem 3.1, $PJC(T)$ is planar if and only if T is a path P_3 and P_4 . Hence it is enough to verify the necessary part of the Theorem for a path P_4 . Assume that T is a path P_4 and the edge set $E(P_4) = e_i$, where $e_i = (v_i, v_{i+1})$, for all $i = 1, 2, 3$. Then the jump graph $J(T)$ is a disconnected graph with two connected components, namely K_1 and K_2 , where $K_1 = e_2$ and $K_2 = (e_1, e_3)$. Let $\{C_1, C_2\}$ be the cut vertex set of T . Hence $JC(T)$ is the cycle $C_5 = \{C_1, e_1, e_3, C_2, e_2, C_1\}$. Furthermore, the path number of T is exactly one. Then the edges joining to $J(T)$ from the corresponding pathos vertex gives $PJC(T)$ such that the inner vertex number of $PJC(T)$ is non-zero, i.e., $i(PJC(T)) \neq 0$, a contradiction.

Conversely, if T is a path P_3 , then $PJC(T)$ is a cycle C_4 , which is an outerplanar. $\square$

Theorem 3.3 *For any tree T , $PJC(T)$ is not maximal outerplanar.*

Proof By Theorem 3.2, $PJC(T)$ is an outerplanar if and only if T is a path P_3 . Moreover, for a path P_3 , $PJC(T)$ is a cycle C_4 , which is not maximal outerplanar, since the addition of an edge between any two vertices of cycle C_4 does not affect the outerplanarity of C_4 . Hence for any tree T , $PJC(T)$ is not maximal outerplanar. $\square$

Theorem 3.4 *The pathos adjacency cut vertex jump graph $PJC(T)$ of a tree T is minimally non-outerplanar if and only if T is (i) a star graph $K_{1,3}$, and (ii) a path P_4 .*

Proof (i) Suppose that $PJC(T)$ is minimally non-outerplanar. If T is a star graph $K_{1,n}$ on $n \geq 7$ vertices, by Theorem 3.1, $PJC(T)$ is nonplanar, a contradiction. Let T be a star graph $K_{1,n}$ on $n=4,5$ and 6 vertices. Now, for $n=4$, $PJC(T)$ is the join of two copies of cycle C_4 with a common vertex and for $n=6$, $PJC(T)$ is the join of three copies of cycle C_4 with a common

vertex. For $n=5$, $PJC(T)$ is $D_4^{(3)} - v$. Finally, on embedding $PJC(T)$ in any plane for the adjacency of pathos vertices corresponding to paths of pathos in T , the inner vertex number of $PJC(T)$ is more than one, i.e., $i(PJC(T)) > 1$, a contradiction.

Conversely, suppose that T is a star graph $K_{1,3}$. Then $J(T)$ is a null graph of order three. Since edge in T lies on exactly one cut vertex C , $JC(T)$ is a star graph $K_{1,3}$. The path number of T is exactly two. By definition, $PJC(T)$ is $D_4^{(2)} - v$. Finally, on embedding $PJC(T)$ in any plane for the adjacency of pathos vertices corresponding to paths of pathos in T , the inner vertex number of $PJC(T)$ is exactly one, i.e., $i(PJC(T)) = 1$. Hence $PJC(T)$ is minimally non-outerplanar.

(ii) Suppose $PJC(T)$ is minimally non-outerplanar. Assume that T is a path on $n \geq 5$ vertices. If T is a path P_5 , by Theorem 3.1, $PJC(T)$ is nonplanar, a contradiction.

Conversely, if T is a path P_4 , by Case 2 of sufficiency part of Theorem 3.1, $PJC(T)$ is $W_n - \{a, b\}$. Clearly, $i(PJC(T)) = 1$. Hence $PJC(T)$ is minimally non-outerplanar. $\square$

Theorem 3.5 *The pathos adjacency cut vertex jump graph $PJC(T)$ of a tree T is Eulerian if and only if the following conditions hold:*

- (i) T is a path P_n on $n = 2i + 1$ vertices, for all $i = 1, 2, \dots$;
- (ii) T is a star graph $K_{1,n}$ on $n = 4j + 2$ vertices, for all $j = 0, 1, 2, \dots$.

Proof (i) Suppose that $PJC(T)$ is Eulerian. If T is a path P_n on $n = 2(i + 1)$ vertices, for all $i = 1, 2, \dots$, then the number of vertices in $J(T)$ is $(2i + 1)$, which is always odd. Since the path number of T is exactly one, by definition, the degree of the corresponding pathos vertex in $PJC(T)$ is odd. By Theorem [A], $PJC(T)$ is non-Eulerian, a contradiction.

For sufficiency, we consider the following two cases.

Case 1 If T is a path P_3 , then $PJC(T)$ is a cycle C_4 , which is Eulerian.

Case 2 Suppose that T is a path P_n on $n = 2i + 1$ vertices, for all $i = 2, 3, \dots$. Let $\{e_1, e_2, \dots, e_{n-1}\}$ be the edge set of T . Then $d(e_1)$ and $d(e_{n-1})$ in $J(T)$ is even and degree of the remaining vertices $e_2, e_3, \dots, e_{n-2}$ is odd. The number of cut vertices in T is $(n - 2)$. By definition, in $JC(T)$ the degree of even and odd degree vertices of $J(T)$ will be incremented by one and two, respectively. Hence the degree of every vertex of $JC(T)$ except cut vertices is odd. Furthermore, the path number of T is exactly one and the corresponding pathos vertex is adjacent to every vertex of $J(T)$. Clearly, every vertex of $PJC(T)$ has an even degree. By Theorem A, $PJC(T)$ is Eulerian.

(ii) Suppose that $PJC(T)$ is Eulerian. We consider the following two cases.

Case 1 Suppose that T is a star graph $K_{1,n}$ on $n = 2j + 1$ vertices, for all $j = 1, 2, \dots$. Then $J(T)$ is a null graph of order n . Since each edge in T lies on exactly one cut vertex C , $JC(T)$ is a star graph $K_{1,n}$ in which $d(C)$ is odd. Moreover, since the degree of a cut vertex C does not change in $PJC(T)$, it is easy to observe that the vertex C remains as an odd degree vertex in $PJC(T)$. By Theorem A, $PJC(T)$ is non-Eulerian, a contradiction.

Case 2 Suppose that T is a star graph $K_{1,n}$ on $n = 4j$ vertices, for all $j = 1, 2, \dots$. Then $J(T)$ is a null graph of order n . Since each edge in T lies on exactly one cut vertex C , $JC(T)$

is a star graph $K_{1,n}$ in which $d(C)$ is even. Since the path number of T is $\lfloor \frac{n}{2} \rfloor$, by definition, $PJC(T)$ is the join of at least two copies of cycle C_4 with a common vertex. Hence for every $v \in PJC(T)$, $d(v)$ is even. Finally, on embedding $PJC(T)$ in any plane for the adjacency of pathos vertices corresponding to paths of pathos in T , there exists at least one pathos vertex, say P_m of odd degree in $PJC(T)$. By Theorem [A], $PJC(T)$ is non-Eulerian, a contradiction.

For sufficiency, we consider the following two cases.

Case 1 For a star graph $K_{1,2}$, T is a path P_3 . Then $PJC(T)$ is a cycle C_4 , which is Eulerian.

Case 2 Suppose that T is a star graph $K_{1,n}$ on $n = 4j + 2$ vertices, for all $j = 1, 2, \dots$. Then the jump graph $J(T)$ is a null graph of order n . Since each edge in T lies on exactly one cut vertex C , $J(T)$ is a star graph $K_{1,n}$ in which $d(C)$ is even. The path number of T is $\lfloor \frac{n}{2} \rfloor$. By definition, $PJC(T)$ is the join of at least three copies of cycle C_4 with a common vertex. Hence for every $v \in PJC(T)$, $d(v)$ is even. Finally, on embedding $PJC(T)$ in any plane for the the adjacency of pathos vertices corresponding to paths of pathos in T , degree of every vertex of $PJC(T)$ is also even. By Theorem A, $PJC(T)$ is Eulerian. $\square$

Theorem 3.6 For any path P_n on $n \geq 3$ vertices, $PJC(T)$ is Hamiltonian.

Proof Suppose that T is a path P_n on $n \geq 3$ vertices with $\{v_1, v_2, \dots, v_n\} \in V(T)$ and $\{e_1, e_2, \dots, e_{n-1}\} \in E(T)$. Let $\{C_1, C_2, \dots, C_{n-2}\}$ be the cut vertex set of T . Also, the path number of T is exactly one and let it be P .

By definition $\{e_1, e_2, \dots, e_{n-1}\} \cup \{C_1, C_2, \dots, C_{n-2}\} \cup P$ form the vertex set in $PJC(T)$. In forming $PJC(T)$, the pathos P becomes a vertex adjacent to every vertex of $\{e_1, e_2, \dots, e_{n-1}\}$ in $J(T)$. Also, the cut vertices C_j , for all $j = 1, 2, \dots, (n-2)$ are adjacent to (e_i, e_{i+1}) for all $i = 1, 2, \dots, (n-1)$ of $J_P(T)$. Clearly, there exist a cycle $(P, e_1, C_1, e_2, C_2, \dots, e_{n-1}, C_{n-2}, e_{n-1}, P)$ containing all the vertices of $PJC(T)$. Hence $PJC(T)$ is Hamiltonian. $\square$

References

- [1] Chartrand H.Hevia and Jarrett M.Schultz, Subgraph distances in graphs defined by edge transfers, *Discrete Math.*, 170, pp-63-79.
- [2] Harary.F., *Graph Theory*, Addison-Wesley, Reading, Mass (1969).
- [3] Harary.F., Converging and packing in graphs-I, *Annals of New York Academy of Science*, 175(1970), pp-198-205.
- [4] Kulli.V.R., On minimally non outerplanar graphs, *Proc.Indian Nat.Sci.Acad.*, 41 (1975), pp-276-280.
- [5] Venkanagouda.M.Goudar, Nagendrappa E. and Murali R., Pathos jump graph of a tree *International Journal of Mathematics Research*, Vol.4, 4(2012), 333-338.
- [6] Venkanagouda.M.Goudar and Nagendrappa G., Cut vertex jump graph, *International Journal of Publications of Applications and Problems in Engineering Research Paper*, Vol.2(2011), 129-133.