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# The Natural Lift Curves and Geodesic Curvatures of the Spherical Indicatrices of The Spacelike-Timelike Bertrand Curve Pair

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**Abstract**: In this paper, when $(\alpha, \alpha^*)$ spacelike-timelike Bertrand curve pair is given, the geodesic curves and the arc-lenghts of the curvatures $(T^*)$, $(N^*)$, $(B^*)$ and the fixed pole curve $(C^*)$ which are generated over the $S_1^2$ Lorentz sphere or the $H_0^2$ hyperbolic sphere by the Frenet vectors $\{T^*,\ N^*,\ B^*\}$ and the unit Darboux vector $C^*$ have been obtained. The condition being the naturel lifts of the spherical indicatrix of the $\alpha^*$ is an integral curve of the geodesic spray has expressed.

**Key Words**: Lorentz space, spacelike-timelike Bertrand curve pair, naturel lift, geodesic spray.

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## §1. Introduction

It is well known that many studies related to the differential geometry of curves have been made. Especially, by establishing relations between the Frenet Frames in mutual points of two curves several theories have been obtained. The best known of these: Firstly, Bertrand Curves discovered by J. Bertrand in 1850 are one of the important and interesting topics of classical special curve theory. A Bertrand curve is defined as a special curve which shares its principal normals with another special curve, called Bertrand mate or Bertrand curve Partner. Secondly, involute–evolute curves discovered by C. Huygens in 1658, who is also known for his work in topics, discovered involutes while trying to build a more accurate clock. The curve $\alpha$ is called evolute of $\alpha^*$ if the tangent vectors are orthogonal at the corresponding points for each $s \in I$: In this case, $\alpha^*$ is called involute of the curve $\alpha$ and the pair of $(\alpha, \alpha^*)$ is called a involute-evolute curve pair. Thirdly, Mannheim curve discovered by A. Mannheim in 1878. Liu and Wang have given a new definition of the curves as known Mannheim curves [8] and [15]. According to the definition given by Liu and Wang, the principal normal vector field of $\alpha$ is linearly dependent on the binormal vector field of $\alpha^*$. Then $\alpha$ is called a Mannheim curve and $\alpha^*$ a Mannheim

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Partner Curve of $\alpha$. The pair $(\alpha, \alpha^*)$ is said to be a Mannheim pair. Furthermore, they showed that the curve is a Mannheim Curve $\alpha$ if and only if its curvature and torsion satisfy the formula $\kappa = \lambda\left(\kappa^2 + \tau^2\right)$, where $\lambda$is a nonzero contant [8], [9] and [15].

In three dimensional Euclidean space $E^3$ and three dimensional Minkowski space $IR_1^3$ the spherical indicatrices of any space curve with the natural lifts and the geodesic sprays of fixed pole curve of any space curve have computed and accordingly, some results related to the curve $\alpha$ for the geodesic spray on the tangent bundle of the natural lifts to be an integral curve have been obtained [5], [11]. On the other hand, the natural lifts and the curvatures of the spherical indicatrices of the Mannheim Pair and the Involute-Evolute curves have been investigated and accordingly, some results related to the curve $\alpha$ for the geodesic spray on the tangent bundle of the natural lifts to be an integral curve have been obtained [2], [4], [5], [7] and [12].

In this paper, arc-lengths and geodesic curvatures of the spherical indicatrix curves with the fixed pole curve of the $(\alpha, \alpha^*)$ spacelike-timelike Bertrand curve pair have been obtained with respect to $IR_1^3$ Lorent space and $S_1^2$ Lorentz sphere or $H_0^2$ Hyperbolic sphere. In addition, the relations among the geodesic curvatures and arc-lengths are given. Finally, the condition being the natural lifts of the spherical indicatrix curves of the $\alpha^*$ timelike curve are an integral curve of the geodesic spray has expressed depending on $\alpha$ spacelike curve.

## §2. Preliminaries

Let Minkowski 3-space $\mathbb{R}_1^3$ be the vector space $\mathbb{R}^3$ equipped with the Lorentzian inner product $g$ given by

$$g(X, X) = x_1^2 + x_2^2 - x_3^2$$

where $X = (x_1, x_2, x_3) \in \mathbb{R}^3$. A vector $X = (x_1, x_2, x_3) \in \mathbb{R}^3$ is said to be timelike if $g(X, X) < 0$, spacelike if $g(X, X) > 0$ and lightlike (or null) if $g(X, X) = 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{R}_1^3$ where $s$ is an arc-length parameter, can locally be timelike, spacelike or null (lightlike), if all of its velocity vectors, $\alpha'(s)$ are respectively timelike, spacelike or null (lightlike) for every $s \in \mathbb{R}$. The norm of a vector $X \in \mathbb{R}_1^3$ is defined by [10]

$$\|X\| = \sqrt{|g(X, X)|}.$$

We denote by $\{T(s),\ N(s),\ B(s)\}$ the moving Frenet frame along the curve $\alpha$. Let $\alpha$ be a timelike curve with curvature $\kappa$ and torsion $\tau$. Let frenet vector fields of $\alpha$ be $\{T,\ N,\ B\}$. In this trihedron, $T$ is a timelike vector field, $N$ and $B$ are spacelike vector fields. Then Frenet formulas are given by ([16])

$$\begin{cases} T' = \kappa N \\ N' = \kappa T - \tau B \\ B' = \tau N. \end{cases} \tag{2.1}$$

Let $\alpha$ be a timelike vector, the frenet vectors $T$ timelike, $N$ and $B$ are spacelike vector, respectively, such that

$$T \times N = -B,\ \mathrm{N} \times B = T,\ B \times T = -N$$

and the frenet instantaneous rotation vector is given by ([14])

$$W = \tau T - \kappa B, \quad \|W\| = \sqrt{\ |\kappa^2 - \tau^2|}.$$

Let $\varphi$ be the angle between $W$ and -$B$ vectors and if $W$ is a spacelike vector, then we can write

$$\begin{cases} \kappa = \|W\| \cosh\varphi, \ \tau = \|W\| \sinh\varphi, \\ \mathrm{C} = \sinh\varphi T - \cosh\varphi B \end{cases} \tag{2.2}$$

and if $W$ is a timelike vector, then we can write

$$\begin{cases} \kappa = \|W\| \sinh\varphi, \ \tau = \|W\| \cosh\varphi, \\ C = \cosh\varphi T - \sinh\varphi B. \end{cases} \tag{2.3}$$

The frenet formulas of spacelike with timelike binormal curve, $\alpha : I \to \mathbb{R}_1^3$ are as followings:

$$\begin{cases} T' = \kappa N \\ N' = \kappa T - \tau B \\ B' = \tau N. \end{cases} \tag{2.4}$$

(see [5] for details), and the frenet instantaneous rotation vector is defined by ([10])

$$W = \tau T - \kappa B, \qquad \|W\| = \sqrt{|\tau^2 - \kappa^2|}.$$

Here, $T \times N = B$ , $N \times B = -T$ , $B \times T = -N$. Let $\varphi$ be the angle between $W$ and $-B$ vectors and if $W$ is taken as spacelike, then the unit Darboux vector can be stated by

$$\begin{cases} \kappa = \|W\| \sinh\varphi, \ \tau = \|W\| \cosh\varphi, \\ C = \cosh\varphi T - \sinh\varphi B. \end{cases} \tag{2.5}$$

and if $W$ is taken as timelike, then it is described by

$$\begin{cases} \kappa = \|W\| \cosh\varphi, \ \tau = \|W\| \sinh\varphi, \\ \mathrm{C} = \sinh\varphi T - \cosh\varphi B \end{cases} \tag{2.6}$$

Let $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ be the vectors in $\mathbb{R}_1^3$. The cross product of $X$ and $Y$ is defined by ([1])

$$X \wedge Y = (x_3 y_2 - x_2 y_3, x_1 y_3 - x_3 y_1, x_1 y_2 - x_2 y_1).$$

The Lorentzian sphere and hyperbolic sphere of radius $r$ and center 0 in $\mathbb{R}_1^3$ are given by

$$S_1^2 = \left\{ X = (x_1, x_2, x_3) \in \mathbb{R}_1^3 \middle| \ g(X, X) = r^2, \ r \in \mathbb{R} \right\}$$

and

$$H_0^2 = \left\{ X = (x_1, x_2, x_3) \in \mathbb{R}_1^3 \middle| \; g(X, X) = -r^2, \; r \in \mathbb{R} \right\}$$

respectively. Let $M$ be a hypersurface in $\mathbb{R}_1^3$. A curve $\alpha : I \to M$ is an integral curve of $X \in \chi(M)$ provided $\alpha' = X_\alpha$; that is

$$\frac{d}{ds}(\alpha(s)) = X(\alpha(s)) \text{ for all } s \in I \qquad ([10]).$$

For any parameterized curve $\alpha : I \to M$, the parameterized curve, $\overline{\alpha} : I \to TM$ given by $\overline{\alpha}(s) = (\alpha(s), \alpha'(s)) = \alpha'(s)\big|_{\alpha(s)}$ is called the natural lift of $\alpha$ on $TM$ ([13]). Thus we can write

$$\frac{d\overline{\alpha}}{ds} = \frac{d}{ds}(\alpha'(s))\big|_{\alpha(s)} = D_{\alpha'(s)}\alpha'(s)$$

where $D$ is standard connection on $\mathbb{R}_1^3$. For $v \in TM$ the smooth vector field $X \in \chi(M)$ defined by

$$X(v) = \varepsilon g(v, S(v))\big|_{\alpha(s)}, \; \varepsilon = g(\xi, \xi) \, [\mathbf{?}]$$

is called the geodesic spray on the manifold $TM$, where $\xi$ is the unit normal vector field of $M$ and $S$ is shape operator of $M$.

Let $\alpha : I \to \mathbb{R}_1^3$ be a spacelike with timelike binormal curve. Let us consider the Frenet frame $\{T, N, B\}$ and the vector $C$. Accordingly, arc-lengths and the geodesic curvatures of the spherical indicatrix curves $(T)$, $(N)$ and $(B)$ with the fixed pole curve $(C)$ with respect to $\mathbb{R}_1^3$, respectively generated by the vectors $T$, $N$ and $B$ with the unit Darboux vector $C$ are as follows:

$$\begin{cases} s_T = \int_0^s |\kappa| ds, & s_N = \int_0^s \|W\| ds, \\ s_B = \int_0^s |\tau| ds, & s_C = \int_0^s |\varphi'| ds. \end{cases} \tag{2.7}$$

if $W$ is a spacelike vector, then we can write

$$\begin{cases} k_T = \dfrac{1}{\sinh \varphi}, & k_N = \sqrt{\left| 1 + (\dfrac{\varphi'}{\|W\|})^2 \right|}, \\ k_B = \dfrac{1}{\cosh \varphi}, & k_C = \sqrt{\left| 1 + (\dfrac{\|W\|}{\varphi'})^2 \right|}. \end{cases} \tag{2.8}$$

if $W$ is a timelike vector, then we have

$$\begin{cases} k_T = \dfrac{1}{\cosh \varphi}, & k_N = \sqrt{\left| 1 - (\dfrac{\varphi'}{\|W\|})^2 \right|}, \\ k_B = \dfrac{1}{\sinh \varphi}, & k_C = \sqrt{\left| -1 + (\dfrac{\|W\|}{\varphi'})^2 \right|} \end{cases} \tag{2.9}$$

(see [3] for details).

**Definition** 2.1([6]) *Let $\alpha$be spacelike with timelike binormal curve and $\alpha^*$ be timelike curve* *in* $\mathbb{R}^3_1$. $\{T,\ N,\ B\}$ *and* $\{T^*,\ N^*,\ B^*\}$ *are Frenet frames, respectively, on these curves.* $\alpha(s)$ *and* $\alpha^*(s)$ *are called Bertrand curves if the principal normal vectors* $N$ *and* $N^*$ *are linearly dependent, and the pair* $(\alpha,\alpha^*)$ *is said to be spacelike-timelike Bertrand curve pair.*

**Theorem** 2.1([6]) *Let* $(\alpha,\alpha^*)$ *be spacelike-timelike Bertrand curve pair. For corresponding* $\alpha(s)$ *and* $\alpha^*(s)$ *points*

$$d\left(\alpha\left(s\right),\alpha^*\left(s\right)\right)=constant,\ \forall s\in I\ .$$

**Theorem** 2.2([6]) *Let* $(\alpha.\alpha^*)$ *be spacelike-timelike Bertrand curve pair.. The measure of the angle between the vector fields of Bertrand curve pair is constant.*

## §3. The Natural Lift Curves And Geodesic Curvatures Of The Spherical Indicatrices Of The Spacelike-Timelike Bertrand Curve Pair

**Theorem** 3.1 *Let* $(\alpha,\alpha^*)$ *be spacelike-timelike Bertrand curve pair. The relations between the Frenet vectors of the curve pair are as follows*

$$\begin{cases} T^*=sinh\theta T-\cosh\theta B\\ N^*=N\\ B^*=\cosh\theta T-\sinh\theta B\end{cases}$$

*Here, the angle* $\theta$ *is the angle between* $T$ *and* $T^*$.

*Proof* By taking the derivative of $\alpha^*(s)=\alpha(s)+\lambda N(s)$ with respect to arc-lenght $s$ and using the equation (2.4), we get

$$T^*\frac{ds^*}{ds}=T\left(1-\lambda\kappa\right)-\lambda\tau B. \tag{3.1}$$

The inner products of the above equation with respect to $T$ and $B$ are respectively defined as

$$\begin{cases}\sinh\theta\dfrac{ds^*}{ds}=1-\lambda\kappa,\\ \cosh\theta\dfrac{ds^*}{ds}=-\lambda\tau.\end{cases} \tag{3.2}$$

and substituting these present equations in (3.1) we obtain

$$T^*=\sinh\theta T-\cosh\theta B. \tag{3.3}$$

Here, by taking derivative and using the equation (2.4) we get

$$N^*=N. \tag{3.4}$$

We can write

$$B^*=\cosh\theta T-\sinh\theta B \tag{3.5}$$

by availing the equation $B^* = -(T^* \times N^*)$. □

**Corollary** 3.1 *Let $(\alpha, \alpha^*)$ be a spacelike-timelike Bertrand curve pair. Between the curvature $\kappa$ and the torsion $\tau$ of the $\alpha$, there is relationship*

$$\mu\tau + \lambda\kappa = 1 \quad \text{and} \quad \mu = -\lambda \tanh\theta, \tag{3.6}$$

*where $\lambda$ and $\mu$ are nonzero real numbers.*

*Proof* From equation (3.2), we obtain

$$\frac{\sinh\theta}{1-\lambda\kappa} = \frac{\cosh\theta}{-\lambda\tau},$$

And by arranging this equation, we get

$$\tanh\theta = \frac{1-\lambda\kappa}{-\lambda\tau}$$

and if we choose $\mu = -\lambda\tanh\theta$ for brevity, then we obtain

$$\mu\tau + \lambda\kappa = 1.$$ □

**Theorem** 3.2 *There are connections between the curvatures $\kappa$ and $\kappa^*$ and the torsions $\tau$ and $\tau^*$ of the spacelike-timelike Bertrand curve pair $(\alpha, \alpha^*)$, which are shown as follows*

$$\begin{cases} \kappa^* = \dfrac{\cosh^2\theta - \lambda\kappa}{\lambda(1-\lambda\kappa)}, \\ \tau^* = -\dfrac{\cosh^2\theta}{\lambda^2\tau}. \end{cases} \tag{3.7}$$

*Proof* If $(\alpha, \alpha^*)$ be a spacelike-timelike Bertrand curve pair, we can write $\alpha(s) = \alpha^*(s) - \lambda N^*(s)$. By taking the derivative of this equation with respect to $s^*$ and using equation (2.1) we obtain

$$T = T^* \frac{ds^*}{ds}(1-\lambda\kappa^*) + \lambda\tau^* B^* \frac{ds^*}{ds}.$$

The inner products of the above equation with respect to $T^*$ and $B^*$ are as following

$$\begin{cases} \sinh\theta = -(1-\lambda\kappa^*)\dfrac{ds^*}{ds}, \\ \cosh\theta = \lambda\tau^*\dfrac{ds^*}{ds}. \end{cases} \tag{3.8}$$

respectively. The proof can easily be completed by using and rearranging the equations (3.2) and (3.8). □

**Corollary** 3.2 *Let $(\alpha, \alpha^*)$ be a spacelike-timelike Bertrand curve pair.*

$$\kappa^* = \frac{\lambda\kappa - \cosh^2\theta}{\lambda^2\tau\tanh\theta}. \tag{3.9}$$

*Proof* By using the equations (3.6) and with substitution of them in 3.7 we get the desired result. □

**Theorem** 3.3 *Let $(\alpha, \alpha^*)$ be a spacelike-timelike Bertrand curve pair. There are following relations between Darboux vector $W$ of curve $\alpha$ and Darboux vector $W^*$ of curve $\alpha^*$*

$$W^* = -\frac{\cosh\theta}{\lambda\tau}W. \tag{3.10}$$

*Proof* For the Darboux vector $W^*$ of timelike curve $\alpha^*$, we can write

$$W^* = \tau^* T^* - \kappa^* B^*.$$

By substituting (3.3), (3.5), (3.7) and (3.9) into the last equation, we obtain

$$W^* = \frac{\cosh\theta}{\lambda\tau}\left[\frac{1}{\lambda}\coth\theta\,(1-\lambda\kappa)\,T + \kappa B\right].$$

By substituting (3.6) into above equation, we get

$$W^* = -\frac{\cosh\theta}{\lambda\tau}W.$$

This completes the proof. □

Now, let compute arc-lengths with the geodesic curvatures of spherical indicatrix curves with the $(T^*)$, $(N^*)$ and $(B^*)$ with the fixed pole curve $(C^*)$ with respect to $\mathbb{R}^3_1$ and $H^2_0$ or $S^2_1$.

Firstly, for the arc-length $s_{T^*}$ of tangents indicatrix $(T^*)$ of the curve $\alpha^*$, we can write

$$s_{T^*} = \int_0^s \left\|\frac{dT^*}{ds}\right\| ds.$$

By taking the derivative of equation (3.3), we have

$$s_{T^*} \leqslant |\sinh\theta| \int_0^s |\kappa|\, ds + |\cosh\theta| \int_0^s |\tau|\, ds.$$

By using equation (2.7) we obtain

$$s_{T^*} \leqslant |\sinh\theta|\, s_T + |\cosh\theta|\, s_B.$$

For the arc-length $s_{N^*}$ of principal normals indicatrix $(N^*)$ of the curve $\alpha^*$, we can write

$$s_{N^*} = \int_0^s \left\|\frac{dN^*}{ds}\right\| ds.$$

By substituting (3.4) into above equation, we get

$$s_{N^*} = s_N.$$

Similarly, for the arc-length $s_{B^*}$ of binormals indicatrix $(B^*)$ of the curve $\alpha^*$, we can write

$$s_{B^*} = \int_0^s \left\| \frac{dB^*}{ds} \right\| ds.$$

By taking the derivative of equation (3.5), we have

$$s_{B^*} \leqslant |\cosh\theta| \int_0^s |\kappa|\, ds + |\sinh\theta| \int_0^s |\tau|\, ds.$$

By using equation (2.7), we obtain

$$s_{B^*} \leqslant |\cosh\theta|\, s_T + |\sinh\theta|\, s_B.$$

Finally, for the arc-length $s_{C^*}$ of the fixed pole curve $(C^*)$, we can write

$$s_{C^*} = \int_0^s \left\| \frac{dC^*}{ds} \right\| ds.$$

If $W^*$ is a spacelike vector, we can write

$$C^* = \sinh\varphi^* T^* - \cosh\varphi^* B^*$$

from the equation (2.2). By taking the derivative of this equation, we obtain

$$s_{C^*} = \int_0^s \left| (\varphi^*)' \right| ds. \tag{3.11}$$

On the other hand, from equation (2.2) and by using

$$\cosh\varphi^* = \frac{\kappa^*}{\|W^*\|} \ \text{ ve } \ \sinh\varphi^* = \frac{\tau^*}{\|W^*\|}$$

we can set

$$\tanh\varphi^* = \frac{\tau^*}{\kappa^*}.$$

By substituting (3.7) and (3.9) into the last equation and differentiating, we obtain

$$(\varphi^*)' = \frac{\lambda\kappa' \sinh\theta\cosh\theta}{\lambda^2\kappa^2 + (1-2\lambda\kappa)\cosh^2\theta}. \tag{3.12}$$

By substituting (3.12) into (3.11), we have

$$s_{C^*} = \int_0^s \left| \frac{\lambda\kappa' \sinh\theta\cosh\theta}{\lambda^2\kappa^2 + (1-2\lambda\kappa)\cosh^2\theta} \right| ds.$$

If $W^*$ is a timelike vector, we have the same result. Thus the following corollary can be drawn.

**Corollary** 3.3 *Let $(\alpha, \alpha^*)$ be a spacelike-timelike Bertrand curve pair and $\{T^*,\ N^*,\ B^*\}$ be the Frenet frame of the curve $\alpha^*$. For the arc-lengths of the spherical indicatrix curves $(T^*)$, $(N^*)$ and $(B^*)$ with the fixed pole curve $(C^*)$ with respect to $\mathbb{R}^3_1$, we have*

(1) $s_{T^*}|\sinh\theta|s_T + |\cosh\theta|s_B$;

(2) $s_{N^*} = s_N$;

(3) $s_{B^*} \leqslant |\cosh\theta|\, s_T + |\sinh\theta|\, s_B$;

(4) $s_{C^*} = \displaystyle\int_0^s \left|\frac{\lambda\kappa'\sinh\theta\cosh\theta}{\lambda^2\kappa^2 + (1-2\lambda\kappa)\cosh^2\theta}\right| ds.$

Now, let us compute the geodesic curvatures of the spherical indicatrix curves $(T^*)$, $(N^*)$ and $(B^*)$ with the fixed pole curve $(C^*)$ with respect to $\mathbb{R}^3_1$. For the geodesic curvature $k_{T^*}$ of the tangents indicatrix $(T^*)$ of the curve $\alpha^*$, we can write

$$k_{T^*} = \|D_{T_{T^*}} T_{T^*}\|\,. \tag{3.13}$$

By differentiating the curve $\alpha_{T^*}(s_{T^*}) = T^*(s)$ with the respect to $s_{T^*}$ and normalizing, we obtain

$$T_{T^*} = N.$$

By taking derivative of the last equation we get

$$D_{T_{T^*}} T_{T^*} = \frac{-\kappa T + \tau B}{|\kappa sinh\theta - \tau\cosh\theta|}. \tag{3.14}$$

By substituting (3.14) into (3.13) we have

$$k_{T^*} = \frac{\|W\|}{|\kappa\sinh\theta - \tau\cosh\theta|}.$$

Here, if $W$ is a spacelike vector, by substituting 2.5 and 2.8 into the last equation we have

$$k_{T^*} = \left|\frac{k_T\cdot k_B}{k_B\cdot\sinh\theta - k_T\cdot\cosh\theta}\right|,$$

if $W$ is a timelike vector, then by substituting 2.6 and 2.9 we have the same result.

Similarly, by differentiating the curve $\alpha_{N^*}(s_{N^*}) = N^*(s)$ with the respect to $s_{N^*}$and by normalizing we obtain

$$T_{N^*} = -\frac{\kappa}{\|W\|}T + \frac{\tau}{\|W\|}B.$$

If $W$ is a spacelike vector, then by using equation (2.5) and (2.8) we have

$$T_{N^*} = -\sinh\varphi T + \cosh\varphi B,$$

$$D_{T_{N^*}} T_{N^*} = \frac{\varphi'}{\|W\|}(-\cosh\varphi T + \sinh\varphi B) + N, \tag{3.15}$$

$$k_{N^*} = k_N = \sqrt{\left|\left(\frac{\varphi'}{\|W\|}\right)^2 + 1\right|}.$$

If $W$ is a timelike vector, then by using of the equations (2.5) and (2.9) we have

$$D_{T_{N^*}} T_{N^*} = \frac{\varphi'}{\|W\|} (\cosh\varphi T - \sinh\varphi B) - \mathrm{N}, \tag{3.16}$$

$$k_{N^*} = k_N = \sqrt{\left|1 - \left(\frac{\varphi'}{\|W\|}\right)^2\right|}.$$

By differentiating the curve $\alpha_{B^*}(s_{B^*}) = B^*(s)$ with the respect to $s_{B^*}$ and by normalizing, we obtain

$$T_{B^*} = N.$$

By taking the derivative of the last equation we get

$$D_{T_{B^*}} T_{B^*} = \frac{-\kappa T + \tau B}{|\kappa \cosh\theta - \tau \sinh\theta|} \tag{3.17}$$

or by taking the norm of equation (3.17), we obtain

$$k_{B^*} = \frac{\|W\|}{|\kappa \cosh\theta - \tau \sinh\theta|}.$$

If $W$ is a spacelike vector, then by substituting (2.5) and (2.8) we have

$$k_{B^*} = \left|\frac{k_T \cdot k_B}{k_B \cdot \cosh\theta - k_T \cdot \sinh\theta}\right|,$$

if $W$ is a timelike vector, then by substituting (2.6) and (2.9) we have the same result. By differentiating the curve $\alpha_{C^*}(s_{C^*}) = C^*(s)$ with the respect to $s_{C^*}$ and normalizing, if $W^*$ is a spacelike vector, then by substituting (2.2) we obtain

$$T_{C^*} = \cosh\varphi^* T^* - \sinh\varphi^* B^*,$$

$$D_{T_{C^*}} T_{C^*} = (\sinh\varphi^* T^* - \cosh\varphi^* B^*) + \frac{\|W^*\|}{(\varphi^*)'} N^*, \tag{3.18}$$

$$k_{C^*} = \sqrt{1 + \left(\frac{\|W^*\|}{(\varphi^*)'}\right)^2}. \tag{3.19}$$

By substituting (3.10) and (3.12) into (3.19) and rearranging we have

$$k_{C^*} = \sqrt{\left|\frac{(\tau^2 - \kappa^2)\left[\lambda^2\kappa^2 + (1 - 2\lambda\kappa)\cosh^2\theta\right]^2}{(\lambda^2\tau\kappa')^2 \sinh^2\theta} + 1\right|}.$$

If $W^*$ is a timelike vector, then by substituting (2.1) and (2.3) we get

$$T_{C^*} = \sinh\varphi^* T^* - \cosh\varphi^* B^*,$$

$$D_{T_{C^*}} T_{C^*} = (\cosh\varphi^* T^* - \sinh\varphi^* B^*) - \frac{\|W^*\|}{(\varphi^*)'} N^*, \tag{3.20}$$

$$k_{C^*} = \sqrt{\left| -1 + \left(\frac{\|W^*\|}{(\varphi^*)'}\right)^2 \right|}. \tag{3.21}$$

By substituting (3.10) and (3.12) into (3.21) we have

$$k_{C^*} = \sqrt{\left| \frac{(\kappa^2 - \tau^2)\left[\lambda^2\kappa^2 + (1-2\lambda\kappa)\cosh^2\theta\right]^2}{(\lambda^2\tau\kappa')^2 \sinh^2\theta} - 1 \right|}.$$

Then the following corollary can be given.

**Corollary** 3.4 *Let $(\alpha, \alpha^*)$ be a spacelike-timelike Bertrand curve cuople and $\{T^*, N^*, B^*\}$ be Frenet frame of the curve $\alpha^*$. For the geodesic curvatures of the spherical indicatrix curves $(T^*)$, $(N^*)$ and $(B^*)$ with the fixed pole curve $(C^*)$ with the respect to $\mathbb{R}^3_1$ we have*

(1)

$$k_{T^*} = \left| \frac{k_T \cdot k_B}{k_B \cdot \sinh\theta - k_T \cdot \cosh\theta} \right|;$$

(2)

$$\begin{cases} k_{N^*} = k_N = \sqrt{\left| \left(\frac{\varphi'}{\|W\|}\right)^2 + 1 \right|}, & W \text{ spacelike ise} \\ k_{N^*} = k_N = \sqrt{\left| 1 - \left(\frac{\varphi'}{\|W\|}\right)^2 \right|}, & W \text{ timelike ise} \end{cases}$$

(3)

$$k_{B^*} = \left| \frac{k_T \cdot k_B}{k_B \cdot \cosh\theta - k_T \cdot \sinh\theta} \right|.$$

(4)

$$\begin{cases} k_{C^*} = \sqrt{\left| \frac{(\tau^2-\kappa^2)[\lambda^2\kappa^2+(1-2\lambda\kappa)\cosh^2\theta]^2}{(\lambda^2\tau\kappa')^2\sinh^2\theta} + 1 \right|}, & W^* \text{ spacelike ise} \\ k_{C^*} = \sqrt{\left| \frac{(\kappa^2-\tau^2)[\lambda^2\kappa^2+(1-2\lambda\kappa)\cosh^2\theta]^2}{(\lambda^2\tau\kappa')^2\sinh^2\theta} - 1 \right|}, & W^* \text{ timelike} \end{cases}.$$

Now let us compute the geodesic curvatures $(T^*)$, $(N^*)$ and $(B^*)$ with the fixed pole curve $(C^*)$ with respect to $H_0^2$ or $S_1^2$.

For the geodesic curvature $\gamma_{T^*}$ of the tangents indicatrix curve $(T^*)$ of the curve $\alpha^*$ with respect to $H_0^2$, we can write

$$\gamma_{T^*} = \left\| \overline{\overline{D}}_{T_{T^*}} T_{T^*} \right\|. \tag{3.22}$$

Here, $\overline{\overline{D}}$ become a covariant derivative operator. By (3.3) and (3.14) we obtain

$$D_{T_{T^*}} T_{T^*} = \overline{\overline{D}}_{T_{T^*}} T_{T^*} + \varepsilon g\left(S\left(T_{T^*}\right), T_{T^*}\right) T^*,$$

$$\overline{\overline{D}}_{T_{T^*}} T_{T^*} = \left(\frac{-\kappa}{|\kappa \sinh\theta - \tau\cosh\theta|} - \sinh\theta\right)\cdot T + \left(\frac{\tau}{|\kappa \sinh\theta - \tau\cosh\theta|} + \cosh\theta\right)\cdot B. \quad (3.23)$$

By substituting (3.23) into (3.22) we get

$$\gamma_{T^*} = \sqrt{\left|\frac{\tau^2-\kappa^2}{\left(\kappa\sinh\theta - \tau\cosh\theta\right)^2} - 1\right|}.$$

If $W$ is a spacelike vector, then by using of the equations (2.5) and (2.8) we have

$$\gamma_{T^*} = \sqrt{\left|\left(\frac{k_T k_B}{k_B\sinh\theta - k_T\cosh\theta}\right)^2 - 1\right|},$$

if $W$ is a timelike vector, then by using of the equations (2.6) and (2.9) we have

$$\gamma_{T^*} = \sqrt{\left|-\left(\frac{k_T k_B}{k_B\sinh\theta - k_T\cosh\theta}\right)^2 - 1\right|}.$$

If the curve $\left(\overline{T^*}\right)$ is an integral curve of the geodesic spray, then $\overline{\overline{D}}_{T_{T^*}} T_{T^*} = 0$. Thus, by (3.23) we can write

$$\begin{cases} \frac{-\kappa}{|\kappa\sinh\theta - \tau\cosh\theta|} - \sinh\theta = 0 \\ \\ \frac{\tau}{|\kappa\sinh\theta - \tau\cosh\theta|} + \cosh\theta = 0 \end{cases}$$

and here, we obtain $\kappa = 0,\ \tau \neq 0$ and $\theta = 0$. So, we can give following corollary.

**Corollary** 3.5 *Let $(\alpha, \alpha^*)$ be a spacelike-timelike Bertrand curve pair. The natural lift $\left(\overline{T^*}\right)$ of the tangent indicatrix $(T^*)$ is never an integral curve of the geodesic spray.*

For the geodesic curvature $\gamma_{N^*}$ of the principal normals indicatrix curve $(N^*)$ of the curve $\alpha^*$ with respect to $S_1^2$ we can write

$$\gamma_{N^*} = \left\|\overline{D}_{T_{N^*}} T_{N^*}\right\|. \quad (3.24)$$

Here, $\overline{D}$ become a covariant derivative operator. If $W$ is a spacelike vector, by using of the equation (3.15) we obtain

$$\overline{D}_{T_{N^*}} T_{N^*} = \frac{\varphi'}{\|W\|}\left(-\cosh\varphi T + \sinh\varphi B\right). \quad (3.25)$$

By substituting (3.25) into (3.24) we get

$$\gamma_{N^*} = \frac{\varphi'}{\|W\|}. \tag{3.26}$$

On the other hand, from the equation (2.5) by using

$$\sinh\varphi = \frac{\kappa}{\|W\|} \text{ ve } \cosh\varphi = \frac{\tau}{\|W\|}$$

we can set

$$\tanh\varphi = \frac{\kappa}{\tau}.$$

By taking the derivative of the last equation we get

$$\varphi' = \frac{\kappa'\tau - \tau'\kappa}{\|W\|^2}.$$

By substituting above the equation into (3.26) we have

$$\gamma_{N^*} = \gamma_N = \frac{\kappa'\tau - \tau'\kappa}{\|W\|^3}.$$

If $W$ is a timelike vector, by using of the equation (3.16) we obtain

$$\overline{D}_{T_{N^*}} T_{N^*} = \frac{\varphi'}{\|W\|}\left(-\sinh\varphi T + \cosh\varphi B\right)\gamma_{N^*} = \frac{\varphi'}{\|W\|}. \tag{3.27}$$

On the other hand, from equation (2.6) by using

$$\cosh\varphi = \frac{\kappa}{\|W\|} \text{ and } \sinh\varphi = \frac{\tau}{\|W\|}$$

we can set

$$\text{tanh}\varphi = \frac{\kappa}{\tau}.$$

By taking the derivative of the last equation we get

$$\varphi' = \frac{\tau'\kappa - \kappa'\tau}{\|W\|^2}$$

or

$$\gamma_{N^*} = \gamma_N = \frac{\tau'\kappa - \kappa'\tau}{\|W\|^3}.$$

If the curve $\left(\overline{N^*}\right)$ is an integral curve of the geodesic spray, then $\overline{D}_{T_{N^*}} T_{N^*} = 0$. Thus, by (3.25) and (3.27) we can write $\varphi' = 0$ and here, we obtain $\frac{\kappa}{\tau}$ =constant. So, we can give following corollary.

**Corollary** 3.6 *Let $(\alpha, \alpha^*)$ be a spacelike-timelike Bertrand curve pair. If the curve $\alpha$ is a helix curve, the natural lift $\left(\overline{N^*}\right)$ of the pirincipal normals indicatrix $(N^*)$ is an integral curve of the* *geodesic spray.*

For the geodesic curvature $\gamma_{B^*}$ of the binormals indicatrix curve $(B^*)$ of the curve $\alpha^*$with respect to $S_1^2$ and substituting (3.5) and (3.17) we obtain

$$D_{T_{B^*}}T_{B^*} = \overline{D}_{T_{B^*}}T_{B^*} + \varepsilon g\left(S\left(T_{B^*}\right), T_{B^*}\right)B^*,$$

$$\overline{D}_{T_{B^*}}T_{B^*} = \left(\frac{-\kappa}{|\kappa\cosh\theta - \tau\sinh\theta|} + \cosh\theta\right)T + \left(\frac{\tau}{|\kappa\cosh\theta - \tau\sinh\theta|} - \sinh\theta\right)B, \quad (3.28)$$

$$\gamma_{B^*} = \sqrt{\left|-1 + \frac{\tau^2 - \kappa^2}{\left(-\kappa\sinh\theta + \tau\cosh\theta\right)^2}\right|}.$$

If $W$ is a spacelike vector, then by using of the equations (2.5) and (2.8) we have

$$\gamma_{B^*} = \sqrt{\left|-1 - \left(\frac{k_T k_B}{k_B.\cosh\theta - k_T.\sinh\theta}\right)^2\right|},$$

if $W$ is a timelike vector, then by using of the equations (2.6) and (2.9) we get

$$\gamma_{B^*} = \sqrt{\left|-1 + \left(\frac{k_T k_B}{k_B.\cosh\theta - k_T.\sinh\theta}\right)^2\right|}.$$

If the curve $\left(\overline{B^*}\right)$ is an integral curve of the geodesic spray, then $\overline{D}_{T_{B^*}}T_{B^*} = 0$. Thus, by (3.28) we can write

$$\begin{cases} \dfrac{-\kappa}{|\kappa\cosh\theta - \tau\sinh\theta|} + \cosh\theta = 0 \\ \dfrac{\tau}{|\kappa\cosh\theta - \tau\sinh\theta|} - \sinh\theta = 0 \end{cases}$$

and here, we obtain $\kappa > 0$, $\tau = 0$ and $\theta = 0$. So, we can give following corollary.

**Corollary** 3.7 *Let $(\alpha, \alpha^*)$ be a spacelike-timelike Bertrand curve pair. If the curve $\alpha$is a planary curve and frames are equivalent, the natural lift $\left(\overline{B^*}\right)$ of the binormals indicatrix $(B^*)$ is an integral curve of the geodesic spray.*

If $W^*$ is a spacelike vector, for the geodesic curvature $\gamma_{C^*}$ of the fixed pole curve $(C^*)$ of the curve $\alpha^*$ with respect to $S_1^2$ and by using of the equations (2.2) and (3.18) we obtain

$$D_{T_{C^*}}T_{C^*} = \overline{D}_{T_{C^*}}T_{C^*} + \varepsilon g\left(S\left(T_{C^*}\right), T_{C^*}\right)C^*,$$

$$\overline{D}_{T_{C^*}}T_{C^*} = \frac{\|W^*\|}{\left(\varphi^*\right)'}N^*, \quad (3.29)$$

$$\gamma_{C^*} = \left\|\frac{\|W^*\|}{\left(\varphi^*\right)'}\right\|.$$

By substituting (3.10) and (3.12) into the last equation we have

$$\gamma_{c^*} = \frac{\|W\| \left[\lambda^2\kappa^2 + (1-2\lambda\kappa)\cosh^2\theta\right]}{\lambda^2\tau\kappa' \sinh\theta}.$$

If $W^*$ is a timelike vector, for the geodesic curvature $\gamma_{C^*}$ of the fixed pole curve $(C^*)$ with respect to $H_0^2$ and by using of the equations (2.3) and (3.20) we have the same result. If the curve $\left(\overline{C^*}\right)$ is an integral curve of the geodesic spray, then $\overline{\overline{D}}_{T_{C^*}} T_{C^*} = 0$. Thus by (3.29) we can write $\|W^*\| = 0$ and here, we get $\kappa^* = \tau^* = 0$ or $\kappa^* = \tau^*$. Thus, by using of the equation (3.7) and (3.9) we obtain

$$\kappa = \frac{\cosh^2\theta - \sinh\theta\cosh\theta}{\lambda}.$$

So, we can give following corollary.

**Corollary** 3.8 *Let $(\alpha, \alpha^*)$ be a spacelike-timelike Bertrand curve pair. If the curve $\alpha$ is a curve that provides the requirement $\kappa = \dfrac{\cosh^2\theta - \sinh\theta\cosh\theta}{\lambda}$, the natural lift $\left(\overline{C^*}\right)$ of the fixed pole curve $(C^*)$ is an integral curve of the geodesic spray.*

**Corollary** 3.9 *Let $(\alpha, \alpha^*)$ be a spacelike-timelike Bertrand curve pair and $\{T^*,\ N^*,\ B^*\}$ be Frenet frame of the curve $\alpha^*$. For the geodesic curvatures of the spherical indicatrix curves $(T^*)$, $(N^*)$ and $(B^*)$ with the fixed pole curve $(C^*)$ with respect to $H_0^2$ or $S_1^2$, we have*

(1)

$$\begin{cases} \gamma_{T^*} = \sqrt{\left|\left(\dfrac{k_T k_B}{k_B \sinh\theta - k_T\cosh\theta}\right)^2 - 1\right|}, & W \text{ spacelike} \\ \gamma_{T^*} = \sqrt{\left|-1 - \left(\dfrac{k_T k_B}{k_B \sinh\theta - k_T\cosh\theta}\right)^2\right|}, & W \text{ timelike} \end{cases}$$

(2)

$$\begin{cases} \gamma_{N^*} = \gamma_N = \dfrac{\kappa'\tau - \tau'\kappa}{\|W\|^3}, & W \text{ spacelike} \\ \gamma_{N^*} = \gamma_N = \dfrac{\tau'\kappa - \kappa'\tau}{\|W\|^3}, & W \text{ timelike} \end{cases}$$

(3)

$$\begin{cases} \gamma_{B^*} = \sqrt{\left|-1 - \left(\dfrac{k_T k_B}{k_B \cosh\theta - k_T\sinh\theta}\right)^2\right|}, & W \text{ spacelike} \\ \gamma_{B^*} = \sqrt{\left|-1 + \left(\dfrac{k_T k_B}{k_B \cosh\theta - k_T\sinh\theta}\right)^2\right|}, & W \text{ timelike} \end{cases}$$

(4)

$$\gamma_{c^*} = \frac{\|W\| \left[\lambda^2\kappa^2 + (1-2\lambda\kappa)\cosh^2\theta\right]}{\lambda^2\tau\kappa' \sinh\theta}.$$

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