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Introduction to Bihypergroups

B.Davvaz

(Department of Mathematics, Yazd University, Yazd, Iran)

A.A.A.Agboola

(Department of Mathematics, Federal University of Agriculture, Abeokuta, Nigeria)

E-mail: davvaz@yazd.ac.ir, aaaola2003@yahoo.com

Abstract: Similar to groups, the union of two sub-hypergroups do not form a hypergroup but they find a nice hyperstructure called bihypergroup. This short note is devoted to the introduction of bihypergroups and illustrate them with examples. A characterization theorem about sub-bihypergroups is given and some of their properties are presented.

Key Words: bigroup, hypergroup, sub-hypergroup, bihypergroup, sub-bihypergroup.

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§1. Introduction

Bigroups are a very nice tool as the answers to a major problem faced by all groups, that is the union of two subgroups do not form any algebraic structure but they find a nice bialgebraic structure as bigroups. The study of bigroups was carried out in 1994-1996. Maggu [7,8] was the first one to introduce the notion of bigroups. However, the concept of bialgebraic structures was recently studied by Vasantha Kandasamy [11]. Agboola and Akinola in [1]studied bicoset of a bivector space.

The theory of hyperstructures was introduced in 1934 by Marty [9] at the 8th Congress of Scandinavian Mathematicians. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Several books have been written on this topic, see [2-4,6,13]. Hyperstructure theory both extends some well-known group results and introduce new topics leading us to a wide variety of applications, as well as to a broadening of the investigation fields.

§2. Basic Facts and Definitions

This section has two parts. In the first part we recall the definition of bigroups. In the second part we recall the notion of hypergroups.

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2.1 Bigroups

Definition 2.1 *Let $*_1$ and $*_2$ be any two binary operations defined on a non-empty set G. Then, G is said to be a bigroup if there exists two proper subsets G_1 and G_2 such that*

(1) $G = G_1 \cup G_2$;
(2) $(G_1, *_1)$ *is a group;*
(3) $(G_2, *_2)$ *is a group.*

Definition 2.2 *Let $G = G_1 \cup G_2$ be a bigroup. A non-empty subset A of G is said to be a sub-bigroup of G if $A = A_1 \cup A_2$, A is a bigroup under the binary operations inherited from G, $A_1 = A \cap G_1$ and $A_2 = A \cap G_2$.*

Example 1([11]) Suppose that $G = \mathbb{Z} \cup \{i, -i\}$ under the operations "+" and "·". We consider $G = G_1 \cup G_2$, where $G_1 = \{-1, 1, i, -i\}$ under the operation "·" and $G_2 = \mathbb{Z}$ under the operation "+" are groups. Take $H = \{-1, 0, 1\} = H_1 \cup H_2$, where $H_1 = \{0\}$ is a group under "+" and $H_2 = \{-1, 1\}$ is a group under "·". Thus, H is a sub-bigroup of G. Note that H is not a group under "+" or "· ".

Definition 2.3 *Let $G = G_1 \cup G_2$ be a bigroup. Then, G is said to be commutative if both G_1 and G_2 are commutative.*

Definition 2.4 *Let $A = A_1 \cup A_2$ be a sub-bigroup of a bigroup $G = G_1 \cup G_2$. Then, A is said to be a normal bi-subgroup of G if A_1 is a normal subgroup of G_1 and A_2 is a normal subgroup of G_2.*

2.2 Hypergroups

In this part, we present the notion of hypergroup and some well-known related concepts. These concepts will be used in the building of bihypergroups, for more details we refer the readers to see [2-4, 6, 13].

Let H be a non-empty set and $\circ : H \times H \to \mathcal{P}^*(H)$ be a hyperoperation. The couple $(H, \circ)$ is called a *hypergroupoid*. For any two non-empty subsets A and B of H and $x \in H$, we define

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \; A \circ x = A \circ \{x\} \;\text{ and }\; x \circ B = \{x\} \circ B.$$

Definition 2.5 *A hypergroupoid $(H, \circ)$ is called a semihypergroup if for all a, b, c of H we have $(a \circ b) \circ c = a \circ (b \circ c)$, which means that*

$$\bigcup_{u \in a \circ b} u \circ c = \bigcup_{v \in b \circ c} a \circ v.$$

A hypergroupoid $(H, \circ)$ is called a quasihypergroup if for all a of H we have $a \circ H = H \circ a = H$. This condition is also called the reproduction axiom.

Definition 2.6 *A hypergroupoid $(H, \circ)$ which is both a semihypergroup and a quasihypergroup is called a hypergroup. For any $x, y \in H$, we define the right and the left extensions as follows: $x/y = \{a \in H \mid x \in a \circ y\}$ and $x \backslash y = \{b \in H \mid y \in x \circ b\}$.*

Example 2 Let $(S, \cdot)$ be a semigroup and let P be a non-empty subset of S. For all x, y of S, we define $x \circ y = xPy$. Then, $(S, \circ)$ is a semihypergroup. If $(S, \cdot)$ is a group. then $(S, \circ)$ is a hypergroup.

Example 3 If G is a group and for all x, y of G, $< x, y >$ denotes the subgroup generated by x and y, then we define $x \circ y =< x, y >$. We obtain that $(G, \circ)$ is a hypergroup.

Definition 2.7 *Let $(H, \circ)$ and $(H', \circ')$ be two hypergroupoids. A map $\phi : H \to H'$, is called*

(1) *an inclusion homomorphism if for all x, y of H, we have $\phi(x \circ y) \subseteq \phi(x) \circ' \phi(y)$;*

(2) *a good homomorphism if for all x, y of H, we have $\phi(x \circ y) = \phi(x) \circ' \phi(y)$.*

A good homomorphism ϕ is called a very good homomorphism if for all $x, y \in H$, $\phi(x/y) = \phi(x)/\phi(y)$ and $\phi(x\backslash y) = \phi(x)\backslash\phi(y)$.

Let $(H, \circ)$ be a semihypergroup and R be an equivalence relation on H. If A and B are non-empty subsets of H, then

$$\begin{aligned} A\overline{R}B &\text{ means that } \forall a \in A, \exists b \in B \text{ such that } aRb \text{ and} \\ &\qquad\qquad\qquad \forall b' \in B, \exists a' \in A \text{ such that } a'Rb'; \\ A\overline{\overline{R}}B &\text{ means that } \forall a \in A, \forall b \in B, \text{ we have } aRb. \end{aligned}$$

Definition 2.8 *The equivalence relation ρ is called*

(1) *regular on the right (on the left) if for all x of H, from $a\rho b$, it follows that $(a \circ x)\overline{\rho}(b \circ x)$ ($(x \circ a)\overline{\rho}(x \circ b)$ respectively);*

(2) *strongly regular on the right (on the left) if for all x of H, from $a\rho b$, it follows that $(a \circ x)\overline{\overline{\rho}}(b \circ x)$ ($(x \circ a)\overline{\overline{\rho}}(x \circ b)$ respectively);*

(3) *ρ is called regular (strongly regular) if it is regular (strongly regular) on the right and on the left.*

Theorem 2.9 *Let $(H, \circ)$ be a semihypergroup and ρ be an equivalence relation on H.*

(1) *If ρ is regular, then H/ρ is a semihypergroup, with respect to the following hyperoperation: $\overline{x} \otimes \overline{y} = \{\overline{z} \mid z \in x \circ y\}$;*

(2) *If the above hyperoperation is well defined on H/ρ, then ρ is regular.*

Corollary 2.10 *If $(H, \circ)$ is a hypergroup and ρ is an equivalence relation on H, then R is regular if and only if $(H/\rho, \otimes)$ is a hypergroup.*

Theorem 2.11 *Let $(H, \circ)$ be a semihypergroup and ρ be an equivalence relation on H.*

(1) *If ρ is strongly regular, then H/ρ is a semigroup, with respect to the following operation: $\overline{x} \otimes \overline{y} = \{\overline{z} \mid z \in x \circ y\}$;*

(2) *If the above operation is well defined on H/ρ, then ρ is strongly regular.*

Corollary 2.12 *If $(H, \circ)$ is a hypergroup and ρ is an equivalence relation on H, then ρ is strongly regular if and only if $(H/\rho, \otimes)$ is a group.*

Definition 2.13 *Let $(H, \circ)$ is a semihypergroup and A be a non-empty subset of H. We say that A is a complete part of H if for any nonzero natural number n and for all $a_1, \ldots, a_n$ of H, the following implication holds:*

$$A \cap \prod_{i=1}^{n} a_i \neq \emptyset \implies \prod_{i=1}^{n} a_i \subseteq A.$$

Theorem 2.14 *If $(H, \circ)$ is a semihypergroup and R is a strongly regular relation on H, then for all z of H, the equivalence class of z is a complete part of H.*

§3. Bihypergroup Structures

In this section, we introduce the concept of bihypergroup and illustrate it with examples.

Definition 3.1 *A set $(H, \circ, \star)$ with two hyperoperations $\circ$ and $\star$ is called a bihypergroup if there exist two proper subsets H_1 and H_2 such that*

(1) *$H = H_1 \cup H_2$;*

(2) *$(H_1, \circ)$ is a hypergroup;*

(3) *$(H_2, \star)$ is a hypergroup.*

Theorem 3.2 *Every hypergroup is a bihypergroup.*

Proof Suppose that $(H, \circ)$ is a hypergroup. If we consider $H = H_1 = H_2$ and $\circ = \star$, then $(H, \circ, \star)$ is a bihypergroup. □

Example 4 Let $H = \{a, b, c, d, e\}$ and let $\circ$ and $\star$ be two hyperoperations on H defined by the following tables:

$\circ$	a	b	c	d	e
a	a	b	c, d	d	b
b	b	a, b	c, d	c, d	e
c	c	c, d	a, b	a, b	e
d	c, d	c, d	a, b	a, b	e
e	b	e	e	e	e

and

$\star$	a	b	c	d	e
a	a	b	c, d	d	e
b	b	a, e	c, d	c, d	b
c	c	c, d	a, b	a, b	e
d	c, d	c, d	a, b	a, b	e
e	e	b	e	e	a

It is not difficult to see that $H_1 = \{a, b, c, d\}$ is a hypergroup together with the hyperoperation $\circ$ and $H_2 = \{a, b, e\}$ is a hypergroup together with the hyperoperation $\star$. Hence, $H = H_1 \cup H_2$ is a bihypergroup.

Example 5 ([5]) Blood groups are inherited from both parents. The ABO blood type is controlled by a single gene (the ABO gene) with three alleles: I^A, I^B and i. The gene encodes glycosyltransferase that is an enzyme that modifies the carbohydrate content of the red blood cell antigens. The gene is located on the long arm of the ninth chromosome (9q34).

People with blood type A have antigen A on the surfaces of their blood cells, and may be of genotype I^AI^A or I^Ai. People with blood type B have antigen B on their red blood cell surfaces, and may be of genotype I^BI^B or I^Bi. People with the rare blood type AB have both antigens A and B on their cell surfaces, and are genotype I^AI^B. People with blood type O have neither antigen, and are genotype ii. A type A and a type B couple can also have a type O child if they are both heterozygous (I^Ai and I^Bi, respectively).

$\otimes$	O	A	B	AB
O	O	O A	O B	A B
A	O A	O A	AB A B O	AB A B
B	O B	AB A B O	O B	AB A B
AB	A B	AB A B	AB A B	AB A B

Now, we consider $H = \{O, A, B\}$. If $H_1 = \{O, A\}$ and $H_2 = \{O, B\}$, then $H = H_1 \cup H_2$ is a bihypergroup.

Definition 3.3 *Let $H = H_1 \cup H_2$ be a bihypergroup. A non-empty subset A of H is said to be a sub- bihypergroup of G if $A = A_1 \cup A_2$, A is a bihypergroup under the binary operations inherited from H, $A_1 = A \cap H_1$ and $A_2 = A \cap H_2$.*

Remark 1 If $(H, \circ, \star)$ is a bihypergroup and K is a sub-bihypergroup of H, then $(K, \circ)$ and $(K, \star)$ in general are not hypergroups.

Theorem 3.4 *Let $H = H_1 \cup H_2$ be a bihypergroup. A non-empty subset $A = A_1 \cup A_2$ of H is a sub-bihypergroup of H if and only if $A_1 = A \cap H_1$ and $A_2 = A \cap H_2$ are sub-hypergroups of H_1 and H_2, respectively.*

Proof Suppose that $A = A_1 \cup A_2$ is a sub-bihypergroup of H. Then, A_i, $i = 1, 2$ are sub-hypergroups of H_i and therefore $A_i = A \cap H_i$ are sub-hypergroups of H_i.

Conversely, suppose that $A_1 = A \cap G_1$ is a sub-hypergroup of H_1 and $A_2 = A \cap H_2$ is a sub-hypergroup of H_2. It can be shown that $A_1 \cup A_2 = (A \cap H_1) \cup (A \cap H_2) = A$. Hence, A is sub-bihypergroup of H. □

Theorem 3.5 *Let H be any hypergroup and let A_1 and A_2 be any two sub-hypergroups of H such that $A_1 \not\subset A_2$ and $A_2 \not\subset A_1$ but $A_1 \cap A_2 \neq \emptyset$. Then, $A = A_1 \cup A_2$ is a bihypergroup.*

Proof The required result follows from the definition of bihypergroup. □

Theorem 3.6 *Let $(H, \circ, \star)$ and $(H', \circ', \star')$ be any two bihypergroups, where $H = H_1 \cup H_2$ and $H' = H'_1 \cup H'_2$. Then, $(H \times H', \odot, \otimes)$ is a bihypergroup, where*

(1) $H \times H' = (H_1 \times H'_1) \cup (H_2 \times H'_2)$;

(2) $(x_1, x'_1) \odot (y_1, y'_1) = \{(z_1, z'_1) \mid z_1 \in x_1 \circ y_1,\ z'_1 \in x'_1 \star y'_1\}$, *for all* $(x_1, x'_1), (y_1, y'_1) \in H_1 \times H'_1$;

(3) $(x_2, x'_2) \odot (y_2, y'_2) = \{(z_2, z'_2) \mid z_2 \in x_2 \circ' y_2,\ z'_2 \in x'_2 \star' y'_2\}$, *for all* $(x_2, x'_2), (y_2, y'_2) \in H_2 \times H'_2$.

Definition 3.7 *Let $(H, \circ, \star)$ be a bihypergroup, where $H = H_1 \cup H_2$. Then, H is said to be commutative if both $(H_1, \circ)$ and $(H_2, \circ)$ are commutative.*

Let $H = H_1 \cup H_2$ and ρ be an equivalence relation on H. The *restriction* of ρ to H_1 and H_2 are the relations on H_1 and H_2 defined as

$$\rho|_{H_1} := \rho \cap (H_1 \times H_1) \text{ and } \rho|_{H_2} := \rho \cap (H_2 \times H_2).$$

Lemma 3.8 *Let $H = H_1 \cup H_2$ and ρ be an equivalence relation on H. Then, $\rho|_{H_1}$ and $\rho|_{H_2}$ are equivalence relations on H_1 and H_2, respectively.*

Definition 3.9 *Let $(H, \circ, \star)$ be a bihypergroup, where $H = H_1 \cup H_2$ and let ρ be an equivalence relation on H. We say that ρ is a (strongly) regular relation on H, if $\rho|_{H_1}$ is a (strongly) regular relation on H_1 and $\rho|_{H_2}$ is a (strongly) regular relation on H_2.*

Theorem 3.10 *Let $(H, \circ, \star)$ be a bihypergroup, where $H = H_1 \cup H_2$, and let ρ be an equivalence relation on H.*

(1) *If ρ is regular, then $H_1/\rho|_{H_1} \cup H_2/\rho|_{H_2}$ is a bihypergroup;*

(2) *If ρ is strongly regular, then $H_1/\rho|_{H_1} \cup H_2/\rho|_{H_2}$ is a bigroup.*

Proof The proof follows from Lemma 3.8 and Theorems 2.9 and 2.11. □

Definition 3.11 *Let $(H, \circ, \star)$ and $(H', \circ', \star')$ be any two bihypergroups, where $H = H_1 \cup H_2$ and*

$H' = H'_1 \cup H'_2$. The map $\phi : H \to H'$ is said to be a bihypergroup (inclusion, good, very good, respectively) homomorphism if ϕ restricted to H_1 is a hypergroup (inclusion, good, very good, respectively) homomorphism from H_1 to H'_1 and ϕ restricted to H_2 is a hypergroup (inclusion, good, very good, respectively) homomorphism from H_2 to H'_2.

Definition 3.12 *Let $\phi = \phi_1 \cup \phi_2 : (H = H_1 \cup H_2, \circ, \star) \to (H' = H'_1 \cup H'_2, \circ', \star')$ be a good homomorphism and $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ be non-empty subsets of H and H', respectively.*

(1) *The image of A under ϕ denoted by $\phi(A) = \phi_1(A_1) \cup \phi_2(A_2)$, is the set $\{\phi_1(a_1), \phi_2(a_2) \mid a_1 \in A_1, a_2 \in A_2\}$;*

(2) *The inverse image of B under ϕ denoted by $\phi^{-1}(B) = \phi^{-1}(B_1) \cup \phi^{-1}(B_2)$, is the set $\{h_1 \in H_1, h_2 \in H_2 \mid \phi_1(h_1) \in B_1, \phi_2(h_2) \in B_2\}$.*

Lemma 3.13 *Let H and H' be two hypergroups and $\phi : H \to H'$ be a good homomorphism.*

(1) *If A is a sub-hypergroup of H, then $\phi(A)$ is a sub-hypergroup of H';*

(2) *If ϕ is a very good homomorphism and B is a subhypergroup of H', then $\phi^{-1}(B)$ is a sub-hypergroup of H.*

Proof The proof of (1) is clear. We prove (2). Suppose that $x, y \in \phi^{-1}(B)$ are arbitrary elements. Then, $\phi(x), \phi(y) \in B$. For every $z \in x \circ y$, $\phi(z) \in \phi(x \circ y) = \phi(x) \star \phi(y) \subseteq B$. So, $z \in \phi^{-1}(B)$. Hence, $\phi^{-1}(B) \circ \phi^{-1}(B) \subseteq \phi^{-1}(B)$.

Now, suppose that $x, a \in \phi^{-1}(B)$ are arbitrary elements. Then, $\phi(x), \phi(a) \in B$. Since B is a sub-hypergroup of H', by reproduction axiom, there exists $u \in B$ such that $\phi(a) \in u \star \phi(x)$. Thus, $u \in \phi(a)/\phi(x)$. Since ϕ is very good homomorphism, $u \in \phi(a/x)$. Hence, there exists $y \in a/x$ such that $u = \phi(y)$. Thus, $y \in \phi^{-1}(B)$ and $a \in y \circ x$. Hence, for every $x, a \in \phi^{-1}(B)$, there exists $y \in \phi^{-1}(B)$ such that $a \in \phi^{-1}(B) \circ x$. This implies that $\phi^{-1}(B) \subseteq \phi^{-1}(B) \circ x$ for all $x \in \phi^{-1}(B)$. Similarly, we can prove that $\phi^{-1}(B) \subseteq x \circ \phi^{-1}(B)$ for all $x \in \phi^{-1}(B)$. □

Proposition 3.14 *Let $\phi = \phi_1 \cup \phi_2 : (H = H_1 \cup H_2, \circ, \star) \to (H' = H'_1 \cup H'_2, \circ', \star')$ be a good homomorphism and $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ be non-empty subsets of H and H', respectively.*

(1) *$\phi(A)$ is a sub-bihypergroup of H';*

(2) *If ϕ is a very good homomorphism, then $\phi^{-1}(B)$ is a sub-bihypergroup of H.*

Proof (1) Suppose that $A = A_1 \cup A_2$ is a sub-bihypergroup of H. By Lemma 3.13(1), $\phi_1(A_1)$ is a sub-hypergroup of H'_1 and $\phi_2(A_2)$ is a sub-hypergroup of H'_2. Thus, $\phi(A) = \phi_1(A_1) \cup \phi_2(A_2)$ is a bihypergroup. Now,

$$\begin{aligned}
\phi(A) \cap H'_1 &= \Big(\phi_1(A_1) \cup \phi_2(A_2)\Big) \cap H'_1 \\
&= \Big(\phi_1(A_1) \cap H'_1\Big) \cup \Big(\phi_2 \cap H'_1\Big) \\
&= \phi(A_1).
\end{aligned}$$

Similarly, it can be shown that $\phi(A) \cap H_2' = \phi_2(A_2)$. Accordingly, $\phi(A)$ is a sub-bihypergroup of H'.

The proof of (2) follows from Lemma 3.13(2) and is similar to the proof of (1). $\square$

Definition 3.15 *Let $(H, \circ, \star)$ be a bihypergroup where $H = H_1 \cup H_2$ and let $A = A_1 \cup A_2$ be a non-empty subset of H. Then, A is said to be a complete part of H if A_1 is a complete part of H_1 and A_2 is a complete part of H_2.*

Theorem 3.16 *Let $(H, \circ, \star)$ be a bihypergroup where $H = H_1 \cup H_2$ and let ρ be an equivalence relation on H. If $\rho|_{H_1}$ and $\rho|_{H_2}$ are strongly regular relations on H_1 and H_2 respectively, then for all $x = x_1 \cup x_2 \in H$, $\overline{x_1} \cup \overline{x_2}$ is a complete part of H.*

Proof It is clear. $\square$

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