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The Kropina-Randers Change of Finsler Metric and Relation Between Imbedding Class Numbers of Their Tangent Riemannian Spaces

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Abstract: In the present paper the relation between imbedding class numbers of tangent Riemannian spaces of (M^n, L) and (M^n, L^*) have been obtained, where the Finsler metric L^* is obtained from L by $L^* = \frac{L^2}{\beta} + \beta$ and M^n is the differentiable manifold.

Key Words: Finsler metric, Randers change, Kropina change, Kropina-Randers change, imbedding class.

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§1. Introduction

Let (M^n, L) be an n-dimensional Finsler space on a differentiable manifold M^n, equipped with the fundamental function $L(x, y)$. In 1971, Matsumoto [2] introduced the transformation of Finsler metric:

$$L^*(x, y) = L(x, y) + \beta(x, y) \tag{1.1}$$

where $\beta(x, y) = b_i(x)y^i$ is a differentiable one-form on M^n. In 1984 Shibata [6] has studied the properties of Finsler space (M^n, L^*) whose metric function $L^*(x, y)$ is obtained from $L(x, y)$ by the relation $L^*(x, y) = f(L, \beta)$ where f is positively homogeneous of degree one in L and β. This change of metric function is called a $\beta-$change. The change (1.1) is a particular case of $\beta-$change called Randers change. The following theorem has importance under Randers change.

Theorem (1.1)([2]) *Let (M^n, L^*) be a locally Minkowskian n-space obtained from a locally Minkowskian n-space (M^n, L) by the change (1.1). If the tangent Riemannian n-space (M^n_x, g_x) to (M^n, L) is of imbedding class r, then tangent Riemannian n-space (M^n_x, g^*_x) to (M^n, L^*) is of imbedding class at most $r + 2$.*

In [5] it has been proved that Theorem (1.1) is valid for Kropina change of Finsler metric

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function given by

$$L^*(x,y) = \frac{L^2(x,y)}{\beta(x,y)}. \tag{1.2}$$

In 1990, Prasad, Shukla and Singh [4] proved that Theorem (1.1) is valid for the transformation given by (1.1) in which $b_i(x)$ in β is replaced by h-vector $b_i(x,y)$ such that $\frac{\partial b_i}{\partial y^j}$ is proportional to angular metric tensor.

Recently Prasad, Shukla and Pandey [3] have proved that Theorem (1.1) is also valid for exponential change of Finsler metric given by

$$L^*(x,y) = Le^{\beta/L}.$$

In the present paper we consider Kropina-Randers change of Finsler metric given by

$$L^* = \frac{L^2}{\beta} + \beta$$

and prove that Theorem (1.1) is valid for this transformation also.

§2. The Finsler Space (M^n, L^*)

Let (M^n, L) be a given Finsler space and let $b_i(x)dx^i$ be a one-form on M^n. We shall define on M^n a function $L^*(x,y)\,(>0)$ by the equation

$$L^* = \frac{L^2}{\beta} + \beta, \tag{2.1}$$

where we put $\beta(x,y) = b_i(x)y^i$. To find the metric tensor g^*_{ij}, the angular metric tensor h^*_{ij}, the Cartan tensor C^*_{ijk} and the v-curvature tensor of (M^n, L^*) we use the following results:

$$\dot\partial_i\beta = b_i \qquad \dot\partial_i L = l_i, \qquad \dot\partial_j l_i = L^{-1}h_{ij}, \tag{2.2}$$

where $\dot\partial_i$ stands for $\frac{\partial}{\partial y^i}$ and h_{ij} are components of angular metric tensor of (M^n, L) given by $h_{ij} = g_{ij} - l_il_j = L\dot\partial_i\dot\partial_j L$.

The successive differentiation of (2.1) with respect to y^i and y^j gives

$$l^*_i = \frac{2L}{\beta}l_i + \left(1 - \frac{L^2}{\beta^2}\right) b_i, \tag{2.3}$$

$$h^*_{ij} = 2\left(\frac{L^2}{\beta^2} + 1\right)\left\{h_{ij} + l_il_j - \frac{L}{\beta}(l_ib_j + l_jb_i) + \frac{L^2}{\beta^2}\, b_ib_j\right\}. \tag{2.4}$$

From (2.3) and (2.4) we get the following relation between metric tensors of (M^n, L) and (M^n, L^*):

$$g^*_{ij} = 2\left(\frac{L^2}{\beta^2} + 1\right) g_{ij} + \frac{4L^2}{\beta^2}l_il_j + \left(\frac{3L^4}{\beta^4} + 1\right) b_ib_j - \frac{4L^3}{\beta^3}(l_ib_j + l_jb_i). \tag{2.5}$$

The contravariant components of the metric tensor of (M^n, L^*) is derived from (2.5) and are given by

$$g^{*ij} = \frac{\beta^2}{2(\beta^2+L^2)}g^{ij} - \frac{\beta^2\{\beta^2(\beta^2+L^2)+\triangle L^2(L^2-\beta^2)\}}{b^2(\beta^2+L^2)^3}l^il^j + \frac{\beta^3 L}{b^2(\beta^2+L^2)^2}(l^ib^j+l^jb^i) - \frac{\beta^2}{2b^2(\beta^2+L^2)}b^ib^j \tag{2.6}$$

where we put $l^i = g^{ij}l_j$, $b^i = g^{ij}b_j$, $b^2 = g^{ij}b_ib_j$ and $\triangle = b^2 - \dfrac{\beta^2}{L^2}$.

Differentiating (2.5) with respect to y^k and using (2.2) we get the following relation between the Cartan tensors of (M^n, L) and (M^n, L^*):

$$\begin{aligned} C^*_{ijk} &= \frac{1}{2}\dot{\partial}_k g^*_{ij} \\ &= 2\left(\frac{L^2}{\beta^2}+1\right)C_{ijk} - \frac{2L^2}{\beta^2}(h_{ij}m_k + h_{jk}m_i + h_{ik}m_j) - \frac{6L^4}{\beta^5}m_im_jm_k, \end{aligned} \tag{2.7}$$

where $m_i = b_i - \dfrac{\beta}{L}l_i$. It is to be noted that

$$m_il^i = 0, \quad m_ib^i = \triangle = m_im^i, \quad h_{ij}l^j = 0, \quad h_{ij}m^j = h_{ij}b^j = m_i, \tag{2.8}$$

where $m^i = g^{ij}m_j = b^i - \dfrac{\beta}{L}l^i$.

The quantities corresponding to (M^n, L^*) will be denoted by putting $*$ on those quantities. To find $C^{*i}_{jk} = g^{*ih}C^*_{jhk}$ we use (2.6), (2.7) and (2.8). We get

$$\begin{aligned} C^{*i}_{jk} &= C^i_{jk} - \frac{L^2}{\beta(\beta^2+L^2)}(h_{jk}m^i + h^i_jm_k + h^i_km_j) \\ &\quad - \frac{\beta^4}{2b^2L^2(\beta^2+L^2)}C_{.jk}n^i - \frac{3L^4}{b^2\beta^3(\beta^2+L^2)}m_jm_km^i \\ &\quad + \frac{\triangle\beta^3}{2b^2(\beta^2+L^2)^2}h_{jk}n^i + \frac{\beta(2\beta^2+3\triangle L^2)}{2b^2(\beta^2+L^2)^2}m_jm_kn^i, \end{aligned} \tag{2.9}$$

where $n^i = \dfrac{2L^2}{\beta^4}\{(\beta^2+L^2)b^i - 2\beta L l^i\}$ and $C_{.jk} = C_{hjk}b^h$.

Throughout this paper we use the symbol . to denote the contraction with b^i. To find the v-curvature tensor of (M^n, L^*) we use the following:

$$\begin{aligned} &C_{ijk}m^i = C_{.ij}, \quad C_{ijk}n^i = \frac{2L^2}{\beta^4}(\beta^2+L^2)C_{.jk}, \\ &m_in^i = \frac{2\triangle L^2}{\beta^4}(\beta^2+L^2), \quad m^im_i = \triangle, \\ &h_{ij}n^i = \frac{2L^2}{\beta^4}(\beta^2+L^2)m_j, \quad C^h_{ij}h_{hk} = C_{ijk}, \quad h^r_jh^i_r = h^i_j. \end{aligned} \tag{2.10}$$

The v-curvature tensor S^*_{hijk} of (M^n, L^*) is defined as

$$S^*_{hijk} = C^{*r}_{hk}C^*_{rij} - C^{*r}_{hj}C^*_{rik} \tag{2.11}$$

From (2.7), (2.8), (2.9), (2.10) and (2.11) we get the following relation between v-curvature tensors of (M^n, L) and (M^n, L^*):

$$S^*_{hijk} = 2\left(\frac{L^2}{\beta^2} + 1\right) S_{hijk} + d_{hj}d_{ik} - d_{hk}d_{ij} + E_{hk}E_{ij} - E_{hj}E_{ik}, \tag{2.12}$$

where

$$d_{ij} = \frac{\sqrt{2(\beta^2 + L^2)}}{b\beta}\left[C_{.ij} + \frac{\beta}{\beta^2 + L^2}h_{ij} + \frac{2L^2}{\beta(\beta^2 + L^2)}m_i m_j\right], \tag{2.13}$$

$$E_{ij} = -\frac{\sqrt{2}L}{\beta\sqrt{\beta^2 + L^2}}\left[h_{ij} + \frac{2L^2}{\beta^2}m_i m_j\right]. \tag{2.14}$$

By direct calculation we get the following results which will be used in the latter section of the paper:

$$\dot\partial_i b^2 = -2C_{..i}, \qquad \dot\partial_i \triangle = -2C_{..i} - \frac{2\beta}{L^2}m_i. \tag{2.15}$$

§3. Imbedding Class Numbers

The tangent vector space M^n_x to M^n at every point x is considered as the Riemannian n-space (M^n_x, g_x) with the Riemannian metric $g_x = g_{ij}(x,y)dy^i dy^j$. Then the components of the Cartan tensor are the Christoffel symbols associated with g_x:

$$C^i_{jk} = \frac{1}{2}g^{ih}(\dot\partial_k g_{jh} + \dot\partial_j g_{hk} - \dot\partial_h g_{jk}).$$

Thus C^i_{jk} defines the components of the Riemannian connection on M^n_x and v-covariant derivative, say

$$X_i|_j = \dot\partial_j X_i - X_h C^h_{ij} \tag{3.0}$$

is the covariant derivative of covariant vector X_i with respect to Riemannian connection C^i_{jk} on M^n_x. It is observed that the v-curvature tensor S_{hijk} of (M^n, L) is the Riemannian Christoffel curvature tensor of the Riemannian space (M^n, g_x) at a point x. The space (M^n, g_x) equipped with such a Riemannian connection is called the tangent Riemannian n-space [2].

It is well known [1] that any Riemannian n-space V^n can be imbedded isometrically in a Euclidean space of dimension $\frac{n(n+1)}{2}$. If $n+r$ is the lowest dimension of the Euclidean space in which V^n is imbedded isometrically, then the integer r is called the imbedding class number of V^n. The fundamental theorem of isometric imbedding ([1] page 190) is that the tangent Riemannian n-space (M^n_x, g_x) is locally imbedded isometrically in a Euclidean $(n+r)-$space if and only if there exist $r-$number $\epsilon_P = \pm 1$, $r-$symmetric tensors $H_{(P)ij}$ and $\frac{r(r-1)}{2}$ covariant vector fields $H_{(P,Q)i} = -H_{(Q,P)i}$; $P, Q = 1, 2, \cdots, r$, satisfying the Gauss equations

$$S_{hijk} = \sum_P \epsilon_P\{H_{(P)hj}H_{(P)ik} - H_{(P)ij}H_{(P)hk}\}, \tag{3.1}$$

The Codazzi equations

$$H_{(P)ij}|_k - H_{(P)ik}|_j = \sum_Q \epsilon_Q\{H_{(Q)ij}H_{(Q,P)k} - H_{(Q)ik}H_{(Q,P)j}\}, \tag{3.2}$$

and the Ricci-Kühne equations

$$\begin{aligned} H_{(P,Q)i}|_j - H_{(P,Q)j}|_i \quad &+ \quad \sum_R \epsilon_R\{H_{(R,P)i}H_{(R,Q)j} - H_{(R,P)j}H_{(R,Q)i}\} \\ &+ \quad g^{hk}\{H_{(P)hi}H_{(Q)kj} - H_{(P)hj}H_{(Q)ki}\} = 0. \end{aligned} \tag{3.3}$$

The numbers $\epsilon_P = \pm 1$ are the indicators of unit normal vector N_P to M^n and $H_{(P)ij}$ are the second fundamental tensors of M^n with respect to the normals N_P.

Proof of Theorem (1.1) In order to prove Theorem (1.1), we put

$$\begin{aligned} &\text{(a)} \quad H^*_{(P)ij} = \left[2\left(\frac{L^2}{\beta^2}+1\right)\right]^{1/2} H_{(P)ij}, \qquad \epsilon^*_P = \epsilon_P, \quad P = 1, 2, \cdots, r \\ &\text{(b)} \quad H^*_{(r+1)ij} = d_{ij}, \qquad \epsilon^*_{r+1} = 1 \\ &\text{(c)} \quad H^*_{(r+2)ij} = E_{ij}, \qquad \epsilon^*_{r+2} = -1. \end{aligned} \tag{3.4}$$

Then it follows from (2.12) and (3.1) that

$$S^*_{hijk} = \sum_{\lambda=1}^{r+2} \epsilon^*_\lambda \{H^*_{(\lambda)hj}H^*_{(\lambda)ik} - H^*_{(\lambda)hk}H^*_{(\lambda)ij}\},$$

which is the Gauss equation of (M^n_x, g^*_x).

Moreover, to verify Codazzi and Ricci Kühne equation of (M^n_x, g^*_x), we put

$$\begin{aligned} &\text{(a)} \quad H^*_{(P,Q)i} = -H^*_{(Q,P)i} = H_{(P,Q)i}, \quad P, Q = 1, 2, , \cdots, r \\ &\text{(b)} \quad H^*_{(P,r+1)i} = -H^*_{(r+1,P)i} = \frac{1}{b}H_{(P).i}, \quad P = 1, 2, \cdots, r \\ &\text{(c)} \quad H^*_{(P,r+2)i} = -H^*_{(r+2,P)i} = 0, \quad P = 1, 2, \cdots, r. \\ &\text{(d)} \quad H^*_{(r+1,r+2)i} = -H^*_{(r+2,r+1)i} = -\frac{L^2\sqrt{2}}{b(\beta^2+L^2)\sqrt{\beta L}} m_i. \end{aligned} \tag{3.5}$$

The Codazzi equations of (M^n_x, g^*_x) consists of the following three equations:

$$\begin{aligned} \text{(a)} \quad H^*_{(P)ij}\overset{*}{|}_k - H^*_{(P)ik}\overset{*}{|}_j = &\sum_Q \epsilon^*_Q\{H^*_{(Q)ij}H^*_{(Q,P)k} - H^*_{(Q)ik}H^*_{(Q,P)j}\} \\ &+ \epsilon^*_{r+1}\{H^*_{(r+1)ij}H^*_{(r+1,P)k} - H^*_{(r+1)ik}H^*_{(r+1,P)j}\} \\ &+ \epsilon^*_{r+2}\{H^*_{(r+2)ij}H^*_{(r+2,P)k} - H^*_{(r+2)ik}H^*_{(r+2,P)j}\} \end{aligned} \tag{3.6}$$

$$\begin{aligned} \text{(b)} \quad H^*_{(r+1)ij}\overset{*}{|}_k - H^*_{(r+1)ik}\overset{*}{|}_j = &\sum_Q \epsilon^*_Q\{H^*_{(Q)ij}H^*_{(Q,r+1)k} - H^*_{(Q)ik}H^*_{(Q,r+1)j}\} \\ &+ \epsilon^*_{r+2}\{H^*_{(r+2)ij}H^*_{(r+2,r+1)k} - H^*_{(r+2)ik}H^*_{(r+2,r+1)j}\} \\ \text{(c)} \quad H^*_{(r+2)ij}\overset{*}{|}_k - H^*_{(r+2)ik}\overset{*}{|}_j = &\sum_Q \epsilon^*_Q\{H^*_{(Q)ij}H^*_{(Q,r+2)k} - H^*_{(Q)ik}H^*_{(Q,r+2)j}\} \\ &+ \epsilon^*_{r+1}\{H^*_{(r+1)ij}H^*_{(r+1,r+2)k} - H^*_{(r+1)ik}H^*_{(r+1,r+2)j}\}. \end{aligned}$$

To prove these equations we note that for any symmetric tensor X_{ij} satisfying $X_{ij}l^i = X_{ij}l^j = 0$, we have from (2.9) and (3.0),

$$\begin{aligned} X_{ij}\overset{*}{|}_k - X_{ik}\overset{*}{|}_j &= X_{ij}|_k - X_{ik}|_j + \frac{1}{b^2}\{C_{.ik}X_{.j} - C_{.ij}X_{.k}\} \\ &\quad + \frac{L^2}{\beta(L^2+\beta^2)}(X_{ij}m_k - X_{ik}m_j) + \frac{L^2}{b^2\beta(L^2+\beta^2)} \\ &\quad \times (X_{.j}m_k - X_{.k}m_j)m_i + \frac{\beta}{b^2(L^2+\beta^2)}(h_{ik}X_{.j} - h_{ij}X_{.k}). \end{aligned} \tag{3.7}$$

In view of (3.4) and (3.5), equation (3.6)a is equivalent to

$$\begin{aligned} &\left(\sqrt{2\left(\frac{L^2}{\beta^2}+1\right)}.H_{(P)ij}\right)\overset{*}{\Big|}_k - \left(\sqrt{2\left(\frac{L^2}{\beta^2}+1\right)}.H_{(P)ik}\right)\overset{*}{\Big|}_j \\ &= \sqrt{2\left(\frac{L^2}{\beta^2}+1\right)}.\sum_Q \epsilon_Q\{H_{(Q)ij}H_{(Q,P)k} - H_{(Q)ik}H_{(Q,P)j}\} \\ &\quad -\frac{1}{b}\{H_{(P).k}d_{ij} - H_{(P).j}d_{ik}\}. \end{aligned} \tag{3.8}$$

Since $\left(\sqrt{2\left(\frac{L^2}{\beta^2}+1\right)}\right)\overset{*}{\Big|}_k = \dot{\partial}_k\left(\sqrt{2\left(\frac{L^2}{\beta^2}+1\right)}\right) = -\dfrac{\sqrt{2}\,L^2}{\beta^2\sqrt{L^2+\beta^2}}\,m_k$, applying formula (3.7) for $H_{(P)ij}$ and using equation (2.13), we get

$$\begin{aligned} &\left(\sqrt{2\left(\frac{L^2}{\beta^2}+1\right)}.H_{(P)ij}\right)\overset{*}{\Big|}_k - \left(\sqrt{2\left(\frac{L^2}{\beta^2}+1\right)}.H_{(P)ik}\right)\overset{*}{\Big|}_j = \sqrt{2\left(\frac{L^2}{\beta^2}+1\right)} \\ &\quad \times\{H_{(P)ij}|_k - H_{(P)ik}|_j\} - \frac{1}{b}\{H_{(P).k}d_{ij} - H_{(P).j}d_{ik}\}, \end{aligned}$$

which after using equation (3.2), gives equation (3.8).

In view of (3.4) and (3.5), equation (3.6)b is equivalent to

$$\begin{aligned} d_{ij}\overset{*}{|}_k - d_{ik}\overset{*}{|}_j &= \frac{\sqrt{2(\beta^2+L^2)}}{b\beta}\sum_Q \epsilon_Q\{H_{(Q)ij}H_{(Q).k} - H_{(Q)ik}H_{(Q).j}\} \\ &\quad -\frac{L^2\sqrt{2}}{b(\beta^2+L^2)\sqrt{\beta L}}\{E_{ij}m_k - E_{ik}m_j\}. \end{aligned} \tag{3.9}$$

To verify (3.9), we note that

$$C_{.ij}|_k - C_{.ik}|_j = b_h S^h_{ijk} \tag{3.10}$$

$$b|_k = -\frac{1}{b}C_{..k}, \qquad h_{ij}|_k - h_{ik}|_j = L^{-1}(h_{ij}l_k - h_{ik}l_j) \tag{3.11}$$

$$m_i|_k = -C_{.ik} - \frac{\beta}{L^2}h_{ik} - L^{-1}l_i m_k. \tag{3.12}$$

The v-covariant differentiation of (2.13) will give the value of $d_{ij}|_k$. Then taking skew-symmetric part of $d_{ij}|_k$ in j and k, we get

$$\begin{aligned} d_{ij}|_k - d_{ik}|_j &= A(C_{.ij}|_k - C_{.ik}|_j) + B(h_{ij}|_k - h_{ik}|_j) + D(m_i|_k m_j \\ &\quad + m_j|_k m_i - m_i|_j m_k - m_k|_j m_i) + (\dot{\partial}_k A)C_{.ij} - (\dot{\partial}_j A)C_{.ik} \\ &\quad + (\dot{\partial}_k B)h_{ij} - (\dot{\partial}_j B)h_{ik} + (\dot{\partial}_k D)m_i m_j - (\dot{\partial}_j D)m_i m_k, \end{aligned} \tag{3.13}$$

where $A = \dfrac{\sqrt{2(\beta^2 + L^2)}}{b\beta}$, $B = \dfrac{\sqrt{2}}{b\sqrt{(\beta^2 + L^2)}}$, $D = \dfrac{4L^2}{b\beta^2\sqrt{2(\beta^2 + L^2)}}$.

Applying formula (3.7) for d_{ij} and using (3.13), (3.10), (3.11), (3.12), (2.15), we get

$$\begin{aligned} d_{ij}\overset{*}{|}_k - d_{ik}\overset{*}{|}_j &= \frac{\sqrt{2(\beta^2 + L^2)}}{b\beta} b_h S^h_{ijk} \\ &+ \frac{2L^3}{b\beta(L^2 + \beta^2)^{3/2}\sqrt{\beta L}}(h_{ij}m_k - h_{ik}m_j). \end{aligned} \tag{3.14}$$

Substituting (3.1) and (2.14) in the right hand side of (3.14), we get equation (3.9).

In view of (3.4) and (3.5), equation (3.6)c is equivalent to

$$E_{ij}\overset{*}{|}_k - E_{ik}\overset{*}{|}_j = \frac{\sqrt{2}L^2}{b(L^2 + \beta^2)\sqrt{\beta L}}(d_{ij}m_k - d_{ik}m_j). \tag{3.15}$$

The v-covariant differentiation of (2.14) and use of (2.15) will give the value of $E_{ij}|_k$. Then taking skew-symmetric part of $E_{ij}|_k$ in j and k and using (3.11), (3.12), we get

$$\begin{aligned} E_{ij}|_k - E_{ik}|_j &= -\frac{2\sqrt{2}L^3}{\beta^3\sqrt{\beta^2 + L^2}}(C_{.ij}m_k - C_{.ik}m_j) \\ &- \frac{2\sqrt{2}L^3}{\beta^2(L^2 + \beta^2)^{3/2}}(h_{ij}m_k - h_{ik}m_j). \end{aligned} \tag{3.16}$$

Applying formula (3.7) for E_{ij} and using (3.16), we get (3.15). This completes the proof of Codazzi equations of (M^n_x, g^*_x).

The Ricci Kühne equations of (M^n_x, g^*_x) consist of the following four equations:

$$\begin{aligned} \text{(a)}\quad & H^*_{(P,Q)i}\overset{*}{|}_j - H^*_{(P,Q)j}\overset{*}{|}_i + \sum_Q \epsilon^*_Q\{H^*_{(R,P)i}H^*_{(R,Q)j} \\ & - H^*_{(R,P)j}H^*_{(R,Q)i}\} + \epsilon^*_{r+1}\{H^*_{(r+1,P)i}H^*_{(r+1,Q)j} \\ & - H^*_{(r+1,P)j}H^*_{(r+1,Q)i}\} + \epsilon^*_{r+2}\{H^*_{(r+2,P)i}H^*_{(r+2,Q)j} \\ & - H^*_{(r+2,P)j}H^*_{(r+2,Q)i}\} + g^{*hk}\{H^*_{(P)hi}H^*_{(Q)kj} \\ & - H^*_{(P)hj}H^*_{(Q)ki}\} = 0, \qquad P, Q = 1, 2, \cdots, r \end{aligned} \tag{3.17}$$

$$\begin{aligned} \text{(b)}\quad & H^*_{(P,r+1)i}\overset{*}{|}_j - H^*_{(P,r+1)j}\overset{*}{|}_i + \sum_R \epsilon^*_R\{H^*_{(R,P)i}H^*_{(R,r+1)j} - H^*_{(R,P)j}H^*_{(R,r+1)i}\} \\ & + \epsilon^*_{r+2}\{H^*_{(r+2,P)i}H^*_{(r+2,r+1)j} - H^*_{(r+2,P)j}H^*_{(r+2,r+1)i}\} \\ & + g^{*hk}\{H^*_{(P)hi}H^*_{(r+1)kj} - H^*_{(P)hj}H^*_{(r+1)ki}\} = 0, \quad P = 1, 2, \cdots, r \end{aligned}$$

$$\begin{aligned} \text{(c)}\quad & H^*_{(P,r+2)i}\overset{*}{|}_j - H^*_{(P,r+2)j}\overset{*}{|}_i + \sum_R \epsilon^*_R\{H^*_{(R,P)i}H^*_{(R,r+2)j} - H^*_{(R,P)j}H^*_{(R,r+2)i}\} \\ & + \epsilon^*_{r+1}\{H^*_{(r+1,P)i}H^*_{(r+1,r+2)j} - H^*_{(r+1,P)j}H^*_{(r+1,r+2)i}\} \\ & + g^{*hk}\{H^*_{(P)hi}H^*_{(r+2)kj} - H^*_{(P)hj}H^*_{(r+2)ki}\} = 0, \quad P = 1, 2, \cdots, r \end{aligned}$$

$$\begin{aligned} \text{(d)}\quad & H^*_{(r+1,r+2)i}\overset{*}{|}_j - H^*_{(r+1,r+2)j}\overset{*}{|}_i + \sum_R \epsilon^*_R\{H^*_{(R,r+1)i}H^*_{(R,r+2)j} - H^*_{(R,r+1)j} \\ & \times H^*_{(R,r+2)i}\} + g^{*hk}\{H^*_{(r+1)hi}H^*_{(r+2)kj} - H^*_{(r+1)hj}H^*_{(r+2)ki}\} = 0. \end{aligned}$$

In view of (3.4) and (3.5), equation (3.17)a is equivalent to

$$H_{(P,Q)i}\overset{*}{|}_j - H_{(P,Q)j}\overset{*}{|}_i + \sum_R \epsilon_R\{H_{(R,P)i}H_{(R,Q)j} - H_{(R,P)j}H_{(R,Q)i}\} \tag{3.18}$$
$$+\frac{1}{b^2}\{H_{(P).i}H_{(Q).j} - H_{(P).j}H_{(Q).i}\} + g^{*hk}\{H_{(P)hi}H_{(Q)kj}$$
$$-H_{(P)hj}H_{(Q)ki}\}.2\left(\frac{L^2}{\beta^2}+1\right) = 0. \qquad P,Q = 1,2,\cdots,r.$$

Since $H_{(P)ij}l^i = 0 = H_{(P,Q)i}l^i$, from (2.6), we get

$$g^{*hk}\{H_{(P)hi}H_{(Q)kj} - H_{(P)hj}H_{(Q)ki}\}\left(\frac{L^2}{\beta^2}+1\right) = g^{hk}\left(\frac{L^2}{\beta^2}+1\right)\{H_{(P)hi}\times$$
$$H_{(Q)kj} - H_{(P)hj}H_{(Q)ki}\} - \frac{1}{2b^2}\{H_{(P).i}H_{(P).j} - H_{(P).j}H_{(P).i}\}.$$

Also, we have $H_{(P,Q)i}\overset{*}{|}_j - H_{(P,Q)j}\overset{*}{|}_i = H_{(P,Q)i}|_j - H_{(P,Q)j}|_i$. Hence equation (3.18) is satisfied identically by virtue of (3.3).

In view of (3.4) and (3.5), equation (3.17)b is equivalent to

$$\left(\frac{1}{b}H_{(P).i}\right)\overset{*}{\Big|}_j - \left(\frac{1}{b}H_{(P).j}\right)\overset{*}{\Big|}_i + \frac{1}{b}\sum_R \epsilon_R\{H_{(R,P)i}H_{(R).j} - H_{(R,P)j}H_{(R).i}\} \tag{3.19}$$
$$+\,g^{*hk}\{H_{(P)hi}d_{kj} - H_{(P)hj}d_{ki}\}\sqrt{2\left(\frac{L^2}{\beta^2}+1\right)} = 0. \quad P,Q = 1,2,\cdots,r.$$

Since $b^h|_j = -g^{hk}C_{.jk}$, $H_{(P)hi}l^i = 0$, we have

$$H_{(P).i}\overset{*}{|}_j - H_{(P).j}\overset{*}{|}_i = H_{(P).i}|_j - H_{(P).j}|_i = [H_{(P)hi}|_j - H_{(P)hj}|_i]b^h \tag{3.20}$$
$$-g^{hk}\{H_{(P)hi}C_{.kj} - H_{(P)hj}C_{.ki}\}$$

$$\frac{1}{b}\overset{*}{\Big|}_j = \dot{\partial}_j\left(\frac{1}{b}\right) = \frac{1}{b^3}C_{..j} \tag{3.21}$$

and

$$g^{*hk}\{H_{(P)hi}d_{kj} - H_{(P)hj}d_{ki}\}\sqrt{2\left(\frac{L^2}{\beta^2}+1\right)} = \frac{\beta}{\sqrt{2(L^2+\beta^2)}}g^{hk}\times \tag{3.22}$$
$$\{H_{(P)hi}d_{kj} - H_{(P)hj}d_{ki}\} - \frac{\beta}{b^2\sqrt{2(L^2+\beta^2)}}\{H_{(P).i}d_{.j} - H_{(P).j}d_{.i}\}.$$

After using (2.13) the equation (3.22) may be written as

$$g^{*hk}\{H_{(P)hi}d_{kj} - H_{(P)hj}d_{ki}\}\sqrt{2\left(\frac{L^2}{\beta^2}+1\right)} = \frac{1}{b}g^{hk}\times \tag{3.23}$$
$$\{H_{(P)hi}C_{.kj} - H_{(P)hj}C_{.ki}\} - \frac{1}{b^3}\{H_{(P).i}C_{..j} - H_{(P).j}C_{..i}\}.$$

From (3.2), (3.20), (3.21) and (3.23) it follows that equation (3.19) holds identically.

In view of (3.4) and (3.5), equation (3.17)c is equivalent to

$$\frac{\sqrt{2}L^2}{b^2(L^2+\beta^2)\sqrt{\beta L}}\{H_{(P).i}m_j - H_{(P).j}m_i) \tag{3.24}$$
$$+\sqrt{2\left(\frac{L^2}{\beta^2}+1\right)}\,g^{*hk}\{H_{(P)hi}E_{kj} - H_{(P)hj}E_{ki}\} = 0,$$

Since $E_{ij}l^i = E_{ij}l^j = 0$, from (2.5), we have

$$\sqrt{2\left(\frac{L^2}{\beta^2}+1\right)}g^{*hk}\{H_{(P)hi}E_{kj} - H_{(P)hj}E_{ki}\} = \frac{\beta}{\sqrt{2(L^2+\beta^2)}}g^{hk}\times$$
$$\{H_{(P)hi}E_{kj} - H_{(P)hj}E_{ki}\} - \frac{\beta}{b^2\sqrt{2(L^2+\beta^2)}}\{H_{(P).i}E_{.j} - H_{(P).j}E_{.i}\}.$$

In view of (2.14) the right hand side of the last equation is equal to

$$-\frac{\sqrt{2}L^2}{b^2(L^2+\beta^2)\sqrt{\beta L}}\{H_{(P).i}m_j - H_{(P).j}m_i\}.$$

Hence equation (3.24) is satisfied identically.

In view of (3.4) and (e3.5), equation (3.17)d is equivalent to

$$\left(-\frac{\sqrt{2}L^2}{b(L^2+\beta^2)\sqrt{\beta L}}m_i\right)\Big|^{*}_{j} - \left(-\frac{\sqrt{2}L^2}{b(L^2+\beta^2)\sqrt{\beta L}}m_j\right)\Big|^{*}_{i} \tag{3.25}$$
$$+\,g^{*hk}(d_{hi}E_{kj} - d_{hj}E_{ki}) = 0.$$

Since $E_{ij}l^i = 0$, $d_{ij}l^i = 0$, from (2.6), it follows that

$$g^{*hk}\{d_{hi}E_{kj} - d_{hj}E_{ki}\} = \frac{\beta^2}{2(L^2+\beta^2)}g^{hk}\{d_{hi}E_{kj} - d_{hj}E_{ki}\}$$
$$-\frac{\beta^2}{2b^2(L^2+\beta^2)}\{d_{.i}E_{.j} - d_{.j}E_{.i}\}.$$

In view of (2.13) the right hand side of the last equation is equal to

$$-\frac{2L}{b^3(L^2+\beta^2)}\{C_{..i}m_j - C_{..j}m_i\}.$$

Also,

$$\left(-\frac{\sqrt{2}L^2}{b(L^2+\beta^2)\sqrt{\beta L}}m_i\right)\Big|^{*}_{j} - \left(-\frac{\sqrt{2}L^2}{b(L^2+\beta^2)\sqrt{\beta L}}m_j\right)\Big|^{*}_{i}$$
$$= -\frac{\sqrt{2}L^2}{b(L^2+\beta^2)\sqrt{\beta L}}(m_i\overset{*}{|}_j - m_j\overset{*}{|}_i) + \dot{\partial}_j\left(-\frac{\sqrt{2}L^2}{b(L^2+\beta^2)\sqrt{\beta L}}\right)m_i$$
$$-\dot{\partial}_i\left(-\frac{\sqrt{2}L^2}{b(L^2+\beta^2)\sqrt{\beta L}}\right)m_j.$$

Since $m_i\overset{*}{|}_j - m_j\overset{*}{|}_i = L^{-1}(l_j m_i - l_i m_j)$ and

$$\dot{\partial}_j\left(-\frac{\sqrt{2}L^2}{b(L^2+\beta^2)\sqrt{\beta L}}\right) = -\frac{\sqrt{2}L}{b(L^2+\beta^2)\sqrt{\beta L}}l_j - \frac{2L}{b^3(L^2+\beta^2)}C_{..j},$$

we have

$$\left(-\frac{\sqrt{2}L^2}{b(L^2+\beta^2)\sqrt{\beta L}}m_i\right)\overset{*}{\Bigg|}_j - \left(-\frac{\sqrt{2}L^2}{b(L^2+\beta^2)\sqrt{\beta L}}m_j\right)\overset{*}{\Bigg|}_i = -\frac{2L}{b^3(L^2+\beta^2)}\{C_{..j}m_i - C_{..i}m_j\}. \tag{3.26}$$

Hence equation (3.25) is satisfied identically. Therefore Ricci-Kühne equations are satisfied for (M_x^n, g_x^*) given in (3.17) are satisfied.

Hence Theorem (1.1) given in introduction is satisfied for Kropina-Randers change of Finsler metric.

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