

Enumeration of k -Fibonacci Paths Using Infinite Weighted Automata

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Abstract: In this paper, we introduce a new family of generalized colored Motzkin paths, where horizontal steps are colored by means of $F_{k,l}$ colors, where $F_{k,l}$ is the l -th k -Fibonacci number. We study the enumeration of this family according to the length. For this, we use infinite weighted automata.

Key Words: Fibonacci sequence, Generalized colored Motzkin path, k -Fibonacci path, infinite weighted automata, generating function.

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§1. Introduction

A lattice path of length n is a sequence of points P_1, P_2, \dots, P_n with $n \geq 1$ such that each point P_i belongs to the plane integer lattice and each two consecutive points P_i and P_{i+1} connect by a line segment. We will consider lattice paths in $\mathbb{Z} \times \mathbb{Z}$ using three step types: a rise step $U = (1, 1)$, a fall step $D = (1, -1)$ and a $F_{k,l}$ -colored length horizontal step $H_l = (l, 0)$ for every positive integer l , such that H_l is colored by means of $F_{k,l}$ colors, where $F_{k,l}$ is the l -th k -Fibonacci number.

Many kinds of generalizations of the Fibonacci numbers have been presented in the literature [10,11] and the corresponding references. Such as those of k -Fibonacci numbers $F_{k,n}$ and the k -Smarandache-Fibonacci numbers $S_{k,n}$. For any positive integer number k , the k -Fibonacci sequence, say $\{F_{k,n}\}_{n \in \mathbb{N}}$, is defined recurrently by

$$F_{k,0} = 0, \quad F_{k,1} = 1, \quad F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad \text{for } n \geq 1.$$

The generating function of the k -Fibonacci numbers is $f_k(x) = \frac{x}{1 - kx - x^2}$, [4,6]. This sequence was studied by Horadam in [9]. Recently, Falcón and Plaza [6] found the k -Fibonacci numbers by studying the recursive application of two geometrical transformations used in the four-triangle longest-edge (4TLE) partition. The interested reader is also referred to [1, 3, 4, 5, 6, 12, 13, 16] for further information about this.

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A *generalized $F_{k,l}$ -colored Motzkin path* or simply *k -Fibonacci path* is a sequence of rise, fall and $F_{k,l}$ -colored length horizontal steps ($l = 1, 2, \dots$) running from $(0, 0)$ to $(n, 0)$ that never pass below the x -axis. We denote by $\mathcal{M}_{F_{k,n}}$ the set of all k -Fibonacci paths of length n and $\mathcal{M}_k = \bigcup_{n=0}^{\infty} \mathcal{M}_{F_{k,n}}$. In Figure 1 we show the set $\mathcal{M}_{F_{2,3}}$.

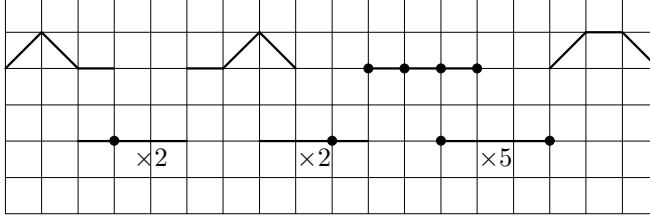


Figure 1 k -Fibonacci Paths of length 3, $|\mathcal{M}_{F_{2,3}}| = 13$

A *grand k -Fibonacci path* is a k -Fibonacci path without the condition that never going below the x -axis. We denote by $\mathcal{M}_{F_{k,n}}^*$ the set of all grand k -Fibonacci paths of length n and $\mathcal{M}_k^* = \bigcup_{n=0}^{\infty} \mathcal{M}_{F_{k,n}}^*$. A *prefix k -Fibonacci path* is a k -Fibonacci path without the condition that ending on the x -axis. We denote by $\mathcal{PM}_{F_{k,n}}$ the set of all prefix k -Fibonacci paths of length n and $\mathcal{PM}_k = \bigcup_{n=0}^{\infty} \mathcal{PM}_{F_{k,n}}$. Analogously, we have the family of *prefix grand k -Fibonacci paths*. We denote by $\mathcal{PM}_{F_{k,n}}^*$ the set of all prefix grand k -Fibonacci paths of length n and $\mathcal{PM}_k^* = \bigcup_{n=0}^{\infty} \mathcal{PM}_{F_{k,n}}^*$.

In this paper, we study the generating function for the k -Fibonacci paths, grand k -Fibonacci paths, prefix k -Fibonacci paths, and prefix grand k -Fibonacci paths, according to the length. We use Counting Automata Methodology (CAM) [2], which is a variation of the methodology developed by Rutten [14] called Coinductive Counting. Counting Automata Methodology uses infinite weighted automata, weighted graphs and continued fractions. The main idea of this methodology is find a counting automaton such that there exist a bijection between all words recognized by an automaton \mathcal{M} and the family of combinatorial objects. From the counting automaton \mathcal{M} is possible find the ordinary generating function (GF) of the family of combinatorial objects [4].

§2. Counting Automata Methodology

The terminology and notation are mainly those of Sakarovitch [13]. An *automaton* \mathcal{M} is a 5-tuple $\mathcal{M} = (\Sigma, Q, q_0, F, E)$, where Σ is a nonempty input alphabet, Q is a nonempty set of states of \mathcal{M} , $q_0 \in Q$ is the initial state of \mathcal{M} , $\emptyset \neq F \subseteq Q$ is the set of final states of \mathcal{M} and $E \subseteq Q \times \Sigma \times Q$ is the set of transitions of \mathcal{M} . The language recognized by an automaton \mathcal{M} is denoted by $L(\mathcal{M})$. If Q, Σ and E are finite sets, we say that \mathcal{M} is a finite automaton [15].

Example 2.1 Consider the finite automaton $\mathcal{M} = (\Sigma, Q, q_0, F, E)$ where $\Sigma = \{a, b\}$, $Q = \{q_0, q_1\}$, $F = \{q_0\}$ and $E = \{(q_0, a, q_1), (q_0, b, q_0), (q_1, a, q_0)\}$. The transition diagram of \mathcal{M} is as shown in Figure 2. It is easy to verify that $L(\mathcal{M}) = (b \cup aa)^*$.

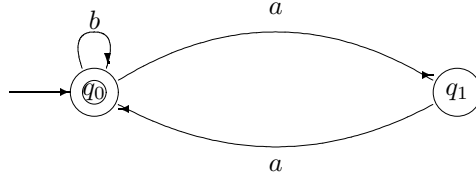


Figure 2 Transition diagram of \mathcal{M} , Example 1

Example 2.2 Consider the infinite automaton $\mathcal{M}_{\mathcal{D}} = (\Sigma, Q, q_0, F, E)$, where $\Sigma = \{a, b\}$, $Q = \{q_0, q_1, \dots\}$, $F = \{q_0\}$ and $E = \{(q_i, a, q_{i+1}), (q_{i+1}, b, q_i) : i \in \mathbb{N}\}$. The transition diagram of $\mathcal{M}_{\mathcal{D}}$ is as shown in Figure 3.

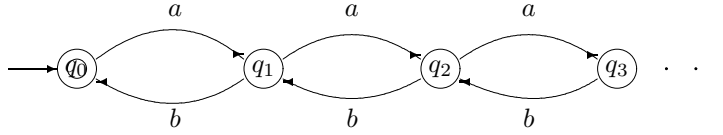


Figure 3 Transition diagram of $\mathcal{M}_{\mathcal{D}}$

The language accepted by $\mathcal{M}_{\mathcal{D}}$ is

$$L(\mathcal{M}_{\mathcal{D}}) = \{w \in \Sigma^* : |w|_a = |w|_b \text{ and for all prefix } v \text{ of } w, |v|_b \leq |v|_a\}.$$

An ordinary generating function $F = \sum_{n=0}^{\infty} f_n z^n$ corresponds to a formal language L if $f_n = |\{w \in L : |w| = n\}|$, i.e., if the n -th coefficient f_n gives the number of words in L with length n .

Given an alphabet Σ and a semiring \mathbb{K} . A *formal power series* or *formal series* S is a function $S : \Sigma^* \rightarrow \mathbb{K}$. The image of a word w under S is called the *coefficient* of w in S and is denoted by s_w . The series S is written as a formal sum $S = \sum_{w \in \Sigma^*} s_w w$. The set of formal power series over Σ with coefficients in \mathbb{K} is denoted by $\mathbb{K} \langle \langle \Sigma^* \rangle \rangle$.

An automaton over Σ^* with weights in \mathbb{K} , or \mathbb{K} -*automaton* over Σ^* is a graph labelled with elements of $\mathbb{K} \langle \langle \Sigma^* \rangle \rangle$, associated with two maps from the set of vertices to $\mathbb{K} \langle \langle \Sigma^* \rangle \rangle$. Specifically, a *weighted automaton* \mathcal{M} over Σ^* with weights in \mathbb{K} is a 4-tuple $\mathcal{M} = (Q, I, E, F)$ where Q is a nonempty set of *states* of \mathcal{M} , E is an element of $\mathbb{K} \langle \langle \Sigma^* \rangle \rangle^{Q \times Q}$ called *transition matrix*. I is an element of $\mathbb{K} \langle \langle \Sigma^* \rangle \rangle^Q$, i.e., I is a function from Q to $\mathbb{K} \langle \langle \Sigma^* \rangle \rangle$. I is the *initial function* of \mathcal{M} and can also be seen as a row vector of dimension Q , called *initial vector* of \mathcal{M} and F is an element of $\mathbb{K} \langle \langle \Sigma^* \rangle \rangle^Q$. F is the *final function* of \mathcal{M} and can also be seen as a column vector of dimension Q , called *final vector* of \mathcal{M} .

We say that \mathcal{M} is a *counting automaton* if $\mathbb{K} = \mathbb{Z}$ and $\Sigma^* = \{z\}^*$. With each automaton, we can associate a counting automaton. It can be obtained from a given automaton replacing every transition labelled with a symbol a , $a \in \Sigma$, by a transition labelled with z . This transition is called a *counting transition* and the graph is called a *counting automaton* of \mathcal{M} . Each transition

from p to q yields an equation

$$L(p)(z) = zL(q)(z) + [p \in F] + \cdots .$$

We use L_p to denote $L(p)(z)$. We also use Iverson's notation, $[P] = 1$ if the proposition P is true and $[P] = 0$ if P is false.

2.1 Convergent Automata and Convergent Theorems

We denote by $L^{(n)}(\mathcal{M})$ the number of words of length n recognized by the automaton \mathcal{M} , including repetitions.

Definition 2.3 We say that an automaton \mathcal{M} is convergent if for all integer $n \geq 0$, $L^{(n)}(\mathcal{M})$ is finite.

The proof of following theorems and propositions can be found in [2].

Theorem 2.4(First Convergence Theorem) *Let \mathcal{M} be an automaton such that each vertex (state) of the counting automaton of \mathcal{M} has finite degree. Then \mathcal{M} is convergent.*

Example 2.5 The counting automaton of the automaton $\mathcal{M}_{\mathcal{D}}$ in Example 2 is convergent.

The following definition plays an important role in the development of applications because it allows to simplify counting automata whose transitions are formal series.

Definition 2.6 Let \mathcal{M} be an automaton, and let $f(z) = \sum_{n=0}^{\infty} f_n z^n$ be a formal power series with $f_n \in \mathbb{N}$ for all $n \geq 0$ and $f_0 = 0$. In a counting automaton of \mathcal{M} the set of counting transitions from state p to state q , without intermediate final states, see Figure 4 (left), is represented by a graph with a single edge labeled by $f(z)$, see Figure 4(right).

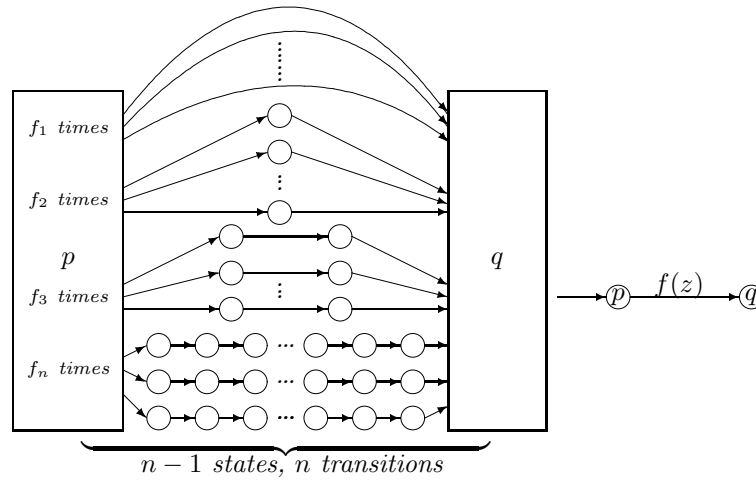


Figure 4 Transitions from the state p to q and its transition in parallel

This kind of transition is called a *transition in parallel*. The states p and q are called *visible states* and the intermediate states are called *hidden states*.

Example 2.7 In Figure 5 (left) we display a counting automaton \mathcal{M}_1 without transitions in parallel, i.e., every transition is label by z . The transitions from state q_1 to q_2 correspond to the series $\frac{1 - \sqrt{1 - 4z}}{2} = z + z^2 + 2z^3 + 5z^4 + 14z^5 + \dots$. However, this automaton can also be represented using transitions in parallel. Figure 5 (right) displays two examples.

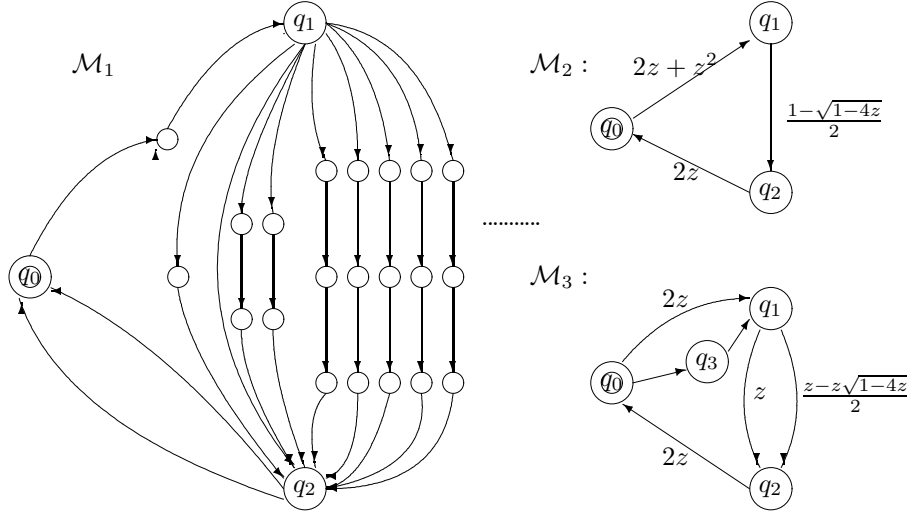


Figure 5 Counting automata with transitions in parallel

Theorem 2.8(Second Convergence Theorem) *Let \mathcal{M} be an automaton, and let $f_1^q(z), f_2^q(z), \dots$, be transitions in parallel from state $q \in Q$ in a counting automaton of \mathcal{M} . Then \mathcal{M} is convergent if the series*

$$F^q(z) = \sum_{k=1}^{\infty} f_k^q(z)$$

is a convergent series for each visible state $q \in Q$ of the counting automaton.

Proposition 2.9 *If $f(z)$ is a polynomial transition in parallel from state p to q in a finite counting automaton \mathcal{M} , then this gives rise to an equation in the system of GFs equations of \mathcal{M}*

$$L_p = f(z)L_q + [p \in F] + \dots$$

Proposition 2.10 *Let \mathcal{M} be a convergent automaton such that a counting automaton of \mathcal{M} has a finite number of visible states q_0, q_1, \dots, q_r , in which the number of transitions in parallel starting from each state is finite. Let $f_1^{q_t}(z), f_2^{q_t}(z), \dots, f_{s(t)}^{q_t}(z)$ be the transitions in parallel from the state $q_t \in Q$. Then the GF for the language $L(\mathcal{M})$ is $L_{q_0}(z)$. It is obtained by solving*

the system of $r + 1$ GFs equations

$$L(q_t)(z) = f_1^{q_t}(z)L(q_{t_1})(z) + f_2^{q_t}(z)L(q_{t_2})(z) + \cdots + f_{s(t)}^{q_t}(z)L(q_{t_{s(t)}})(z) + [q_t \in F],$$

with $0 \leq t \leq r$, where q_{t_k} is the visible state joined with q_t through the transition in parallel $f_k^{q_t}$, and $L(q_{t_k})$ is the GF for the language accepted by \mathcal{M} if q_{t_k} is the initial state.

Example 2.11 The system of GFs equations associated with \mathcal{M}_2 , see Example 2.7, is

$$\begin{cases} L_0 &= (2z + z^2)L_1 + 1 \\ L_1 &= \frac{1 - \sqrt{1 - 4z}}{2}L_2 \\ L_2 &= 2zL_0. \end{cases}$$

Solving the system for L_0 , we find the GF for the language \mathcal{M}_2 and therefore of \mathcal{M}_1 and \mathcal{M}_3

$$L_0 = \frac{1}{1 - (2z^2 + z^3)(1 - \sqrt{1 - 4z})} = 1 + 4z^3 + 6z^4 + 10z^5 + 40z^6 + 114z^7 + \cdots$$

2.2 An Example of the Counting Automata Methodology (CAM)

A counting automaton associated with an automaton \mathcal{M} can be used to model combinatorial objects if there is a bijection between all words recognized by the automaton \mathcal{M} and the combinatorial objects. Such method, along with the previous theorems and propositions constitute the *Counting Automata Methodology (CAM)*, see [2].

We distinguish three phases in the CAM:

- (1) Given a problem of enumerative combinatorics, we have to find a convergent automaton \mathcal{M} (see Theorems 2.4 and 2.8), whose GF is the solution of the problem.
- (2) Find a general formula for the GF of \mathcal{M}' , where \mathcal{M}' is an automaton obtained from \mathcal{M} truncating a set of states or edges see Propositions 2.9 and 2.10. Sometimes we find a relation of iterative type, such as a continued fraction.
- (3) Find the GF $f(z)$ to which converge the GFs associated to each \mathcal{M}' , which is guaranteed by the convergences theorems.

Example 2.12 A *Motzkin path* of length n is a lattice path of $\mathbb{Z} \times \mathbb{Z}$ running from $(0, 0)$ to $(n, 0)$ that never passes below the x -axis and whose permitted steps are the up diagonal step $U = (1, 1)$, the down diagonal step $D = (1, -1)$ and the horizontal step $H = (1, 0)$. The number of Motzkin paths of length n is the n -th *Motzkin number* m_n , sequence A001006¹. The number of words of length n recognized by the convergent automaton \mathcal{M}_{Mot} , see Figure 6, is the n th Motzkin number and its GF is

$$M(z) = \sum_{i=0}^{\infty} m_i z^i = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z^2}.$$

¹Many integer sequences and their properties are found electronically on the On-Line Encyclopedia of Sequences [17].

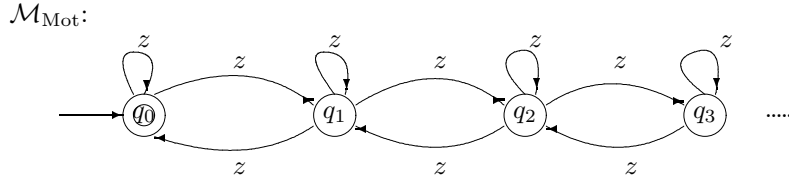


Figure 6 Convergent automaton associated with Motzkin paths

In this case the edge from state q_i to state q_{i+1} represents a rise, the edge from the state q_{i+1} to q_i represents a fall and the loops represent the level steps, see Table 1.

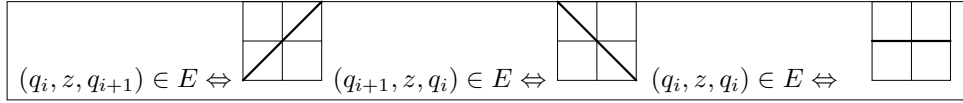


Table 1 Bijection between \mathcal{M}_{Mot} and Motzkin paths

Moreover, it is clear that a word is recognized by \mathcal{M}_{Mot} if and only if the number of steps to the right and to the left coincide, which ensures that the path is well formed. Then

$$m_n = |\{w \in L(\mathcal{M}_{\text{Mot}}) : |w| = n\}| = L^{(n)}(\mathcal{M}_{\text{Mot}}).$$

Let $\mathcal{M}_{\text{Mot}s}$, $s \geq 1$ be the automaton obtained from \mathcal{M}_{Mot} , by deleting the states q_{s+1}, q_{s+2}, \dots . Therefore the system of GFs equations of $\mathcal{M}_{\text{Mot}s}$ is

$$\begin{cases} L_0 = zL_0 + zL_1 + 1, \\ L_i = zL_{i-1} + zL_i + zL_{i+1}, & 1 \leq i \leq s-1, \\ L_s = zL_{s-1} + zL_s. \end{cases}$$

Substituting repeatedly into each equation L_i , we have

$$L_0 = \frac{H}{1 - \frac{F^2}{1 - \frac{F^2}{\vdots \frac{1}{1 - F^2}}}} \quad \left. \vphantom{\frac{H}{1 - \frac{F^2}{1 - \frac{F^2}{\vdots \frac{1}{1 - F^2}}}}} \right\} s \text{ times},$$

where $F = \frac{z}{1-z}$ and $H = \frac{1}{1-z}$. Since \mathcal{M}_{Mot} is convergent, then as $s \rightarrow \infty$ we obtain a convergent continued fraction M of the GF of \mathcal{M}_{Mot} . Moreover,

$$M = \frac{H}{1 - F^2 \left(\frac{M}{H} \right)}.$$

Hence $z^2 M^2 - (1 - z)M + 1 = 0$ and

$$M(z) = \frac{1 - z \pm \sqrt{1 - 2z - 3z^2}}{2z^2}.$$

Since $\epsilon \in L(\mathcal{M}_{\text{Mot}})$, $M \rightarrow 0$ as $z \rightarrow 0$. Hence, we take the negative sign for the radical in $M(z)$.

§3. Generating Function for the k -Fibonacci Paths

In this section we find the generating function for k -Fibonacci paths, grand k -Fibonacci paths, prefix k -Fibonacci paths and prefix grand k -Fibonacci paths, according to the length.

Lemma 3.1([2]) *The GF of the automaton \mathcal{M}_{Lin} , see Figure 7, is*

$$E(z) = \frac{1}{1 - h_0(z) - \frac{f_0(z)g_0(z)}{1 - h_1(z) - \frac{f_1(z)g_1(z)}{\ddots}}},$$

where $f_i(z), g_i(z)$ and $h_i(z)$ are transitions in parallel for all integer $i \geq 0$.

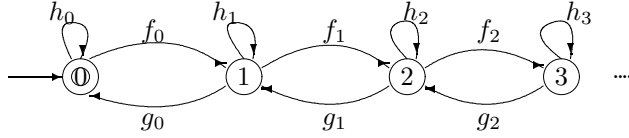


Figure 7 Linear infinite counting automaton \mathcal{M}_{Lin}

The last lemma coincides with Theorem 1 in [7] and Theorem 9.1 in [14]. However, this presentation extends their applications, taking into account that $f_i(z), g_i(z)$ and $h_i(z)$ are GFs, which can be GFs of several variables.

Corollary 3.2 *If for all integers $i \geq 0$, $f_i(z) = f(z), g_i(z) = g(z)$ and $h_i(z) = h(z)$ in \mathcal{M}_{Lin} , then the GF is*

$$B(z) = \frac{1 - h(z) - \sqrt{(1 - h(z))^2 - 4f(z)g(z)}}{2f(z)g(z)} \quad (1)$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_n \binom{m+2n}{m} (f(z)g(z))^n (h(z))^m \quad (2)$$

$$= \frac{1}{1 - h(z) - \frac{f(z)g(z)}{1 - h(z) - \frac{f(z)g(z)}{1 - h(z) - \frac{f(z)g(z)}{\ddots}}}}, \quad (3)$$

where C_n is the n th Catalan number, sequence A000108.

Theorem 3.3 *The generating function for the k -Fibonacci paths according to the their length is*

$$T_k(z) = \sum_{i=0}^{\infty} |\mathcal{M}_{F_{k,i}}| z^i \quad (4)$$

$$= \frac{1 - (k+1)z - z^2 - \sqrt{(1 - (k+1)z - z^2)^2 - 4z^2(1 - kz - z^2)^2}}{2z^2(1 - kz - z^2)} \quad (5)$$

$$= \frac{1}{1 - \frac{z}{1 - kz - z^2} - \frac{z^2}{1 - \frac{z}{1 - kz - z^2} - \frac{z^2}{1 - \frac{z}{1 - kz - z^2} - \frac{z^2}{\ddots}}}} \quad (6)$$

and

$$[z^t] T_k(z) = \sum_{n=0}^t \sum_{m=0}^{t-2n} \binom{m+2n}{m} C_n F_{k,t-2n-m+1}^{(m)},$$

where C_n is the n -th Catalan number and $F_{k,j}^{(r)}$ is a convolved k -Fibonacci number.

Convolved k -Fibonacci numbers $F_{k,j}^{(r)}$ are defined by

$$f_k^{(r)}(x) = (1 - kx - x^2)^{-r} = \sum_{j=0}^{\infty} F_{k,j+1}^{(r)} x^j, \quad r \in \mathbb{Z}^+.$$

Note that

$$F_{k,m+1}^{(r)} = \sum_{j_1+j_2+\dots+j_r=m} F_{k,j_1+1} F_{k,j_2+1} \cdots F_{k,j_r+1}.$$

Moreover, using a result of Gould[8, p.699] on Humbert polynomials (with $n = j, m = 2, x = k/2, y = -1, p = -r$ and $C = 1$), we have

$$F_{k,j+1}^{(r)} = \sum_{l=0}^{\lfloor j/2 \rfloor} \binom{j+r-l-1}{j-l} \binom{j-l}{l} k^{j-2l}.$$

Ramírez [13] studied some properties of convolved k -Fibonacci numbers.

Proof Equations (5) and (6) are clear from Corollary 3.2 taking $f(z) = z = g(z)$ and $h(z) = \frac{z}{1 - kz - z^2}$. Note that $h(z)$ is the GF of k -Fibonacci numbers. In this case the edge from state q_i to state q_{i+1} represents a rise, the edge from the state q_{i+1} to q_i represents a fall and the loops represent the $F_{k,l}$ -colored length horizontal steps ($l = 1, 2, \dots$). Moreover, from

Equation (2), we obtain

$$\begin{aligned}
T_k(z) &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_n \binom{m+2n}{m} z^{2n} \left(\frac{z}{1-kz-z^2} \right)^m \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_n \binom{m+2n}{m} z^{2n+m} \left(\frac{1}{1-kz-z^2} \right)^m \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_n \binom{m+2n}{m} z^{2n+m} \sum_{i=0}^{\infty} F_{k,i+1}^{(m)} z^i \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} C_n F_{k,i+1}^{(m)} \binom{m+2n}{m} z^{2n+m+i},
\end{aligned}$$

taking $s = 2n + m + i$

$$T_k(z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{s=2n+m}^{\infty} C_n F_{k,s-2n-m+1}^{(m)} \binom{m+2n}{m} z^s.$$

Hence

$$[z^t] T_k(z) = \sum_{n=0}^t \sum_{m=0}^{t-2n} C_n F_{k,t-2n-m+1}^{(m)} \binom{m+2n}{m}.$$

□

In Table 2 we show the first terms of the sequence $|\mathcal{M}_{F_{k,i}}|$ for $k = 1, 2, 3, 4$.

k	Sequence
1	1, 1, 3, 8, 23, 67, 199, 600, 1834, 5674, 17743, ...
2	1, 1, 4, 13, 47, 168, 610, 2226, 8185, 30283, 112736, ...
3	1, 1, 5, 20, 89, 391, 1735, 7712, 34402, 153898, 690499, ...
4	1, 1, 6, 29, 155, 820, 4366, 23262, 124153, 663523, 3551158, ...

Table 2 Sequences $|\mathcal{M}_{F_{k,i}}|$ for $k = 1, 2, 3, 4$

Definition 3.4 For all integers $i \geq 0$ we define the continued fraction $E_i(z)$ by:

$$E_i(z) = \frac{1}{1 - h_i(z) - \frac{f_i(z)g_i(z)}{1 - h_{i+1}(z) - \frac{f_{i+1}(z)g_{i+1}(z)}{\ddots}}},$$

where $f_i(z), g_i(z), h_i(z)$ are transitions in parallel for all integers positive i .

Lemma 3.5([2]) *The GF of the automaton $\mathcal{M}_{\text{BLin}}$, see Figure 8, is*

$$E_b(z) = \frac{1}{1 - h_0(z) - f_0(z)g_0(z)E_1(z) - f'_0(z)g'_0(z)E'_1(z)},$$

where $f_i(z), f'_i(z), g_i(z), g'_i(z), h_i(z)$ and $h'_i(z)$ are transitions in parallel for all $i \in \mathbb{Z}$.

$\mathcal{M}_{\text{BLin}}$:

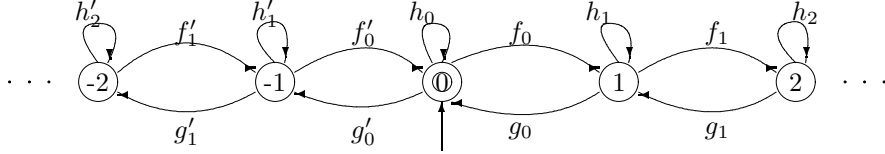


Figure 8 Linear infinite counting automaton $\mathcal{M}_{\text{BLin}}$

Corollary 3.6 *If for all integers i , $f_i(z) = f(z) = f'_i(z), g_i(z) = g(z) = g'_i(z)$ and $h_i(z) = h(z) = h'_i(z)$ in $\mathcal{M}_{\text{BLin}}$, then the GF*

$$B_b(z) = \frac{1}{\sqrt{(1 - h(z))^2 - 4f(z)g(z)}} \quad (7)$$

$$= \frac{1}{1 - h(z) - \frac{2f(z)g(z)}{1 - h(z) - \frac{f(z)g(z)}{1 - h(z) - \frac{f(z)g(z)}{\ddots}}}} \quad (8)$$

where $f(z), g(z)$ and $h(z)$ are transitions in parallel. Moreover, if $f(z) = g(z)$, then the GF

$$B_b(z) = \frac{1}{1 - h(z)} + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} 2^n \frac{n}{n+2k} \binom{n+2k}{k} \binom{l+2n+2k}{l} f(z)^{2n+2k} h(z)^l. \quad (9)$$

Theorem 3.7 *The generating function for the grand k -Fibonacci paths according to the their length is*

$$T_k^*(z) = \sum_{i=0}^{\infty} |\mathcal{M}_{F_{k,i}}^*| z^i = \frac{1 - kz - z^2}{\sqrt{(1 - (k+1)z - z^2)^2 - 4z^2(1 - kz - z^2)^2}} \quad (10)$$

$$= \frac{1}{1 - \frac{z}{1 - kz - z^2} - \frac{2z^2}{1 - \frac{z}{1 - kz - z^2} - \frac{z^2}{1 - \frac{z}{1 - kz - z^2} - \frac{z^2}{\ddots}}}} \quad (11)$$

and

$$[z^t] T_k^*(z) = F_{k+1,t}^{(1)} + \sum_{n=1}^t \sum_{m=0}^t \sum_{l=0}^{t-2n-2m} 2^n \frac{n}{n+2m} \binom{n+2m}{m} \binom{l+2n+2m}{l} F_{k,t-2n-2m-l+1}^{(l)}, \quad (12)$$

with $t \geq 1$.

Proof Equations (10) and (11) are clear from Corollary 3.6, taking $f(z) = z = g(z)$ and $h(z) = \frac{z}{1-kz-z^2}$. Moreover, from Equation (9), we obtain

$$\begin{aligned} T_k^*(z) &= \frac{1}{1 - \frac{z}{1-kz-z^2}} + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} 2^n \frac{n}{n+2m} \binom{n+2m}{m} \binom{l+2n+2m}{l} z^{2n+2m} \left(\frac{z}{1-kz-z^2} \right)^l \\ &= 1 + \sum_{j=0}^{\infty} F_{k+1,j}^{(1)} z^j + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \sum_{u=0}^{\infty} 2^n \frac{n}{n+2m} \binom{n+2m}{m} \binom{l+2n+2m}{l} F_{k,u}^{(l)} z^{2n+2m+u+1}, \end{aligned}$$

taking $s = 2n + 2m + l + u$

$$\begin{aligned} T_k^*(z) &= 1 + \sum_{j=0}^{\infty} F_{k+1,j}^{(1)} z^j + \\ &\quad \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \sum_{s=2n+2m+l}^{\infty} 2^n \frac{n}{n+2m} \binom{n+2m}{m} \binom{l+2n+2m}{l} F_{k,s-2n-2m-l}^{(l)} z^s. \end{aligned}$$

Therefore, Equation (12) is clear. \square

In Table 3 we show the first terms of the sequence $|\mathcal{M}_{F_{k,i}}^*|$ for $k = 1, 2, 3, 4$.

k	Sequence
1	1, 4, 11, 36, 115, 378, 1251, 4182, 14073, 47634, \dots
2	1, 5, 16, 63, 237, 920, 3573, 14005, 55156, 218359, \dots
3	1, 6, 23, 108, 487, 2248, 10371, 48122, 223977, 1046120, \dots
4	1, 7, 32, 177, 949, 5172, 28173, 153963, 842940, 4624581, \dots

Table 3 Sequences $|\mathcal{M}_{F_{k,i}}^*|$ for $k = 1, 2, 3, 4$ and $i \geq 1$

In Figure 9 we show the set $\mathcal{M}_{F_{2,3}}^*$.

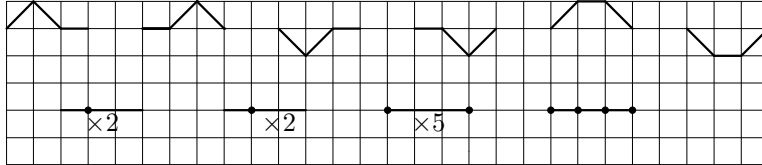


Figure 9 Grand k -Fibonacci Paths of length 3, $|\mathcal{M}_{F_{2,3}}^*| = 16$

Lemma 3.8([2]) *The GF of the automaton $\text{FIN}_{\mathbb{N}}(\mathcal{M}_{Lin})$, see Figure 10, is*

$$G(z) = E(z) + \sum_{j=1}^{\infty} \left(\prod_{i=0}^{j-1} (f_i(z)E_i(z))E_j(z) \right),$$

where $E(z)$ is the GF in Lemma 3.1.

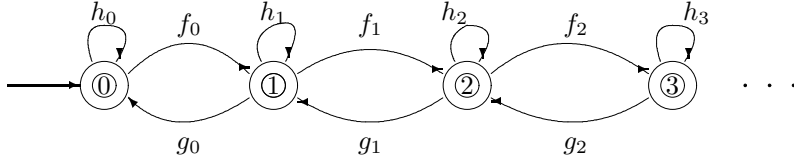


Figure 10 Linear infinite counting automaton $\text{FIN}_{\mathbb{N}}(\mathcal{M}_{Lin})$

Corollary 3.9 *If for all integer $i \geq 0$, $f_i(z) = f(z)$, $g_i(z) = g(z)$ and $h_i(z) = h(z)$ in $\text{FIN}_{\mathbb{N}}(\mathcal{M}_{Lin})$, then the GF is:*

$$G(z) = \frac{1 - 2f(z) - h(z) - \sqrt{(1 - h(z))^2 - 4f(z)g(z)}}{2f(z)(f(z) + g(z) + h(z) - 1)} \quad (13)$$

$$= \frac{1}{1 - f(z) - h(z) - \frac{f(z)g(z)}{1 - h(z) - \frac{f(z)g(z)}{1 - h(z) - \frac{f(z)g(z)}{\ddots}}}}, \quad (14)$$

where $f(z)$, $g(z)$ and $h(z)$ are transitions in parallel and $B(z)$ is the GF in Corollary 3.2. Moreover, if $f(z) = g(z)$ and $h(z) \neq 0$, then we obtain the GF

$$G(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{n+1}{n+k+1} \binom{n+2k+l}{k, l, k+n} f^{2k+n}(z) h^l(z). \quad (15)$$

Theorem 3.10 *The generating function for the prefix k -Fibonacci paths according to the their length is*

$$\begin{aligned} PT_k(z) &= \sum_{i=0}^{\infty} |\mathcal{PM}_{F_{k,i}}| z^i \\ &= \frac{(1 - 2z)(1 - kz - z^2) - z - \sqrt{(1 - z(k+1) - z^2)^2 + 4z^2(1 - kz - z^2)^2}}{2z((1 - kz - z^2)(2z - 1) + z)} \end{aligned}$$

and

$$[z^t] PT_k(z) = \sum_{n=0}^t \sum_{m=0}^t \sum_{l=0}^{t-2m-n} \frac{n+1}{n+m+1} \binom{n+2m+l}{m, l, m+n} F_{k, t-2m-n-l+1}^{(l)}, \quad t \geq 0.$$

Proof The proof is analogous to the proof of Theorem 3.3 and 3.7. \square

In Table 4 we show the first terms of the sequence $|\mathcal{PM}_{F_{k,i}}|$ for $k = 1, 2, 3, 4$.

k	Sequence
1	1, 2, 6, 19, 62, 205, 684, 2298, 7764, 26355, 89820, \dots
2	1, 2, 7, 26, 101, 396, 1564, 6203, 24693, 98605, 394853, \dots
3	1, 2, 8, 35, 162, 757, 3558, 16766, 79176, 374579, 1775082, \dots
4	1, 2, 9, 46, 251, 1384, 7668, 42555, 236463, 1315281, 7322967, \dots

Table 4 Sequences $|\mathcal{PM}_{F_{k,i}}|$ for $k = 1, 2, 3, 4$

In Figure 11 we show the set $\mathcal{MP}_{F_{2,3}}$.

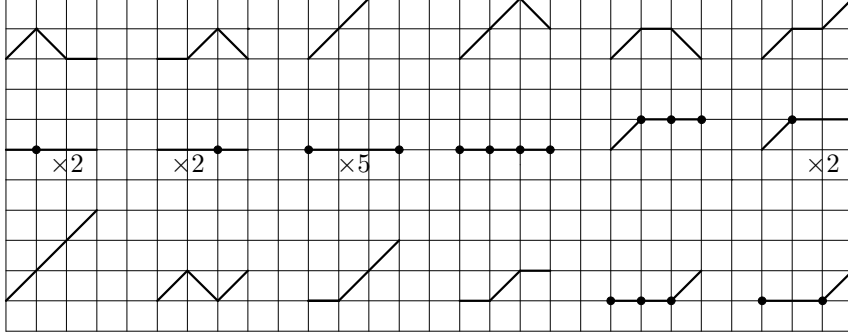


Figure 11 Prefix k -Fibonacci paths of length 3, $|\mathcal{PM}_{F_{2,3}}| = 26$

Lemma 3.11 The GF of the automaton $\text{FIN}_{\mathbb{Z}}(\mathcal{M}_{BLin})$, see Figure 12, is

$$\begin{aligned} H(z) &= \frac{EE'}{E + E' - EE'(1 - h_0)} \left(1 + \sum_{j=1}^{\infty} \prod_{k=1}^{j-1} f_k E_k f_0 E_j + \sum_{j=1}^{\infty} \prod_{k=1}^{j-1} g'_k E'_k g'_0 E'_j \right) \\ &= \frac{E'(z)G(z) + E(z)G'(z) - E(z)E'(z)}{E(z) + E'(z) - E(z)E'(z)(1 - h_0(z))}, \end{aligned}$$

where $G(z)$ is the GF in Lemma 3.8 and $G'(z), E'(z)$ are the GFs obtained from $G(z)$ and $E(z)$ changing $f(z)$ to $g'(z)$ and $g(z)$ to $f'(z)$.

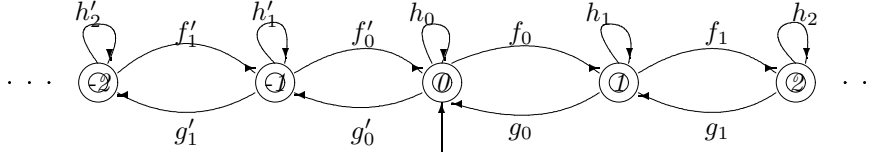


Figure 12 Linear infinite counting automaton $\text{FIN}_{\mathbb{Z}}(\mathcal{M}_{BLin})$

Moreover, if for all integer $i \geq 0$, $f_i(z) = f(z) = f'_i(z)$, $g_i(z) = g(z) = g'_i(z)$ and $h_i(z) = h(z) = h'_i(z)$ in $\text{FIN}_{\mathbb{Z}}(\mathcal{M}_{BLin})$, then the GF is

$$H(z) = \frac{1}{1 - f(z) - g(z) - h(z)}. \quad (16)$$

Theorem 3.12 The generating function for the prefix grand k -Fibonacci paths according to the their length is

$$PT_k^*(z) = \sum_{i=0}^{\infty} |\mathcal{PM}_{F_{k,i}^*}| z^i = \frac{1 - kz - z^2}{1 - (k+3)z - (1-2k)z^2 + 2z^3}.$$

it Proof The proof is analogous to the proof of Theorem 3.3 and 3.7. \square

In Table 5 we show the first terms of the sequence $|\mathcal{PM}_{F_{k,i}^*}|$ for $k = 1, 2, 3, 4$.

k	Sequence
1	1, 3, 10, 35, 124, 441, 1570, 5591, 19912, 70917, 252574, ...
2	1, 3, 11, 44, 181, 751, 3124, 13005, 54151, 225492, 938997, ...
3	1, 3, 12, 55, 264, 1285, 6280, 30727, 150392, 736157, 3603528, ...
4	1, 3, 13, 68, 379, 2151, 12268, 70061, 400249, 2286780, 13065595 ...

Table 4 Sequences $|\mathcal{PM}_{F_{k,i}^*}|$ for $k = 1, 2, 3, 4$

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