International J.Math. Combin. Vol.2(2013), 84-90

The Bisector Surface of Rational Space Curves in Minkowski 3-Space

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Abstract: The aim of this paper is to compare bisector surfaces of rational space curves in Euclidean and Minkowski 3-spaces.

Key Words: Minkowski 3-space, bisector surface, medial surface, Voronoi surface.

AMS(2010): 53B40, 53C60

§1. Introduction

Bisector construction plays an important role in many geometric computations, such as Voronoi diagrams construction, medial axis transformation, shape decomposition, mesh generation, collision-avoidance motion planning, and NC tool path generation (Dutta 1993, Elber 1998, Pottmann 1995, Peternell 2000, Farouki 1994b).

Let $\mathbb{R}^3_1$ be a Minkowski 3-space with Lorentzian metric

$$ds^2 = dx^2 + dy^2 - dz^2 \tag{1}$$

If $\langle X, Y\rangle = 0$ for all X and Y, the vectors X and Y are called perpendicular in the sense of Lorentz, where $\langle , \rangle$ is the induced inner product in $\mathbb{R}^3_1$. The norm of $X \in \mathbb{R}^3_1$ is denoted by $\|X\|$ and defined as

$$\|X\| = \sqrt{|\langle X, X\rangle|} \tag{2}$$

We say that a Lorentzian vector X is spacelike, lightlike or timelike if $\langle X, X\rangle > 0$ and $X = 0$, $\langle X, X\rangle = 0$, $\langle X, X\rangle < 0$, respectively. A smooth regular curve is said to be a timelike, spacelike or lightlike curve if the tangent vector is a timelike, spacelike, or lightlike vector, respectively (Turgut 1998, Turgut 1997, O'Neill 1983) .

For any $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in \mathbb{R}^3_1$, the Lorentz vector product of X and Y is defined as follows:

$$X \wedge Y = (x_2y_3 - x_3y_2,\ x_1y_3 - x_3y_1,\ x_2y_1 - x_1y_2).$$

This yields

[1]Received November 16, 2012, Accepted June 22, 2013.

$$e_1 \wedge e_2 = -e_3, \qquad e_3 \wedge e_1 = -e_2, \qquad e_2 \wedge e_3 = e_1$$

where $\{e_1, e_2, e_3\}$ are the base of the space $\mathbb{R}^3_1$.

§2. Bisector Surface of Two Space Curves in Minkowski 3-Space

To introduce the subject of bisector in Minkowski 3-space, we deal with an elementary example. Let $A = (a, b, c)$ and $N = (m, n, l)$ be two points in $\mathbb{R}^3_1$. Since the bisector $B = (x, y, z)$ is the set of points equidistant from the two points A and N, we have

$$\left|(x-a)^2 + (y-b)^2 - (z-c)^2\right| = \left|(x-m)^2 + (y-n)^2 - (z-l)^2\right| \tag{3}$$

There are two cases in Equation (3). Now, let us discuss the following two cases.

Case 1 If $(x-a)^2 + (y-b)^2 - (z-c)^2 = (x-m)^2 + (y-n)^2 - (z-l)^2$ then, we have a plane equation in $\mathbb{R}^3_1$ given by

$$x(m-a) + y(n-b) + z(c-l) + \frac{1}{2}(a^2 + b^2 - c^2 + m^2 + n^2 - l^2) = 0 \tag{4}$$

Case 2 If $(x-a)^2 + (y-b)^2 - (z-c)^2 = (z-l)^2 - (x-m)^2 - (y-n)^2$ then, we have a hyperboloid equation in $\mathbb{R}^3_1$ given by

$$x^2 + y^2 - z^2 - x(a+m) - y(b+n) + z(c+l) + \frac{1}{2}(a^2 + b^2 - c^2 + m^2 + n^2 - l^2) = 0 \tag{5}$$

We now investigate the bisector surface of two rational space curves. Let

$$\begin{aligned} C_1(s) &= (x_1(s), y_1(s), z_1(s)) \\ C_2(t) &= (x_2(t), y_2(t), z_2(t)) \end{aligned} \tag{6}$$

be two regular parametric C^1-continuous space curves in Minkowski 3-space. The tangent vectors of $C_1(s)$ and $C_2(t)$ are determined by, respectively

$$\begin{aligned} T_1(s) &= (x'_1(s), y'_1(s), z'_1(s)) \\ T_2(t) &= (x'_2(t), y'_2(t), z'_2(t)) \end{aligned} \tag{7}$$

When a point P is on the bisector of two curves, there exist (at least) two points $C_1(s)$ and $C_2(t)$ such that point P is simultaneously contained in the normal planes $L_1(s)$ and $L_2(t)$. As a result, the point P satisfies the following two linear equations:

$$L_1(s) :< P - C_1(s), T_1(s) >= 0 \tag{8}$$

$$L_2(t) :< P - C_2(t), T_2(t) >= 0 \tag{9}$$

Moreover, point P is also contained in the bisector plane $L_{12}(s,t)$ between the two points $C_1(s)$ and $C_2(t)$. The plane $L_{12}(s,t)$ is orthogonal to the vector $C_1(s) - C_2(t)$ and passes through the mid point $[C_1(s) + C_2(t)]/2$ of $C_1(s)$ and $C_2(t)$. Therefore, the bisector plane $L_{12}(s,t)$ is defined by the following linear equation:

$$L_{12}(s,t) :< P - \frac{C_1(s) + C_2(t)}{2}, C_1(s) - C_2(t) >= 0 \tag{10}$$

Any bisector point P must be a common intersection point of the three planes of $L_1(s)$, $L_2(t)$, and $L_{12}(s,t)$, for some s and t. Therefore, the point P can be computed by solving the following simultaneous linear equations in P:

$$\left.\begin{array}{ll} L_1(s): & < P, T_1(s) >=< C_1(s), T_1(s) > \\ L_2(t): & < P, T_2(t) >=< C_2(t), T_2(t) > \\ L_{12}(s,t): & < P, C_1(s) - C_2(t) >= \dfrac{C_1(s)^2 - C_2(t)^2}{2} \end{array}\right\} \tag{11}$$

Using Equations (6), we have

$$C_1(s) - C_2(t) = (x_{12}(s,t), y_{12}(s,t), z_{12}(s,t)) \tag{12}$$

where $x_{12}(s,t) = x_1(s) - x_2(t)$, $y_{12}(s,t) = y_1(s) - y_2(t)$ and $z_{12}(s,t) = z_1(t) - z_2(s)$.

Substituting Equations (12), (1) and (7) into Equation (11) then, we obtain the implicit equations of the planes $L_1(s)$, $L_2(t)$, and $L_{12}(s,t)$ as

$$\left.\begin{array}{lcl} L_1(s): & = & x_1'(s)P_x + y_1'(s)P_y - z_1'(s)P_z = d_1(s) \\ L_2(t): & = & x_2'(t)P_x + y_2'(t)P_y - z_2'(t)P_z = d_2(t) \\ L_{12}(s,t): & = & x_{12}(s,t)P_x + y_{12}(s,t)P_y - z_{12}(s,t)P_z = m(s,t) \end{array}\right\} \tag{13}$$

where $P = (P_x, P_y, P_z)$ is the bisector point, and $d_1(s)$, $d_2(t)$ and $m(s,t)$ are given by

$$d_1(s) =< C_1(s), T_1(s) >, \quad d_2(t) =< C_2(t), T_2(t) > \tag{14}$$

$$m(s,t) = \frac{C_1(s)^2 - C_2(t)^2}{2} \tag{15}$$

We may express results in the matrix form as

$$\begin{bmatrix} x_1'(s) & y_1'(s) & -z_1'(s) \\ x_2'(t) & y_2'(t) & -z_2'(t) \\ x_{12}(s,t) & y_{12}(s,t) & -z_{12}(s,t) \end{bmatrix} \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} d_1(s) \\ d_2(t) \\ m(s,t) \end{bmatrix} \tag{16}$$

By Cramer's rule, Equation (13) can be solved as follows:

$$P_x(s,t) = \frac{\begin{vmatrix} d_1(s) & y_1'(s) & -z_1'(s) \\ d_2(t) & y_2'(t) & -z_2'(t) \\ m(s,t) & y_{12}(s,t) & -z_{12}(s,t) \end{vmatrix}}{\begin{vmatrix} x_1'(s) & y_1'(s) & -z_1'(s) \\ x_2'(t) & y_2'(t) & -z_2'(t) \\ x_{12}(s,t) & y_{12}(s,t) & -z_{12}(s,t) \end{vmatrix}}, \tag{17}$$

$$P_y(s,t) = \frac{\begin{vmatrix} x_1'(s) & d_1(s) & -z_1'(s) \\ x_2'(t) & d_2(t) & -z_2'(t) \\ x_{12}(s,t) & m(s,t) & -z_{12}(s,t) \end{vmatrix}}{\begin{vmatrix} x_1'(s) & y_1'(s) & -z_1'(s) \\ x_2'(t) & y_2'(t) & -z_2'(t) \\ x_{12}(s,t) & y_{12}(s,t) & -z_{12}(s,t) \end{vmatrix}}, \tag{18}$$

and

$$P_z(s,t) = \frac{\begin{vmatrix} x_1'(s) & y_1'(s) & d_1(s) \\ x_2'(t) & y_2'(t) & d_2(t) \\ x_{12}(s,t) & y_{12}(s,t) & m(s,t) \end{vmatrix}}{\begin{vmatrix} x_1'(s) & y_1'(s) & -z_1'(s) \\ x_2'(t) & y_2'(t) & -z_2'(t) \\ x_{12}(s,t) & y_{12}(s,t) & -z_{12}(s,t) \end{vmatrix}}. \tag{19}$$

The bisector surface $P(s,t) = (P_x(s,t), P_y(s,t), P_z(s,t))$ has a simple rational representation as long as the common denominator of P_x, P_y and P_z in equation (13) does not vanish.

Example 2.1 Let $C_1(s)$ and $C_2(t)$ be two non-intersecting orthogonal straight lines in Minkowski space given by parametrization

$$C_1(s) = (1, s, 0), \quad C_2(t) = (0, 0, t) \tag{20}$$

By using Equations (20), (14) and (15), we have

$$d_1(s) = s, \quad d_2(t) = -t, \quad m(s,t) = \frac{1 + s^2 + t^2}{2} \tag{21}$$

Substituting Equation (21) into Equations (17), (18) and (19). Finally, we have the bisector surface $P(s,t)$ given by parametrization

$$P(s,t) = (\frac{1 - s^2 - t^2}{2}, s, t)$$

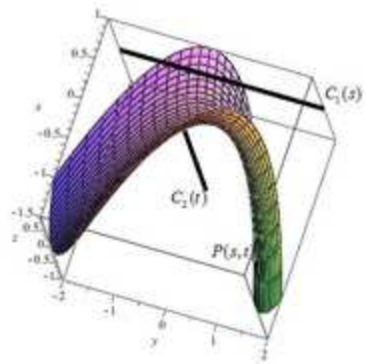

Figure 2.1a

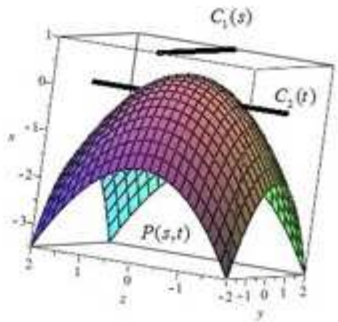

Figure 2.1b

where, Figure 2.1a shows the bisector surface of two lines in Euclidean space, Figure 2.1b shows that the bisector surface of two lines in Minkowski space.

We observe that the bisector of $C_1(s)$ and $C_2(t)$ lines, shown in Fig. 1(a), is a hyperbolic paraboloid of one sheet in Euclidean 3-space (Elber 1998). On the other hand, the bisector surface of $C_1(s)$ and $C_2(t)$ lines, shown in Fig. 1(b), is elliptic paraboloid in Minkowski 3-space.

Example 2.2 Figure 2(a) and Figure 2(b) illustrates the bisector surfaces of a Euclidean circle and a line, given by parametrization

$$C_1(s) = (\cos(s), \sin(s), 0), \quad C_2(t) = (0, 0, t) \tag{22}$$

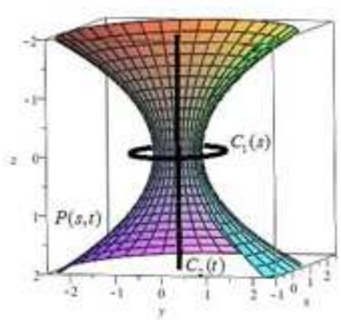

Figure 2.2a

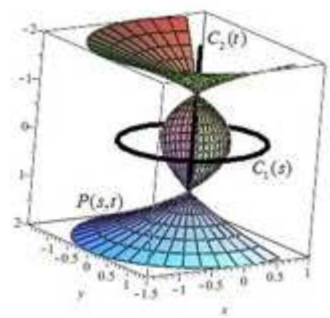

Figure 2.2b

where, Figure 2.2a shows the bisector surface of a circle and a line in Euclidean space, Figure 2.2b shows that the bisector surface of a circle and a line in Minkowski space.

From (22), (12) and (13), we get

$$d_1(s) = 0, \quad d_2(t) = -t, \quad m(s,t) = \frac{1+t^2}{2} \tag{23}$$

$$(x_{12}(s,t), y_{12}(s,t), z_{12}(s,t)) = (\cos(s), \sin(s), -t) \tag{24}$$

Substituting above equations into Equations (17), (18) and (19), we have the bisector surface given by parametrization

$$P(s,t) = (\frac{1-t^2}{2}\cos(s), \frac{1-t^2}{2}\sin(s), t) \tag{25}$$

Consequently, Fig.3(a) and Fig.3(b) shows an example of the bisector surface of a non-planar curve (Euclidean helix) and a line.

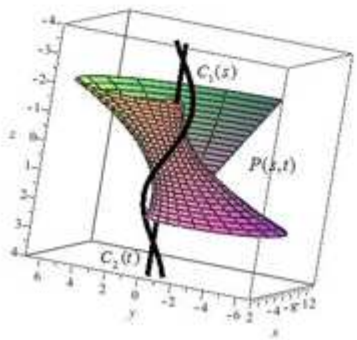

Figure 2.3a

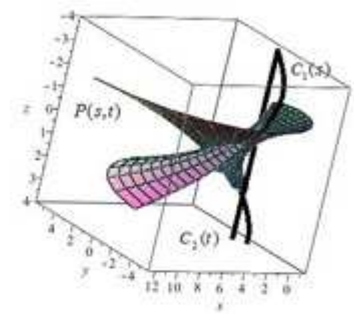

Figure 2.3b

where, Figure 2.3a shows the bisector surface of a helix and a line in Euclidean space, Figure 2.3 shows that the bisector surface of a helix and a line in Minkowski space.

§3. Conclusions

In this paper, we have shown that the bisector surface of curve/curve in Minkowski 3-space. Bisector surface of point/curve and surface/surface are not included in this paper. The different studies on bisector surface in Minkowski 3-space may be presented in a future publication.

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