International J.Math. Combin. Vol.2(2014), 63-73

Antisotropic Cosmological Models of Finsler Space With (γ, β)-Metric

Arunesh Pandey[1], V.K.Chaubey[2] and T.N.Pandey[3]

1.Department of Mathematics, SLIET Deemed University, Longowal-148106, Sangrur Punjab, India

2.Department of Applied Sciences, Ansal Technical Campus, Lucknow (U.P.)-226030, India

3.Department of Mathematics and Statistics, D.D.U.Gorakhpur University, Gorakhpur(U.P.)-273009, India

E-mail: ankpandey11@rediffmail.com, vkcoct@hotmail.com, tnp1952@gmail.com

Abstract: In this paper we have studied an anisotropic model of space time with Finslerian spaces with (γ, β)-metrics as suggested by one of the co-author in his paper [21] with an extra requirement of $\gamma^3 = a_{ijk}(x)y^iy^jy^k$. Here γ, is a cubic metric and $\beta = b_i(x)y^i$, is a one form metric. The observed anisotropy of the microwave background radiation is incorporated in the Finslerian metric of space time.

Key Words: Cosmology, Finsler space with (γ, β)-metric, cubic metric and one form metric.

AMS(2010): 53B20, 53C40 83F05

§1. Introduction

The concept of cubic metric on a differentiable manifold with the local co-ordinate x^i, defined by

$$L(x, y) = \{a_{ijk}(x)y^iy^jy^k\}^{\frac{1}{3}}$$

was introduced by M. Matsumoto in the year 1979 ([1]), where,$a_{ijk}(x)$ are components of a symmetric tensor field of (0, 3)-type, depending on the position x alone, and a Finsler space with a cubic metric (called the cubic Finsler space). During investigation of some interesting results we have gone through papers/research outcomes regarding cubic Finsler spaces as referred in the papers [3, 4, 5, 6, 7, 8]. It has been observed that there are various interesting results on geometry of spaces with a cubic metric as a generalization of Euclidean or Riemannian geometry have been published in recent years. It is further noticed that one of the paper published by one of the coauthor of this paper [21] in the year 2011, define the concept of (γ, β)-metric considering γ is a cubic metric and β is a one-form and discussed various important results in stand-point of the Finsler Geometry in this paper. Here we wish to mention that in paper [2] concerned with the unified field theory of gravitation and electromagnetism Randers wrote that

[1]Received February 8, 2013, Accepted May 28, 2014.

Perhaps the most characteristic property of the physical world is the uni direction of time-like intervals. Since there is no obvious reason why this asymmetry should disappear in the mathematical description it is of interest to consider the possibility of a metric with asymmetrical property.

It is also noticed that many researchers are interested to investigate something new result in physics and those possible application in modern cosmology and other reference of the same kind [9, 10, 11].We are fully agreed with the theory expressed in the above referred publication regarding, as it is based on tangent bundle on space time manifold are positively with local Lorentz violations which may be related with dark energy and dark matter models with variable in cosmology. Certainly this may also be one of the most recent hidden connections between Finsler geometry and cosmology. Recently many researchers are constructing suitable cosmological models with variable Lambda term including our own research group [12, 13, 14, 15, 16]. So in extension of our research work, we have decided to study Cosmological model of General Theory of Relativity based on the frame work of Finsler geometry in this communication.

§2. **Results**

In the present paper we try to generalize the above changes by defining a Lagrangian which expresses this anisotropy as such

$$L = L(\gamma, \beta), \tag{2.1}$$

where $\gamma = \{a_{ijk}(x)y^i y^j y^k\}^{\frac{1}{3}}$ is a cubic metric and $\beta = \phi(x)\hat{b}_i y^i$is a one-form and for this metric. The purpose of the present study is to obtain the relationship between the anisotropic cosmological models of space time with above generalized Finslerian metric motivated by the work of Stavrions and Diakogiannis [16].

Let us consider an n- dimensional Finsler space (M^n, L) and an adaptable 1-form on M^n we shall use a Lagrangian function on M^n , given by the equation:

$$L = L\{(a_{ijk}(x)y^i y^j y^k)^{\frac{1}{3}}, \varphi(x)\hat{b}_i y^i\}, \tag{2.2}$$

where $b_i(x) = \phi(x)\hat{b}_i$, the vector $\hat{b}_i$ represents the observed an isotropic of the microwave background radiation. A coordinate transformation on the total space TM may be expressed as

$$\bar{x^i} = \bar{x^i}(x^0, x^1, x^2, x^3), \tag{2.3}$$

A fundamental function or a Finsler metric is a scalar field L(x, y) which satisfies the following three conditions:

(1) It is defined and differentiable for any point of $TM^n - (0)$;

(2) It is positively homogeneous of first degree in y^i, that is, $L(x, py) = pL(x, y)$ for any positive number p;

(3) It is regular, that is, $g_{ij}(x)\dfrac{\partial^2}{\partial y^i \partial y^j}\left(\dfrac{L^2}{2}\right)$ constitute the regular matrix (g_{ij}).The inverse matrix of g^{ij} is indicated by (g_{ij}). The homogeneity condition (2) enables us to consider the

integral $s = \int_b^a L(\frac{dx}{dt})dt$ along an arc , independently of the choice of parameter t except the orientation. The manifold M^n equipped with a fundamental function $L(x, y)$ is called a Finsler space $F^n = (M^n, L)$ and the s is called the length of the arc. Thus the following two conditions are desirable for $L(x, y)$ from the geometrical point of view.

(4) It is positive-valued for any point $TM^n - (0)$;

(5) $g_{ij}(x, y)$ define a positive-definite quadratic form.

Here we have to remark that there are some cases where the conditions (1), (4) and (5) should be restricted to some domain of $TM^n - (0)$.The value $L(x, y)$ is called the length of the tangent vector y at a point x. We get $L^2 = g_{ij}(x, y)y^i y^j$. The set $\left(\frac{y}{L(x,y)} = 1\right)$ in the tangent space at x or geometric-cally the set of all the end points of such y is called the indicatrix at x. If we have an equation f(x, y) =0 of the indicatrix at x,then the fundamental function L is defined by $\frac{(f(x,y))}{L} = 0$. The tensor g_{ij} is called the fundamental tensor. From L we get two other important tensors

$$l_i = \dot{\partial}_i L h_{ij} = L\dot{\dot{\partial}}_i \dot{\dot{\partial}}_j .$$

The former is called the normalized supporting element, because $l^i = g^{ir} l_r$ is written as $\frac{y^i}{L(x,y)}$ and satisfies $L(x, y) = 1$. The latter is called angular metric tensor. It satisfies $h_{ij} y^j = 0$ and the rank of h_{ij} is equal to $(n - 1)$.

The Cartan torsion coefficients C_{ijk} are given by

$$C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}. \tag{2.4}$$

The torsions and curvatures which we use are given by [17, 18, 19, 20]

$$P_{ijk} = C_{ijk|l} y^l, \tag{2.5}$$

$$S_{jikh} = C_{iks} C^s_{jh} - C_{ihs} C^s_{jk}, \tag{2.6}$$

$$P_{ihkj} = C_{ijk|h} - C_{hjk|i} + C^r_{hj} C_{rik|l} y^l - C^r_{ij} C_{rkh|l} y^l, \tag{2.7}$$

$$S^l_{ikh} = g^{lj} S_{jikh}, \tag{2.8}$$

$$P^l_{ikh} = g^{lj} P_{jikh}, \tag{2.9}$$

Differentiating equation (2.1) with respect to y^i, the normalized supporting elements $l_i = \dot{\partial}_i L$is given by

$$l_i = \dot{\partial}_i L = \frac{\partial L}{\partial y^i}, \tag{2.10}$$

$$l_i = \dot{\partial}_i L = \frac{\partial L}{\partial y}\left(\frac{a_{ijk} y^j y^k}{\gamma^2}\right) + \frac{\partial L}{\partial y}\phi(x)\hat{b_i},$$

$$l_i = \dot{\partial}_i L = L_\gamma \left(\frac{a_i(x,y)}{\gamma^2}\right) + L_\beta \phi(x)\hat{b^i}, \tag{2.11}$$

where

$$(a_{ijk}y^jy^k) = a_i(x,y). \tag{2.12}$$

Again differentiating equation (2.11) with respect to y^j, we have

$$\dot{\partial}_i\dot{\partial}_j L = \dot{\partial}_j\dot{\partial}_i L = \dot{\partial}_j\left\{\frac{L_\gamma}{\gamma^2}a_i(x,y)\right\} + \dot{\partial}_j\{L_\beta\phi(x)\hat{\beta}_i\},$$

$$\dot{\partial}_i\dot{\partial}_j L = \left[\frac{L_\gamma}{\gamma^2}a_{ij} + L_{\beta\beta}\phi^2\hat{b_i}\hat{b_j} + \frac{L_{\gamma\beta}}{\gamma^2}\phi(a_i\hat{b_j}a_j\hat{b_i} + a_ia_j)\right], \tag{2.13}$$

$$\dot{\partial}_j\phi(x)\hat{b}_i = 0,$$

where

$$2a_{ijk}y^k = a_{ij}(x,y). \tag{2.14}$$

The angular metric tensor $h_{ij} = L\dot{\partial}_i\dot{\partial}_j L$ as

$$h_{ij} = L\dot{\partial}_i\dot{\partial}_j L = L\left[\frac{L_\gamma}{\gamma^2}a_{ij} + L_{\beta\beta}\phi^2\hat{b_i}\hat{b_j} + \frac{L_{\gamma\beta}}{\gamma^2}\phi(a_i\hat{b_j} + a_j\hat{b_i}) + \frac{(L_{\gamma\gamma} - \frac{2L_\gamma}{\gamma})}{\gamma^4}a_ia_j\right], \tag{2.15}$$

$$h_{ij} = L\dot{\partial}_i\dot{\partial}_i L = \left[u_{-1}a_{ij} + u_0\phi^2\hat{b_i}\hat{b_j} + u_{-2}\phi(a_i\hat{b_j} + a_j\hat{b_i}) + u_{-4}a_ia_j\right], \tag{2.16}$$

where

$$u_{-1} = \frac{LL_\gamma}{L^2}, u_0 = LL_{\beta\beta}, u_{-2} = \frac{LL_{\gamma\beta}}{L^2}, u_{-4} = \frac{L(L_{\gamma\gamma} - \frac{2L_\gamma}{\gamma})}{\gamma}. \tag{2.17}$$

§3. Anisotropic Cosmological Model with Finsler Space of $(\gamma\beta)$ Metric

The Lagrangian function on M_n, given by the equation $L = L(\gamma, \phi(x)\hat{b_i}y^i)$, where $\gamma = (a_{ijk}y^iy^jy^k)^{\frac{1}{3}}$. For the anisotropy, we must insert an additional term to the cubic metric line element. This additional term fulfills the following requirements:

(1) It must give absolute maximum contribution for the direction of movement parallel to the anisotropy axis;

(2) The new line element must coincide with the cubic metric one for the direction vertical to the anisotropy axis;

(3) It must not symmetric with respect to replacement $y^i = -y^i$;

(4) We see that a term which satisfies the above conditions is $\beta = \phi(x)b^i$, where $b_i(x)$ reveals this anisotropic axis.

Now let $b_i(x) = \phi(x)\hat{b^i}$ where $\hat{b^i}$ the unit vector in the direction is $b_i(x)$. Then $\phi(x)$ plays the role of length of the vector $b_i(x)$, $\phi(x) \in R$. β is the Finslerian line element and γ is cubic one.

We have

$$\gamma = cd\tau = \mu d(ct) = \mu dx^0, \tag{3.1}$$

where $\mu\sqrt{1-\frac{v^2}{c^2}}$ and v is the 3- velocity in cubic space-time. One possible explanation of the anisotropy axis could be that it represents the resultant of spin densities of the angular momenta of galaxies in a restricted area of space ($b_i(x)$ space like).

The Finsler metric tensor g_{ij} is

$$g_{ij} = \dot{\partial}_i \dot{\partial}_j \frac{L^2}{2} = h_{ij} + l_i l_j, \tag{3.2}$$

Thus,

$$\begin{aligned} g_{ij} &= h_{ij} + l_i l_j \\ &= L\left[\frac{L_\gamma}{\gamma^2} a_{ij} + L_{\beta\beta}\phi^2 \hat{b_i}\hat{b_j} + \frac{L_{\gamma\beta}}{\gamma^2}\phi(a_i\hat{b_j} + a_j\hat{b_i}) + \frac{(L_{\gamma\gamma} - \frac{2L_\gamma}{\gamma})}{\gamma^4} a_i a_j\right] \\ &+ \left(\frac{L_\gamma}{\gamma^2} a_i + L_\beta \phi b^i\right)\left(\frac{L_\gamma}{\gamma^2}) a_j + L_\beta \phi \hat{b_j}\right) \end{aligned}$$

$$g_{ij} = h_{ij} + l_i l_j = [u_{-1} a_{ij} + m_0 \phi^2 \hat{b_i}\hat{b_j} + m_{-2}\phi(a_i\hat{b_j} + a_j\hat{b_i}) + m_{-4} a_i a_j, \tag{3.3}$$

where

$$m_0 = LL_{\beta\beta} + (L_\beta^2), m_(-2) = \left(\frac{LL_{\gamma\beta}}{\gamma^2}\right) + \frac{L_\gamma L_\beta}{\gamma^2}, m_{-4} = \frac{L}{\gamma^4}\left(L_{\gamma\gamma} - \frac{(2L_\gamma)}{\gamma}\right) + \frac{L_\gamma^2}{\gamma^4}, \tag{3.4}$$

where, we put $y^i = a_{ij} y^j$ and a_{ij} is the fundamental tensor for the Finsler space F^n. It will be easy to see that the determinant $\parallel g_{ij} \parallel$ does not vanish, and the reciprocal tensor with components g^{ij} is given by

$$g^{ij} = \left[\frac{1}{(u_{-1}} a^i j - z_2 \phi^2 \hat{B}_i \hat{B}_j - z_0 \phi(a^i \hat{b^j} + a^j \hat{b^i}) - z_{-2} a^i a^j\right], \tag{3.5}$$

where

$$\begin{aligned} z_2 &= \frac{\phi_0 u_{-1}^2 \phi^2 (\eta_{-2} + \phi^2 m_{-4} \bar{a}^2 - 2m_{-2}\bar{a})}{\eta_{-2} u_{-1}(u_{-1} + c^2)} \\ z_0 &= \frac{m_{-2} u_{-1} - \phi^2 m_{-4} \bar{a}}{\eta_{-2} u_{-1}}, \\ z_{-2} &= \frac{m_{-4} u_{-1} - c^2 m_{-4}}{\eta_{-2} u_{-1}}. \end{aligned} \tag{3.6}$$

As

$$\begin{aligned} &c^2 = \phi^2 b^2, \bar{a} = a_i B^j = a^{im} a_i b_m = a^i b_j, \\ &b^2 = B^i b_i = a^{im} b_m b_i, \\ &\phi \hat{B^i} = a^{ij} \phi \hat{b_j}, \phi \hat{a^i} = \hat{a^{ij}} \phi \hat{a^j}, \end{aligned} \tag{3.7}$$

where g^{ij} is the reciprocal tensor of g_{ij} and a^{ij} is the inverse matrix of a_{ij} as it may be verified by direct calculation, where $b^2 = \hat{b_i}\hat{b^i} = 0, \pm 1$ according whether $\hat{b^i}$ is null, space like or time like. It is interesting to observe that, that if y^i represents the velocity of a particle (time like) then $\hat{b^i}$ is bound to be space like. This follows from the fact that one possible value of $\hat{b^i}y^i$ is zero. Therefore we have decided to calculate Cartan covariant tensor C.

§4. The Cartan Covariant Tensor C

The Cartan covariant tensor C with the components C_{ijk} is obtained by Differentiating equation (3.3) with respect to y^k. We get

$$C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij},$$

$$C_{ijk} = \frac{1}{2}\left[2u_{-1}a_{ijk} + m_{o\beta}\phi^3 b_i b_j b_k + \prod_{(ijk)}\left(K_i a_{jk} + m_{-2}\phi^2 a_i \hat{b_j}\hat{b_k} + \frac{m_{-2\gamma}}{\gamma^2}\phi a_i a_j \hat{b_k}\right) + \frac{m_{-4\gamma}}{\gamma^2} a_i a_j a_k\right], \tag{4.1}$$

where $\prod_{(ijk)}$ represent the sum of cyclic permutation of i, j, k.

$$K_i = m_{-4}a_i + m_{-2}\phi\hat{b_i}. \tag{4.2}$$

If $\phi = 0$, i.e., absence of anisotrophy, then $K_i = m_{-4}a_i$.

From equations (4.1)and (4.2), we have

$$\begin{aligned} C_{ijk} \quad = \quad & \frac{1}{2u_{-1}}[2u_{-1}^2 a_{ijk} + d_{-2}\phi^3 b_i b_j b_k + \prod_{(ijk)}(K_i h_{jk} + d_{-4}\phi^2 a_i b_j b_k \\ & + d_{-6}\phi a_i a_j a_k) + d_{-8} a_i a_j a_k], \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} & d_{-2} = u_{-1}m_{0\beta} - 3m_{-2}u_0,\ d_{-4} = u_{-1}m_{-2\beta} - u_0 m_{-4} - 2m_{-2}u_{-2} \\ & d_{-6} = u_{-1}m_{-4\beta} - 2m_{-4}u_{-2} - m_{-2}u_{-4},\ d_{-8} = u_{-1}\frac{m_{-4\gamma}}{\gamma^2} - 3m_{-4}u_{-4}. \end{aligned} \tag{4.4}$$

After simplification, we have

$$C_{ijk} = \frac{1}{2u_(-1)}\left[2^2_{u_{-1}} a_{ijk} + \prod_{(ijk)}(H_{jk}K_i)\right], \tag{4.5}$$

where

$$H_{ij} = h_{ij} + \frac{d_{-2}}{3(m_{-2})^3}K_i K_j. \tag{4.6}$$

In equation (4.6), we replace the covariant indices j by k and k by s, we have

$$C_{iks} = \frac{1}{2u_{-1}^2}\left[a_{iks} + \prod_{(iks)}(H_{ks}K_i)\right]. \tag{4.7}$$

Now $C_{ijk}g^{jh}=C^h_{ik}$ Multiplying equation (3.5) and equation (4.1) and after simplification, we have

$$C^h_{ik} = \frac{1}{2u^2_{-1}}[2u_(-1^2a^h_{ik}+(\delta^h_iK_k-l^hl_iK_k)+(\delta^h_kK_i-l^hl_kK_i)+ \frac{d_{(-2)}}{(m_{-2})^3}K^hK_iK_k+h_{ik}K^h], \tag{4.8}$$

where $K^i=K_hg^{hk}$, $l^h=g^{hi}l_i$, $a^h_{ik}=a_{jik}g^{jh}$.

Now in equation (4.8) interchange the covariant indices h by s, i by j and k by h, we have

$$C^s_{jh}=\frac{1}{2u_{-1}}\left[2u^2_{-1}a^s_{jh}+(\delta^s_jK_h-l^sl_jK_h)+(\delta^s_hK_j-l^sl_hK_j)+\frac{d_{-2}}{(m_(-2)^3}K^sK_jK_h+h_{jh}K^s\right]. \tag{4.9}$$

Therefore, $S_{jikh}=C_{iks}C^s_{jh}-C_{ihs}C^s_{jk}$ yields

$$S_{jikh} = \frac{1}{4(u_{-1})^2}\theta_{(kh)}[(4(u_{-1})^4a^s_{jh}a_{sik}+2(u_{-1})^2(a^s_{ik}K_sH_{jh} +a^s_{jh}K_sH_{ik})-(l_jK_h+l_hK_j)A_{ik}-(l_iK_k+l_kK_i)A_{jh} +H^{'}_{ik}K_jK_h+H^{'}_{jh}K_iK_k)], \tag{4.10}$$

where $A_{ik}=2(u_{-1})^2a_{ik}-\frac{\bar{K}}{L^2}h_{ik}$, $H^{'}_{ik}=2(u_{-1})^2\frac{2d_{-2}}{3(m^3_{-2}}a_{sik}K_s+(1+\frac{K^2}{L^4})h_{ik}$ and

$$\frac{K^2}{L^4}=K^sK_s,\quad \frac{\bar{K}}{L^2}=K^sl_s,\quad K^sg_{is}=K_i,\quad a_{isk}l^s=\frac{a_{ik}}{L}.$$

Thus S-curvature as defined in the equation (2.6) above represents the anisotropy of matter.

If $b_{i|h}=0$ then for $L(\gamma,\beta)$- metric, we have $a_{i|j}=0$, $a_{ij|k}=0$. Because of $l_{i|j}=0h_{ij|k}=0$, differentiating covariant derivative of equation (4.6) with respect to h, we get $C_{ijk|h}=u_{-1}a_{ijk|h}$. Therefore the v torsion tensor P_{ijk} is written as

$$P_{ijk}=C_{ij}k|\ hy^h=C_{ijk|0}=u_{-1}a_{ijk|0}. \tag{4.11}$$

Therefore $S_{jikh}=C_{iks}C^s_{jh}-C_{ihs}C^s_{jk}$ yields

$$S_{jikh} = \frac{1}{4(u_{-1})^2}\theta_{(kh)}[4(u_{-1})^4a^s_{jh}a_{sik}+2(u_{-1})^2(a^s_{ik}K_sH_{jh} +a^s_{jh}K_sH_{ik}-(l_jK_h+l_hK_j)A_{ik}-(l_iK_k+l_kK_i)A_{jh} +H^{'}_{ik}K_jK_h+H^{'}_{jh}K_iK_k)], \tag{4.12}$$

where $A_{ik}=2(u_{-1})^2a_{ik}-\frac{\bar{K}}{L^2}h_{ik}$ and

$$H^{'}_{ik}=2(u_{-1})^2\frac{2d_{-2}}{3m^3_{-2}}a_{sik}K_s+\left(1+\frac{K^2}{L^4}\right)h_{ik},$$

$$\frac{K^2}{L^4}=K^sK_s,\quad \frac{\bar{K}}{L^2}=K^sl_s,\quad K^sg_{is}=K_i,\quad a_{isk}l^s=\frac{a_{ik}}{L}.$$

Thus S-curvature as defined in the equation (2.6) above represents the anisotropy of matter. If $b_{i|h} = 0$ then for $L(\gamma, \beta)$- metric we have

$$a_{i|j} = 0, a_{ij|k} = 0. \tag{4.13}$$

Because of $l_{i|j} = 0h_{ij|k} = 0$, differentiating covariant derivative of equation (4.6) with respect to h we get

$$C_{ijk|h} = u_{-1}a_{ijk|h}. \tag{4.14}$$

Therefore, the v torsion tensor P_{ijk} is written as

$$P_{ijk} = C_{ijk|h}y^h = C_{ijk|0} = u_{-1}a_{ijk|0}. \tag{4.15}$$

Now the v curvature tensor P_{hijk} ([19, 20]) is written as

$$P_{hijk} = \theta_{(hi)}(C_{ijk|h} + C^r_{hj}C_{rik|0}),$$

$$\begin{aligned} C^r_{hj}C_{rik|0} &= (u_{-1})^2 a^r_{hj}a_{rik|0} + \frac{1}{2}a_{hik|0}K_j - \frac{1}{2}La_{ik|0}(l_hK_j \\ &+l_jK_h) + \frac{1}{2}a_{jik|0}K_h + \frac{d_{-2}}{2(m_{-2})^3}a_{rik|0}K^rK_jK_h + \frac{1}{2}h_{jh}K^ra_{rik}. \end{aligned} \tag{4.16}$$

Thus

$$\begin{aligned} P_{hijk} &= \theta_{(hi)}[a_{ijk|h} + \frac{1}{2}a_{ijk|0}K_h \\ &-\frac{1}{2L}a_{ik|0}(l_hK_j + l_jK_h) + a_{rik|0}K^rH_{jh} + a_{rik|0}A^r_{hj}], \end{aligned} \tag{4.17}$$

where $A^r_{hj} = (u_{-1})^2a^r_{hj} + \frac{d_{-2}}{b(m_{-2})^3}a_{rik|0}K^rK_jK_h$

$$\begin{aligned} S_{jikh} &= \frac{1}{4(u_{-1})^2}\theta_{(kh)}[4(u_{-1})^4a^s_{jh}a_{sik} + 2(u_{-1})^2(a_{sik}K_sH_{jh} \\ &+a^s_{jh}K_sH_{ik} - (l_jK_h + l_hK_j)A_{ik} - (l_iK_k + l_kK_i)A_{jh} \\ &+H^{'}_{ik}K_jK_h + H^{'}_{jh}K_iK_k)]. \end{aligned} \tag{4.18}$$

From equation (3.5) and equation (4.12), we have

$$\begin{aligned} S^h_{ijk} &= S_{sijk}g^{sh} = \frac{1}{4u^3_{-1}}\theta_{(jk)}\{4u^4_{-1}a^r_{hk}a_{rij} + 2u^2_{-1}(H^h_ka_{rij}K^r + a^{rh}_kK_rH_{ij}) \\ &-(l^hK_k + l_kK^h)A_{ij} - (l_iK_j + l_jK_i)A^h_k + H^{'}_{ij}K_kK^h + H^{'h}_kK_iK_j\} \\ &-[\frac{1}{4u^2_{-1}\{z_2\phi^2}\hat{B}^s\hat{B}^h + z_0\phi(\hat{B}^sa^h + \hat{B}^ha^s) \\ &+z_2a^sa_h\}\theta_{(jk)}[4u^4_{-1}a^r_{sk}a_{rij} + 2u^2_{-1}(a_{rij}K_rH_{sk} + a^r_{sk}K_rH_{ij}) \\ &-(l_sK_k + l_kK_s)A_{ij} - (l_iK_j + l_jK_i)A_{sk} \\ &+H^{'}_{sk}K_sK_k + H^{'}_{hk}K_iK_j]\}], \end{aligned} \tag{4.19}$$

$$\begin{aligned} S^h_{ijk} &= \frac{1}{(4(u_{-1})^3)}\theta_{(jk)}4(u_{-1})^4 a^r_{hk}a_{rij} + 2(u_{-1})^2(H^h_k a_{rij}K^r \\ &+a^{rh}_k K_r H_{ij}) - (l^h K_k + l_k K^h)A_{ij} - (l_i K_j + l_j K_i A^h_k + H^{'}_{ij}K_k K^h \\ &+H^{'h}_k K_i K_j - \frac{1}{(4(u_(- 1)^2)}M^h_{ijk}, \end{aligned} \tag{4.20}$$

where

$$\begin{aligned} M^h_{ijk} &= [\theta_{(jk)}\{z_2\phi^2\hat{B}^s\hat{B}^h + z_0\phi(\hat{B}^s a^h + \hat{B}^h a^s) \\ &+z_2 a^s a_h\}\{4(u_{-1})^4 a^r_{sk}a_{rij} + 2(u_{-1})^2(a_{rij}K_r H_{sk} \\ &+a^r_{sk}K_r H_{ij}) - (l_s K_k + l_k K_s)A_{ij} - (l_i K_j + l_j K_i)A_{sk} \\ &+H^{'}_{sk}K_s K_k + H^{'}_{hk}K_i K_j\}], \end{aligned} \tag{4.21}$$

$$K_s a^{sh} = K^h, l_s a^{sh} = l^h, H_{sk}a^{sh} = H^h_k, A_{sk}a^{hs} = A^h_k. \tag{4.22}$$

From equation (4.17) and equation (3.5), we have

$$\begin{aligned} P^h_{ijk} &= P_{sijk}g^{sh} = \frac{1}{u_{(-1)}}\theta_{(hi)}[a_{ijk|s}a^{sh} + \frac{1}{2}a_{ijk|0}K^h - \frac{1}{2}La_{ik|0}(l^h K_j \\ &+K_j K^h) + a_{rik|0}K^r H^h_j + a_{rik|0}A^{rh}_j] - [\theta_{(hi)}\{z_2\phi^2\hat{B}^s\hat{B}^h + z_0\phi(\hat{B}^s a^h \\ &+\hat{B}^h a^s) + z_2 a^s a_h\}\{a_{ijk|s} + \frac{1}{2}a_{ijk|0}K_s - \frac{1}{2}La_{ik|0}(l_s K_j + l_j K_s \\ &+a_{ris|0}K^r H_{js} + a_{rik|0}A^r_{hj}\}], \end{aligned} \tag{4.23}$$

$$\begin{aligned} P^h_{ijk} &= \frac{1}{u_{-1}}\theta_{(hi)}[a_{ijk|s}a^{sh} + \frac{1}{2}a_{ijk|0}K^h - \frac{1}{2L}a_{ik|0}(l^h K_j + K_j K^h) \\ &+a_{rik|0}K^r H^h_j + a_{rik|0}A^{rh}_j] - N^h_{ijk}, \end{aligned} \tag{4.24}$$

where,

$$\begin{aligned} N^h_{ijk} &= [\theta_{(hi)}z_2\phi^2\hat{B}^s\hat{B}^h + z_0\phi(\hat{B}^s a^h + \hat{B}^h a^s) + z_2 a^s a_h a_{ijk|s} \\ &+\frac{1}{2}a_{ijk|0}K_s - \frac{1}{2L}a_{ik|0}(l_s K_j + l_j K_s) + a_{ris|0}K^r H_{js} + a_{rik|0}A^r_{hj}]. \end{aligned} \tag{4.25}$$

§5. **Concluding Remarks**

The above discussed applications may be considered as Finslerian extension of the Cubic root structure of space-time. The important results and properties associated with Cartan's tensor have been presented in section 4. Here we may observe that when C_{ijk} is equal to zero then the metric tensor g_{ij} is reduced to the Reimannian one. Historically we may say that y-dependent as discussed in the above sections has been combined with the concept of anisotropy. As we know that the cosmological constant problem of general relativity can be extended to locally anisotropic spaces with Finslerian structure. According to S. Weinberg [22] everything that contributes to the energy density at the vacuum acts just like a cosmological constant. In the Finslerian framework of space-time the anisotropic form of the microwave background radiation may contribute to that content, if we consider a metric of the form of Eq. (2.1).

The field equations in a Finslerian space-time are to be obtained from a variational principle. We observed that for the similar metric Stavrions and Diakogiannis [16] have also obtained the relationship between the anisotropic cosmological models of space time and Randers Finslerian metric. Here it is further mentioned that the Finslerian geodesics satisfy the Euler-Lagrange equations of geodesics

$$\frac{d^2x^b}{ds^2} + \Gamma^b_{ij}y^iy^j + \sigma(x)r^{ae}(\partial_j\hat{b}_e - \partial_e\hat{b}_j)y^j = 0.$$

In this equation we observe the additional term $r^{ae}(\partial_j\hat{b}_e - \partial_e\hat{b}_j)y^j = 0$.
where $\sigma = \sqrt{r_{if}y^iy^j}$ and Γ^b_{ij} are the cubic Christoffel symbols. This term expresses a rotation of the anisotropy. We may say that the equations of geodesics of the cubic space-time may be generalized as shown in the above calculations. It is also mentioned that if y^i represents the velocity of a particle (time like) then $\hat{b}^i$ is bound to be space like. This follows from the fact that one possible value of $\hat{b}^iy^i$is zero. All the above connections, in which the matter density is hidden, can be considered as a property of the field itself. A weak F^n space-time is proposed for the study and detection of gravitational waves, in virtue of the equation of deviation of geodesics. We have already considered an interesting class of F^n.

References

[1] Matsumoto M. and Numata S., On Finsler spaces with cubic metric, *Tensor N. S.*, 33(1979), 153-162.

[2] Randers G., On an asymmetric metric in the four-space of general relativity, *Phy. Rev.*, (2) 59(1941), 195-199.

[3] Pandey T. N., Chaubey V. K. and Prasad B. N., Scalar curvature of two-dimensional cubic Finsler spaces, *J. International Acad. Of Phys. Sciences*, 12(2008), 127-137.

[4] Kropina V. K., Projective two-dimensional Finsler spaces with specal metric (Rus-sian), *Trudy Sem. Vektor. Tenzor. Anal.*, 11(1961), 277-292.

[5] Wagner V. V., Two- dimensional space with the metric defined by a cubic differential form (Russian and English), *Abh.Tschemy*, Staatuniv, Saratow 1,(1938), 29-40.

[6] Wagner V. V., On generalized Berwald spaces, *C. R. Dokl. Acad. Sci., URSS, N.S.*, 39, (1943), 3-5.

[7] Wegener J. M., Untersuchung der zwei-und dreidimensionalen Finslerschen Raume mit der Grundform $L = (a_{ikl}x^ix^kx^l)^{\frac{1}{3}}$, *Akad. Wetensch. Proc.*, 38, (1935), 949-955.

[8] Wegener J. M., Untersuchung uber Finslersche Raume, *Lotos Prag*, 84, (1936), 4-7.

[9] Zhe Chang and Xin Li, Modified Friedmann model in Randers- Finsler space of approviate Berwald type as a possible alternative to dark energy hypothesis, *Pys.lett.*, B676(2009), 173-176; arXiv :0901.1023.

[10] Kouretsis A.P., Stathakopoulos M. and Stavrinos P.C., The general very special relativity

in Finsler cosmology, *Phys.Rev.*, D.79 104011;(2009),1-17, ar Xiv: 0810.3267v3.

[11] Stavrinos P.C., Kouretsis A.P. and Stathakopoulos M., Friedamnn Robertson- Walker model in generalised metric space-time with weak anisotrophy, *Gen.Rel.Grav.*, 40, 2008, 1403-1425 gr-qc / 0612157.

[12] Pande H.D, Chandra R.C.and Mishra R.K, Cosmological models with variable cosmological constant and gravitational constant, *Indian Journal of Pure. Appl. Math.* 31(2), Feb 2000, 161-175,

[13] Chawla Chanchal and Mishra R.K., Bianchi type-I viscous fluid cosmological models with variable deceleration parameter, *Rom. Journ. Phys.*, Vol.58, No. 1-2, (2013), 75-85.

[14] Mishra R.K, Pradhan, A. and Chawla Chanchal, Anisotropic viscous fluid cosmological models from deceleration to acceleration in string cosmology, *Int. J. Theor. Phys*, 52(2013), 2546-2559.

[15] Chawla Chanchal, Mishra R.K. and Pradhan A., String cosmological models from early deceleration to current acceleration phase with varying G and Lambda, *Eur. Phys. J. Plus* , 2012, 127-137, Doi: 10.1140/epjp/i2012-12137-4.

[16] Stavrinos P.C. and Diakogiannis F.I., A geometrical anisotropic model of space- time based on Finslerian metric, *Proceeding of the conference of Applied Differential Geometry- general relativity and the workshop on Global analysis , Differential Geometry and Lie Algebras*, 2001, 165-178.

[17] Miron R. and Anastasiei M., *The Geometry of Lagrange spaces: Theory and applications*, Kluwer academic Publishers, 1994.

[18] Miron R., Watanabe S. and Ikeda S., Some connections on tangent bundle and their applications to the general relativity, *Tensor, N.S.*, 46, 1987.

[19] Matsumoto M., *Foundations of Finsler geometry and special Finsler spaces*, Kaiseisha Press, Saikawa, Otsu, 1986, Japan.

[20] Antonelli P.L., Ingarden R. S. and Matsumoto, M., *The theory of sprays and Finsler spaces with applications in physics and biology*, Kluwer Academic Publishers, 1993, Dor-drecht.

[21] Pandey T.N. and Chaubey V.K., Theory of Finsler space with (γ, β)- metrics, *Bulletin of Transilvania University of Brasov*, Vol 4(53), No. 2, 2011, 43-56.

[22] Weinberg S, The cosmological constant problem, *Reviews of Modern Physics*, 61, No.1 (1989), 1-23.