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Graph Folding and Incidence Matrices

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Abstract: In this paper we described the graph folding of two given graphs under some known operations such as, intersection, union, join product, Cartesian product, tensor product, normal product and composition product by using incidence matrices.

Key Words: Incidence matrices, Cartesian product, tensor product, normal product, composition product, join product

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§1. Introduction

By a simple graph G, we mean that a graph with no loops or multiple edges. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be simple graphs. Then

(1) The simple graph $G = (V, E)$, where $V = V_1 \bigcup V_2$ and $E = E_1 \bigcup E_2$ is called the union of G_1 and G_2, and is denoted by $G_1 \bigcup G_2$ ([2]). When G_1 and G_2 are vertex disjoint, $G_1 \bigcup G_2$ is denoted by $G_1 + G_2$ and is called the sum of the graphs G_1 and G_2.

(2) If $V_1 \bigcap V_2 \neq \emptyset$, the graph $G = (V, E)$, where $V = V_1 \bigcap V_2$ and $E = E_1 \bigcap E_2$ is called the intersection of G_1, G_2 and is written as $G_1 \bigcap G_2$ ([2]).

(3) If G_1 and G_2 are vertex-disjoint graphs. Then the join, $G_1 \vee G_2$ is the supergraph of $G_1 + G_2$, in which each vertex of G_1 is adjacent to every vertex of G_2.

(4) The cartesian product $G_1 \times G_2$ is the simple graph with vertex set $V(G_1 \times G_2) = V_1 \times V_2$ and edge set $E(G_1 \times G_2) = (E_1 \times V_2) \bigcup (V_1 \times E_2)$ such that two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \times G_2$ iff either

(i) $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 , or

(ii) u_1 is adjacent to v_1 in G_1 and $u_2 = v_2$ ([1]).

(5) The composition, or lexicographic product $G_1[G_2]$ is the simple graph with $V_1 \times V_2$ as the vertex set in which the vertices (u_1, u_2) , (v_1, v_2) are adjacent if either u_1 is adjacent to v_1

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or $u_1 = v_1$ and u_2 is adjacent to v_2.

The graph $G_1[G_2]$ need not to be isomorphic to $G_2[G_1]$ ([2]).

(6) The normal product, or the strong product $G_1 \circ G_2$ is the simple graph with $V(G_1 \circ G_2) = V_1 \times V_2$, where (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \circ G_2$ iff either

(i) $u_1 = v_1$ and u_2 is adjacent to v_2 , or

(ii) u_1 is adjacent to v_1 and $u_2 = v_2$, or

(iii) u_1 is adjacent to v_1 and u_2 is adjacent to v_2 ([2]).

(7) The tensor product or Kronecher product $G_1 \otimes G_2$ is the simple graph with $V(G_1 \otimes G_2) = V_1 \times V_2$, where (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \otimes G_2$ iff u_1 is adjacent to v_1 in G_1 and u_2 is adjacent to v_2 in G_2.

Notice that $G_1 \circ G_2 = (G_1 \otimes G_2) \bigcup (G_1 \otimes G_2)$ ([2]).

§2. Graph Folding

Let G_1 and G_2 be graphs and $f : G_1 \to G_2$ be a continuous function. Then f is called a graph map, if

(i) for each vertex $v \in V(G_1)$, $f(v)$ is a vertex in $V(G_2)$;

(ii) for each edge $e \in E(G_1)$, $\dim(f(e)) \leq \dim(e)$.

A graph map $f : G_1 \to G_2$ is called a graph folding iff f maps vertices to vertices and edges to edges, i.e., for each $v \in V(G_1)$, $f(v) \in V(G_2)$ and for $e \in E(G_1)$, $f(e) \in E(G_2)$ ([3]).

The set of graph foldings between graphs G_1 and G_2 is denoted by $\mu(G_1, G_2)$ and the set of graph foldings of G_1 into itself by $\mu(G_1)$.

§3. Incidence Matrices

Let G be a finite graph with the set of vertices $V(G) = \{v_1, \cdots, v_{r_1}, v_{r_1+1}, \cdots, v_r\}$ and the set of edges $E(G) = \{e_{12}, \cdots, e_{1r_1}, \cdots, e_{1r}, e_{23}, \cdots, e_{2r}, \cdots, e_{(r-1)r}\}$.

The incidence matrix denoted by $I = (\lambda_{kd})$ is defined by $\lambda_{kd} = 1$ if v_k, $k = 1, \cdots, r_1, \cdots, r$ is a face of e_d, $d = 12, \cdots, 1r_1, \cdots, 1r, 23, \cdots, 2r, \cdots, r(r-1)$ in G, $\lambda_{kd} = 0$ if v_k is not a face of e_d in G. The matrix I has order $s \times r$, where s is the number of edges of G and r is the number of vertices in G.

Let G_1, G_2 be finite graphs and $f \in \mu(G_1, G_2)$. Then $f(G_1)$ is a subgraph of G_2. In particular, if $f \in \mu(G_1)$ with $f(G_1) = G_1' \neq G_1$, then G_1' is a subgraph of G_1. This suggests that the incidence matrix I' of $f(G_1) = G_1'$ is a submatrix of the incidence matrix I of G_1 possibly after rearranging its rows and columns.

We claim that the matrix I can be partition into four blocks, such that I' appears in the upper left corner block and a zero matrix O in the upper right one. The matrix R, the complement of I' will be a submatrix of I' possibly after deleting the rows and columns of I' which are not images of any of the edges $e_{1(r_1+1)}, \cdots, e_{1r}, e_{2(r_1+1)}, \cdots, e_{2r}, \cdots, e_{r_1r}$ and the vertices $v_{r_1+1}, \cdots, v_r$, respectively.

The zero matrix O is due to the fact that non of the vertices $v_{r_1+1}, \cdots, v_r$ is incidence with any edge of the image.

$$I = \begin{array}{c c} \begin{array}{cc} \quad\overline{v}_1 & \quad\quad\overline{v}_2 \end{array} & \\ \left[\begin{array}{c|c} I' & O \\ \hline Q & R \end{array}\right] & \begin{array}{c} \overline{e}_1 \\ \\ \overline{e}_2 \end{array} \end{array}$$

where

$\overline{v}_1 = (v_1, v_2, \cdots, , v_{r_1})$,
$\overline{v}_2 = (v_{r_1+1}, \cdots, v_r)$,
$\overline{e}_1 = (e_{12}, \cdots, e_{1r_1}, e_{23}, \cdots, e_{2r_1}, \cdots, e_{(r_1-1)r_1})^T$ and
$\overline{e}_2 = (e_{1(r_1+1)}, \cdots, e_{1r}, e_{2(r_1+1)}, \cdots, e_{2r}, \cdots, e_{r_1(r_1+1)}, \cdots, e_{r_1 r})^T$.

Conversely, if the incidence matrix I of a graph G_1 can be partitioned into four blocks with a zero matrix at the right hand corner block. Then a graph folding may be defined, if there is any, as a map f of G_1 to an image $f(G_1)$ characterized by the incidence matrix I' which lie in the upper left corner of I. This map can be defined by mapping:

(i) the vertices $v_j,\ j = r_1 + 1, \cdots, r$ to the vertices $v_i, i = 1, \cdots, r_1$ if the j^{th} column in R is the same as the i^{th} column in I', after deleting the zero from i^{th} column;

(ii) the edges $e_k,\ k = 1(r_1 + 1), \cdots, r_1 r$ to the edges $e_l, l = 12, \cdots, (r_1 - 1)r_1$ if e_k and e_l are incidence.

Example 3.1 Let G be a graph whose $V(G) = \{v_1, v_2, \cdots, v_8\}$, $E(G) = \{e_{12}, e_{13}, e_{15}, e_{24}, e_{26}, e_{34}, e_{37}, e_{48}, e_{56}, e_{57}, e_{68}, e_{78}\}$ and $f:\ G \to G'$ be a graph folding, see Fig.1.

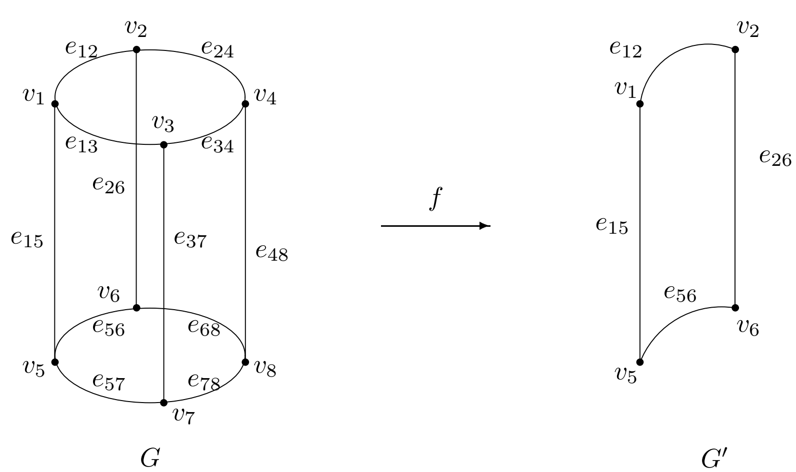


Fig.1

Then

$$I = \begin{array}{c} \begin{array}{cccccccc} v_1 & v_2 & v_5 & v_6 & v_3 & v_4 & v_7 & v_8 \end{array} \\ \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ e_{12} \\ e_{13} \\ e_{15} \\ e_{24} \\ e_{26} \\ e_{34} \\ e_{37} \\ e_{48} \\ e_{56} \\ e_{57} \\ e_{68} \\ e_{78} \end{array}$$

§4. Incidence Matrices and Operations on Graph Foldings

Let G_1 and G_2 be finite graphs with $V_1 = V(G_1) = \{v_1, v_2, \cdots, v_{r_1}, v_{r_1+1}, \cdots, v_r\}$, $V_2 = V(G_2) = \{v_1, v_2, \cdots, v_{s_1}, v_{s_1+1}, \cdots, v_s\}$, $E_1 = E(G_1) = \{e_{12}, \cdots, e_{1r}, e_{23}, \cdots, e_{2r}, \cdots e_{r_1 r}, \cdots, e_{(r-1)r}\}$ and $E_2 = E(G_2) = \{e_{12}, \cdots, e_{1s}, e_{23}, \cdots, e_{2s}, \cdots, e_{s_1 s}, \cdots, e_{(s-1)s}\}$.

Let $f \in \mu(G_1)$ such that $f(G_1) = G_1' \neq G_1$ and $g \in \mu(G_2)$ such that $g(G_2) = G_2' \neq G_2$, and the incidence matrices $I(G_1) = (\lambda_{k_1 d_1})$, where $k_1 = \{1, 2, \cdots, r_1, r_1+1, \cdots, r\}$, $d_1 = \{12, \cdots, 1r, 23, \cdots, 2r, \cdots, r_1 r, \cdots, (r-1)r\}$ and $I(G_2) = (\lambda_{k_2 d_2})$, where $k_2 = \{1, 2, \cdots, s_1, s_1 + 1, \cdots, s\}$ and $d_2 = \{12, \cdots, 1s, 23, \cdots, 2s, \cdots, s_1 s, \cdots, (s-1)s\}$, respectively.

Then the graph maps $f \bigcup g: \ G_1 \bigcup G_2 \to G_1' \bigcup G_2'$ defined by

$$(i) \quad \forall v \in V_1 \bigcup V_2, \ (f \bigcup g)(v) = \begin{cases} f(v) & \text{if } \ v \in V_1, \\ g(v) & \text{if } \ v \in V_2. \end{cases}$$

$$(ii) \quad \forall e \in E_1 \bigcup E_2, \ (f \bigcup g)(e) = \begin{cases} f(e) & \text{if } \ e \in E_1, \\ g(e) & \text{if } \ e \in E_2 \end{cases}$$

and $f \bigcap g: \ G_1 \bigcap G_2 \to G_1' \bigcap G_2'$ defined by:

(i) $\forall v \in V_1 \bigcap V_2$, $(f \bigcap g)(v) = f(v)$ or $g(v)$, where $V_1 \bigcap V_2 \neq \emptyset$.

(ii) $\forall e \in E_1 \bigcap E_2$, $(f \bigcap g)(e) = f(e)$ or $g(e)$,

are graph foldings iff f and g are graph foldings.

The incidence matrices $I(G_1 \bigcup G_2)$ and $I(G_1 \bigcap G_2)$ can be obtained from $I(G_1)$ and $I(G_2)$ as follows:

$I(G_1 \bigcup G_2) = (\lambda_{kd})$, where $k = \{1, 2, \cdots, r_1, r_1 + 1, \cdots, r\} \bigcup \{1, 2, \cdots, s_1, s_1 + 1, \cdots, s\}$ and $d = \{12, \cdots, 1r, 23, \cdots, 2r, \cdots, r_1 r, \cdots, (r_1 - 1)r\} \bigcup \{12, \cdots, 1s, 23, \cdots, 2s, \cdots, s_1 s, \cdots, (s_1 -$

$1)s\}$ such that

$\lambda_{kd} = 1$ *if* $\lambda_{k_1 d_l} = 1$ *in* $I(G_1)$ *or* $\lambda_{k_2 d_2} = 1$ *in* $I(G_2)$ *and* $\lambda_{kd} = 0$ *if* $\lambda_{k_1 d_l} = 0$ *in* $I(G_1)$ *or* $\lambda_{k_2 d_2} = 0$ *in* $I(G_2)$

and $I(G_1 \bigcap G_2) = (\lambda_{kd})$, where $k = \{1, 2, \cdots, r_1, r_1{+}1, \cdots, r\} \bigcap \{1, 2, \cdots, s_1, s_1{+}1, \cdots, s\}$, $d = \{12, \cdots, 1r, 23, \cdots, 2r, \cdots, r_1 r, \cdots, (r_1 - 1)r\} \bigcap \{12, \cdots, 1s, 23, \cdots, 2s, \cdots, s_1 s, \cdots, (s_1 - 1)s\}$ such that

$\lambda_{kd} = 1$ *if* $\lambda_{k_1 d_l} = 1$ *in* $I(G_1)$ *and* $\lambda_{k_2 d_2} = 1$ *in* $I(G_2)$ *and* $\lambda_{kd} = 0$ *if* $\lambda_{k_1 d_l} = 0$ *in* $I(G_1)$ *and* $\lambda_{k_2 d_2} = 0$ *in* $I(G_2)$.

Example 4.1 Let G_1, G_2 be two graphs with $V(G_1) = \{v_1, v_2, v_3, v_4\}$, $E(G_1) = \{e_{12}, e_{14}, e_{23}, e_{24}, e_{34}\}$, $V(G_2) = \{v_1, v_2, v_3, v_5\}$ and $E(G_2) = \{v_1, v_2, v_3, v_5\}$ and let $f : G_1 \to G_1'$, $g : G_2 \to G_2'$ be graph foldings.

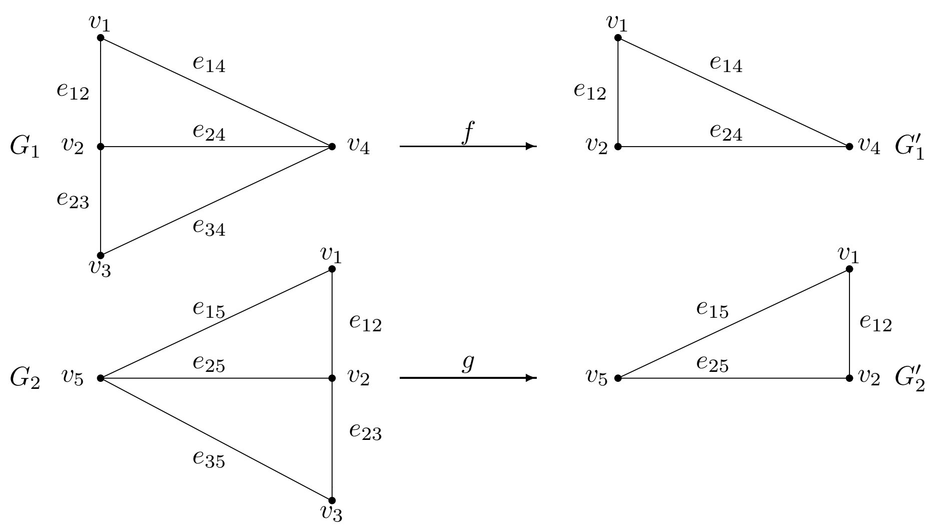


Fig.2

Then

$$I(G_1) = \begin{array}{cc} \begin{array}{cccc} v_1 & v_2 & v_4 & v_3 \end{array} & \\ \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] & \begin{array}{c} e_{12} \\ e_{14} \\ e_{24} \\ e_{23} \\ e_{34} \end{array} \end{array} \qquad I(G_2) = \begin{array}{cc} \begin{array}{cccc} v_1 & v_2 & v_5 & v_3 \end{array} & \\ \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] & \begin{array}{c} e_{12} \\ e_{15} \\ e_{25} \\ e_{23} \\ e_{35} \end{array} \end{array}$$

$$I(G_1\bigcup G_2)=\begin{array}{c|c}\begin{array}{ccccc} v_1 & v_2 & v_4 & v_5 & v_3\end{array} & \\ \left[\begin{array}{cccc|c} 1&1&0&0&0\\ 1&0&1&0&0\\ 1&0&0&1&0\\ 0&1&1&0&0\\ 0&1&0&1&0\\ \hline 0&1&0&0&1\\ 0&0&1&0&1\\ 0&0&0&1&1\end{array}\right] & \begin{array}{c} e_{12}\\ e_{14}\\ e_{15}\\ e_{24}\\ e_{25}\\ e_{23}\\ e_{34}\\ e_{35}\end{array}\end{array}\qquad I(G_1\bigcap G_2)=\begin{array}{cc}\begin{array}{ccc} v_1 & v_2 & v_3\end{array} & \\ \left[\begin{array}{cc|c} 1&1&0\\ \hline 0&1&1\end{array}\right] & \begin{array}{c} e_{12}\\ e_{23}\end{array}\end{array}$$

Let G_1, G_2 be finite graphs such that $V_1 = V(G_1) = \{v_1, v_2, \cdots, v_{r_1}, v_{r_1+1}, \cdots, v_r\}$, $E_1 = E(G_1) = \{e_{12}, \cdots, e_{1r_1}, \cdots, e_{1r}, e_{23}, \cdots, e_{2r_1}, \cdots, e_{2r}, \cdots, e_{r_1 r}, \cdots, e_{(r-1)r}\}$, $V_2 = V(G_2) = \{v_{r+1}, \cdots, v_{s_1}, v_{s_1+1}, \cdots, v_s\}$, $E_2 = E(G_2) = \{e_{(r+1)(r+2)}, \cdots, e_{(r+1)s_1}, \cdots, e_{(r+1)s}, e_{(r+2)(r+3)}, \cdots, e_{(r+2)s_1}, \cdots, e_{(r+2)s}, \cdots, e_{s_1 s}, \cdots, e_{(s-1)s}\}$, where e_{ij} is the edge incidence with v_i and v_j, $e_{ij} = e_{ji}$. Let $f \in \mu(G_1)$ such that $f(G_1) = G_1' \neq G_1$ and $g \in \mu(G_2)$ such that $g(G_2) = G_2' \neq G_2$ with incidence matrices are $I(G_1)$ and $I(G_2)$ respectively. Then

(1) The join graph map $f \vee g : G_1 \vee G_2 \to G_1' \vee G_2'$ defined by

(i) $\forall v \in V_1 \bigcup V_2$, $(f \vee g)\{v\} = \begin{cases} f\{v\} \text{ if } v \in V_1, \\ g\{v\} \text{ if } v \in V_2. \end{cases}$

(ii) $\forall e = (v_1, v_2), v_1 \in V_1, v_2 \in V_2$,

$$(f \vee g)\{e\} = (f \vee g)\{(v_1, v_2)\} = \{(f(v_1), f(v_2))\} \in G_1' \vee G_2'.$$

(iii) if $e = (u_1, v_1) \in E_1$, then $(f \vee g)\{e\} = (f \vee g)\{(u_1, v_1)\} = \{(f(u_1), g(v_1))\}$. Also, if $e = (u_2, v_2) \in E_2$, then $(f \vee g)\{e\} = (f \vee g)\{(u_2, v_2)\} = \{(g(u_2), g(v_2))\}$. Note that if $f\{u_1\} = f\{v_1\}$, then the image of the join graph map $(f \vee g)\{e\}$ will be a vertex of $G_1' \vee G_2'$, otherwise it will be an edge of $G_1' \vee G_2'$ ([4]), is a graph folding iff f and g are graph foldings.

The incident matrix $I(G_1 \bigcup G_2)$ can be defined from $I(G_1)$ and $I(G_2)$ as follows:

$$I(G_1)=\begin{array}{cc}\begin{array}{cc} \overline{v}_1 & \overline{v}_1'\end{array} & \\ \left[\begin{array}{c|c} I(G_1') & O\\ \hline Q & R\end{array}\right] & \begin{array}{c}\overline{e}_1\\ \overline{e}_1'\end{array}\end{array}\qquad I(G_2)=\begin{array}{cc}\begin{array}{cc} \overline{v}_2 & \overline{v}_2'\end{array} & \\ \left[\begin{array}{c|c} I(G_2') & O\\ \hline Q & R\end{array}\right] & \begin{array}{c}\overline{e}_2\\ \overline{e}_2'\end{array}\end{array}$$

where, $\overline{v}_1 = (v_1, v_2, \cdots, v_{r_1})$, $\overline{v}_1' = (v_{r_1+1}, \cdots, v_r)$, $\overline{e}_1 = \left(e_{12}, \cdots, e_{1r_1}, e_{23}, \cdots, e_{2r_1}, \cdots, e_{(r_1-1)r_1}\right)^T$, $\overline{e}_1' = \left(e_{1(r_1+1)}, \cdots, e_{1r}, e_{2(r_1+1)}, \cdots, e_{2r}, \cdots, e_{r_1(r_1+1)}, \cdots, e_{r_1 r}\right)^T$, $\overline{v}_2 = \{v_{r+1}, \cdots, v_{s_1}\}$, $\overline{v}_2' =$

$\{v_{s_1+1},\cdots,v_s\}$, $\overline{e}_1=\big(e_{(r+1)(r+2)},\cdots,e_{(r+1)s_1},e_{(r+2)(r+3)},\cdots,e_{(r+2)s_1},\cdots,e_{(s_1-1)s_1}\big)^T$, $\overline{e}'_1=\big(e_{(r+1)(s_1+1)},\cdots,e_{(r+1)s},e_{(r+2)(s_1+1)},\cdots,e_{(r+2)s},\cdots,e_{s_1(s_1+1)},\cdots,e_{s_1s}\big)^T$.

Then

$$I(G_1\vee G_2)=\begin{array}{cc} \begin{array}{cc} \overline{v}_\vee & \overline{v}'_\vee \end{array} & \\ \left[\begin{array}{c|c} I(G'_1\vee G'_2) & O \\ \hline Q & R \end{array}\right] & \begin{array}{c} \overline{e}_\vee \\ \overline{e}'_\vee \end{array} \end{array}$$

where,

$\overline{v}_\vee=(v_1,v_2,\cdots,v_{r_1},v_{r_1+1},\cdots,v_{s_1})$, $\overline{v}'_\vee=\{v_{r_1+1},\cdots,v_r,v_{s_1+1},\cdots,v_s\}$,

$\overline{e}_\vee=(e_{12},\cdots,\ e_{1r_1},e_{23},\cdots,e_{2r_1},\cdots,e_{(r_1-1)r_1},e_{(r+1)(r+2)},\cdots,e_{(r+1)s_1},e_{(r+2)(r+3)},\cdots,$
$e_{(r+2)s_1},\cdots,\ e_{(s_1-1)s_1},e_{1(r+1)},\cdots,e_{1s_1},e_{2(r+1)},\cdots,e_{2s_1},\cdots,e_{r_1(r+1)},\cdots.e_{r_1s_1}\big)^T$ and

$\overline{e}'_\vee=\big(e_{1(r_1+1)},\cdots,e_{1r},e_{2(r_1+1)},\cdots,e_{2r},\cdots,\ e_{r_1(r_1+1)},\cdots,e_{r_1r},e_{(r+1)(s_1+1)},\cdots,e_{(r+1)s},$
$e_{(r+2)(s_1+1)},\ \cdots,e_{(r+2)s},\cdots,e_{s_1(s_1+1)},\cdots,e_{s_1s},\ e_{1(s_1+1)},\cdots,e_{1s},e_{2(s_1+1)},\cdots,e_{2s},$
$\cdots,e_{r_1(s_1+1)},\cdots,e_{r_1s},\ e_{(r_1+1)(r+1)},\cdots,e_{(r_1+1)s},\cdots,e_{r(r+1)},\cdots,e_{rs_1},e_{(r_1+1)(s_1+1)},$
$\cdots,e_{(r_1+1)s},\cdots,e_{r(s_1+1)},\cdots,e_{rs}\big)^T$.

Thus, $I(G_1\vee G_2)=(\lambda_{kd})$, where $k=1,2,\cdots,r_1,r_1+1,\cdots,r,r+1,\cdots,s_1,s_1+1,\cdots,s$, $d=ij$, $i\neq j$ and $i,j=1,2,\cdots,r_1,r_1+1,\cdots,r,r+1,\cdots,s_1,s_1+1,\cdots,s$. It is clear that if $I(G_1)$ has order $m_1\times n_1$ and $I(G_2)$ has order $m_2\times n_2$, then $I(G_1\vee G_2)$ has order $(m_1+m_2+n_1n_2)(n_1+n_2)$.

(2) The cartesian product graph map $f\times g:\ G_1\times G_2\to G'_1\times G'_2$ defined by

(i) if $v=(v_1,v_2)\in V_1\times V_2,v_1\in V_1,v_2\in V_2$, then $(f\times g)\{(v_1,v_2)\}=\{(f\{v_1\},g\{v_2\})\}\in G'_1\times G'_2$.

(ii) if $e=\{(\{v_1\}_i,\{v_2\}_j),(\{v_1\}_i,\{v_2\}_k)\},\{v_1\}_i\in V(G_1)$ and $\{v_2\}_j,\{v_2\}_k\in V(G_2)$, then

$$f\times g\{(\{v_1\}_i,\{v_2\}_j),(\{v_1\}_i,\{v_2\}_k)\}=\{(\{v_1\}_i,g\{v_2\}_j),(\{v_1\}_i,g\{v_2\}_k)\}.$$

But if $e=\{(\{v_1\}_i,\{v_2\}_j),(\{v_1\}_k,\{v_2\}_j)\}$, where $\{v_1\}_i,\{v_1\}_k\in V(G_1)$, $\{v_2\}_j\in V(G_2)$, then

$$(f\times g)\{(\{v_1\}_i,\{v_2\}_j),(\{v_1\}_k,\{v_2\}_j)\}=\{(f\{v_1\}_i,\{v_2\}_j),(f\{v_1\}_k,\{v_2\}_j)\}.$$

Note that if $g\{v_2\}_j=g\{v_2\}_k$, or $f\{v_1\}_i=f\{v_1\}_k$, the image of the edge e will be a vertex ([4]), is a graph folding iff f and g are graph foldings. The incidence matrix $I(G_1\times G_2)$ can be defined from $I(G_1)$ and $I(G_2)$ as follows:

$$
I(G_1 \times G_2) = \begin{array}{cc} \begin{array}{cc} \overline{v}_\times & \overline{v}'_\times \end{array} & \\ \left[\begin{array}{c|c} I(G'_1 \vee G'_2) & O \\ \hline Q & R \end{array}\right] & \begin{array}{c} \overline{e}_\times \\ \overline{e}'_\times \end{array} \end{array}
$$

where,

$\overline{v}_\times = ((v_1, v_{r+1}), \cdots, (v_1, v_{s_1}), \cdots, (v_{r_1}, v_{r+1}), \cdots, (v_{r_1}, v_{s_1}))$,
$\overline{v}'_\times = ((v_1, v_{s_1+1}), \cdots, (v_1, v_s), \cdots, (v_{r_1}, v_s), (v_{r_1+1}, v_{r+1}), \cdots, (v_{r_1+1}, v_{s_1}), \cdots, (v_r, v_{s_1}))$,
$\overline{e}_\times = \big((e_{12}, v_{r+1}), \cdots, (e_{12}, v_{s_1}), \cdots, (e_{1r_1}, v_{r+1}), \cdots, (e_{1r_1}, v_{s_1}), \cdots (e_{(r_1-1)r_1}, v_{r+1}), \cdots, (e_{(r_1-1)r_1}, v_{s_1}), (v_1, e_{(r+1)(r+2)}), \cdots, (v_{r_1}, e_{(r+1)(r+2)}), \cdots, (v_1, e_{(r+1)s_1}), \cdots, (v_{r_1}, e_{(r+1)s_1}), \cdots, (v_1, e_{(s_1-1)s_1}), \cdots, (v_{r_1}, e_{(s_1-1)s_1})\big)^T$ and
$\overline{e}'_\times = (\ (e_{1(r_1+1)}, v_{r+1}), \cdots, (e_{1(r_1+1)}, v_{s_1}), \cdots, (e_{1r}, v_{r+1}), \cdots, (e_{1r}, v_{s_1}), \cdots, e_{r_1r}, v_{r+1}), \cdots, (e_{r_1r}, v_{s_1}), (v_1, e_{(r+1)(s_1+1)}), \cdots, (v_{r_1}, e_{(r+1)(s_1+1)}), \cdots, (v_1, e_{(r+1)s}), \cdots, (e_{(r+1)s}, v_{r_1}), \cdots, (v_1, e_{s_1s}), \cdots, (v_{r_1}, e_{s_1s}), (e_{12}, v_{s_1+1}), \cdots, (e_{12}, v_s), \cdots, (e_{1r}, v_{s_1+1}), \cdots, (e_{1r}, v_s), \cdots, (e_{(r_1-1)r_1}, v_{s_1+1}), (e_{(r_1-1)r_1}, v_s), (v_{r_1+1}, e_{(r+1)(r+2)}), \cdots, (v_r, e_{(r+1)(r+2)}), \cdots, (v_{r_1+1}, e_{(r+1)s_1}), \cdots, (v_r, e_{(r+1)s_1}), \cdots, (v_{r+1}, e_{(s_1-1)s_1}), \cdots, (v_r, e_{(s_1-1)s_1}), (e_{1(r_1+1)}, v_{s_1+1}), \cdots, (e_{1(r_1+1)}, v_{s+1}), \cdots, (e_{1r}, v_{s_1+1}), \cdots, (e_{1r}, v_s), \cdots, (e_{r_1r}, v_{s_1+1}), \cdots, (e_{r_1r}, v_s), (v_{r_1+1}, e_{(r+1)(s_1+1)}), \cdots, (v_{r_1+1}, e_{(r+1)s_1}), \cdots, (v_{r_1+1}, e_{s_1s}), \cdots, (v_r, e_{(r+1)(s_1+1)}), \cdots, (v_r, e_{(r+1)s}), \cdots, (v_r, e_{s_1s})\big)^T$.

It is clear that if $I(G_1)$ has order $m_1 \times n_1$ and $I(G_2)$ has order $m_2 \times n_2$, then $I(G_1 \times G_2)$ has order $(m_1n_2 + m_2n_1) \times (n_1n_2)$.

(3) The tensor product graph map $f \otimes g : G_1 \otimes G_2 \to G'_1 \otimes G'_2$ defined by:

(i) if $v = (v_1, v_2) \in V(G_1 \otimes G_2) = V_1 \times V_2$, then $(f \otimes g)\{(v_1, v_2)\} = \{(f\{v_1\}, g\{v_2\})\} \in V(G'_1 \otimes G'_2)$;

(ii) let $e = \{(\{v_1\}_i, \{v_2\}_j), (\{v_1\}_k, \{v_2\}_l)\}$, where $\{v_1\}_i$ is adjacent to $\{v_2\}_k$ and $\{v_2\}_j$ is adjacent to $\{v_2\}_l$, then $(f \otimes g)\{e\} = f\{(\{v_1\}_i, \{v_1\}_k)\} \otimes g\{(\{v_2\}_j, \{v_2\}_l$, i.e., $(f \otimes g)(G_1 \otimes G_2) = f(G_1) \times g(G_2)$ ([4]) is a graph folding is a graph folding iff f and g are graph foldings.

The incidence matrix $I(G_1 \otimes G_2)$ can be defined from $I(G_1)$ and $I(G_2)$ as follows:

$$
I(G_1 \otimes G_2) = \begin{array}{cc} \begin{array}{cc} \overline{v}_\otimes & \overline{v}'_\otimes \end{array} & \\ \left[\begin{array}{c|c} I(G'_1 \otimes G'_2) & O \\ \hline Q & R \end{array}\right] & \begin{array}{c} \overline{e}_\otimes \\ \overline{e}'_\otimes \end{array} \end{array}
$$

where,

$$\overline{v}_{\otimes} = ((v_1, v_{r+1}), \cdots, (v_1, v_{s_1}), \cdots, (v_{r_1}, v_{r+1}), \cdots, (v_{r_1}, v_{s_1})),$$
$$\overline{v}'_{\otimes} = ((v_1, v_{s_1+1}), \cdots, (v_1, v_s), \cdots, (v_{r_1}, v_s), (v_{r_1+1}, v_{r+1}), \cdots, (v_{r_1+1}, v_{s_1}), \cdots, (v_r, v_{s_1})),$$
$$\overline{e}_{\otimes} = \big(e_{(1,r+1)(2,r+2)}, e_{1,r+2)(2,r+1)}, \cdots, e_{(1,s_1-1)(2,s_1)}, e_{(1,s_1)(2,s_1-1)}, \cdots, e_{(r_1-1,r+1)(r_1,r+2)}, e_{(r_1-1,r+2)(r_1,r+1)}, \cdots, e_{(r_1-1,s_1-1)(r_1,s_1)}, e_{(r_1-1,s_1)(r_1,s_1-1)}\big)^T \text{ and}$$
$$\overline{e}'_{\otimes} = \big(e_{(1,r+1)(2,s_1+1)}, e_{(1,s_1+1)(2,r+1)}, \cdots, e_{(1,s_1)(2,s)}, e_{(1,s)(2,s_1)}, \cdots, e_{(r_1-1,r+1)(r_1,s_1+1)}, e_{(r_1-1,s_1+1)(r_1,r+1)}, \cdots, e_{(r_1-1,s_1)(r_1,s)}, e_{(r_1-1,s)(r_1,s_1)}, e_{(1,r+1)(r_1+1,r+2)}, e_{(1,r+2)(r_1+1,r+1)}, \cdots, e_{(1,s_1-1)(r_1+1,s_1)}, e_{(1,s_1)(r_1+1,s_1-1)} \cdots, e_{(r_1,r+1)(r,r+2)}, e_{(r_1,r+2)(r,r+1)}, \cdots, e_{(r_1,s_1-1)(r,s_1)}, e_{(r_1,s_1)(r,s_1-1)}, e_{(1,r+1)(r_1+1,s_1+1)}, e_{(1,s_1+1)(r_1+1,r+1)}, \cdots, e_{(1,s_1)(r_1+1,s)}, e_{(1,s)(r_1+1,s_1)}, \cdots, e_{(r_1,r+1)(r,s_1+1)}, e_{(r_1,s_1+1)(r,r+1)}, \cdots, e_{(r_1,s_1)(r,s)}, e_{(r_1,s)(r,s_1)}\big)^T.$$

It is clear that if $I(G_1)$ has order $m_1 \times n_1$ and $I(G_2)$ has order $m_2 \times n_2$, then $I(G_1 \otimes G_2)$ has order $(2m_1m_2) \times (n_1n_2)$.

(4) The normal product graph map $f \circ g : G_1 \circ G_2 \to G'_1 \circ G_2$ defined by

(i) for any vertex $v = (v_1, v_2) \in V(G_1 \circ G_2) = V_1 \times G_2$, then

$$(f \circ g)\{(v_1, v_2)\} = \{(f\{v_1\}, g\{v_2\})\} \in V(G'_1 \circ G'_2);$$

(ii) for any edge $e = \{(\{v_1\}_i, \{v_2\}_j), (\{v_1\}_k, \{v_2\}_l)\}$, then

$$\begin{aligned}(f \circ g)\{e\} &= (f \circ g)\{(\{v_1\}_i, \{v_2\}_j), (\{v_1\}_k, \{v_2\}_l)\}\\ &= f\{(\{v_1\}_i, g\{v_2\}_j), (\{v_1\}_k, g\{v_2\}_l)\}\\ &= \{(f\{v_1\}_i, g\{v_2\}_j), (f\{v_1\}_k, g\{v_2\}_l)\}.\end{aligned}$$

Note that if $f\{v_1\}_i = f\{v_1\}_k$ and $g\{v_2\}_j = g\{v_2\}_l$, then $(f \circ g)\{e\}$ will be a vertex ([4]) is a graph folding is a graph folding if f and g are graph foldings.

The incidence matrix $I(G_1 \circ G_2)$ can be obtained from $I(G_1 \times G_2)$ and $I(G_1 \otimes G_2)$, since $G_1 \circ G_2 = (G_1 \times G_2) \bigcup (G_1 \otimes G_2)$.

It is clear that if $I(G_1)$ has order $m_1 \times n_1$ and $I(G_2)$ has order $m_2 \times n_2$, then $I_{f \circ g}$ has order $(m_1n_2 + m_2n_1 + 2m_1m_2) \times (n_1n_2)$.

(5) The composition product graph map $f[g]:\ G_1[G_2] \to G'_1[G'_2]$ defined by:

(i) if $v = (v_1, v_2) \in V(G_1[G_2]) = V1 \times V_2$, then $f[g]\{(v_1, v_2)\} = \{(f\{v_1\}, g\{v_2\})\} \in (G'_1[G'_2]);$

(ii) let $e = \{(\{v_1\}_i, \{v_2\}_j), (\{v_1\}_k, \{v_2\}_l)\}$. If $\{v_1\}_i$ is adjacent to $\{v_1\}_k$, then $f[g]\{e\} = \{(\{v_1\}_i, g\{v_2\}_j), (f\{v_1\}_k, g\{v_2\}_l)\}$. If $\{v_1\}_i = \{v_1\}_k$ and $\{v_2\}_j$ is adjacent to $\{v_2\}_l$, then $f[g]\{e\} = \{(\{v_1\}_i, g\{v_2\}_j), (f\{v_1\}_i, g\{v_2\}_l)\}$.

Note that if $f\{v_1\}_i = f\{v_1\}_k$ and $g\{v_2\}_j = g\{v_2\}_l$, then $f[g]\{e\}$ will be a vertex, also if $g\{v_2\}_j = g\{v_2\}_l$, then $f[g]$ will be a vertex ([4]) is a graph folding is a graph folding if f, g are graph foldings and the incidence matrix $I(G_1[G_2])$ can be obtained from $I(G_1)$ and $I(G_2)$ as follows:

$$
\begin{array}{ccc}
 & \begin{array}{cc} \overline{v}_\square & \overline{v}'_\square \end{array} & \\
I(G_1[G_2]) = & \left[\begin{array}{c|c} I(G'_1[G'_2]) & O \\ \hline Q & R \end{array}\right] & \begin{array}{c} \overline{e}_\square \\ \overline{e}'_\square \end{array}
\end{array}
$$

where,

$$
\begin{aligned}
\overline{v}_\square &= ((v_1, v_{r+1}), \cdots, (v_1, v_{s_1}), \cdots, (v_{r_1}, v_{r+1}), \cdots, (v_{r_1}, v_{s_1})),\\
\overline{v}'_\square &= ((v_1, v_{s_1+1}), \cdots, (v_1, v_s), \cdots, (v_{r_1}, v_s), (v_{r_1+1}, v_{r+1}), \cdots, (v_{r_1+1}, v_{s_1}), \cdots, (v_r, v_{s_1})),\\
\overline{e}_\square &= \big((v_1, e_{(r+1)(r+2)}, \cdots, (v_1, e_{(s_1-1)s_1}), \cdots, (v_{r_1}, e_{(r+1)(r+2)}), \cdots, (v_{r_1}, e_{(s_1-1)s_1}),\\
&\quad e_{(1,r+1)(2,r+1)}, \cdots, e_{(r_1-1,r+1)(r_1,r+1)}, e_{(1,r+1)(2,r+2))}, e_{(1,r+2)(2,r+1)}, \cdots,\\
&\quad e_{(r_1-1,r+1)(r_1,r+2)}, e_{(r_1-1,r+2)(r_1,r+1)}, \cdots, e_{(1,r+1)(2,s_1)}, e_{(1,s_1)(2,r+1)}, \cdots,\\
&\quad e_{(r_1-1,r+1)(r_1,s_1)}, e_{(r_1-1,s_1)(r_1,r+1)}, e_{(1,r+2)(2,r+2)}, \cdots, e_{(r_1-1,r+2)(r_1,r+2)},\\
&\quad \cdots, e_{(1,r+2)(2,s_1)}, e_{(1,s_1)(2,r+2)}, \cdots, e_{(r_1-1,r+2)(r_1,s_1)}, e_{(r_1-1,s_1)(r_1,r+2)}, \cdots,\\
&\quad e_{(1,s_1-1)(2,s_1)}, e_{(1,s_1)(2,s_1-1)}, \cdots, e_{(r_1-1,s_1-1)(r_1,s_1)}, e_{(r_1-1,s_1)(r_1,s_1-1)}, e_{(1,s_1)(2,s_1)},\\
&\quad \cdots, e_{(r_1-1,s_1)(r_1,s_1)}\big)^T \text{ and}\\
\overline{e}'_\square &= \big((v_1, e_{(r+1)(s_1+1)}), \cdots, (v_1, e_{s_1 s}), \cdots, (v_{r_1}, e_{(r+1)(s_1+1)}), \cdots, (v_{r_1}, e_{s_1 s}),\\
&\quad (v_{r_1+1}, e_{(r+1)(r+2)}), \cdots, (v_{r_1+1}, e_{(r+1)(r+2)}), \cdots, (v_{r_1+1}, e_{(s_1-1)s_1}),\\
&\quad (v_{r_1+1}, e_{(r+1)(s_1+1)}), \cdots, (v_{r_1+1}, e_{s_1 s}), \cdots, (v_r, e_{(r+1)(r+2)}), \cdots, (v_r, e_{(s_1-1)s_1}),\\
&\quad (v_r, e_{(r+1)(s_1+1)}), \cdots, (v_r, e_{s_1 s}), e_{(1,r+1)(r_1+1,r+1)}, \cdots, e_{(r_1,r+1)(r,r+1)},\\
&\quad e_{(1,r+1)(r_1+1,r+2)}, e_{(1,r+2)(r_1+1,r+1)}, \cdots, e_{(r_1,r+1)(r,r+2)}, e_{(r_1,r+2)(r,r+1)}, \cdots,\\
&\quad e_{(1,r+1)(r_1+1,s_1)}, e_{(1,s_1)(r_1+1,r+1)}, \cdots, e_{(r_1,r+1)(r,s_1)}, e_{(r_1,s_1)(r,r+1)}, \cdots,\\
&\quad e_{(1,r+1)(r_1+1,s)}, e_{(1,s)(r_1+1,r+1)}, \cdots, e_{(r_1,r+1)(r,s)}, e_{(r_1,s)(r,r+1)}, e_{(1,r+2)(r_1+1,r+2)}, \cdots,\\
&\quad e_{(r_1,r+2)(r,r+2)}, \cdots, e_{(1,r+2)(r_1+1,s)}, e_{(1,s)(r_1+1,r+2)}, \cdots, e_{(r_1,r+2)(r,s)}, e_{(r_1,s)(r,r+2)},\\
&\quad \cdots, e_{(1,s_1)(r_1+1,s_1)}, \cdots, e_{(r_1,s_1)(r,s_1)}, \cdots, e_{(1,s_1)(r,s_1)}, e_{(1,s)(r_1+1,s_1)}, \cdots, e_{(r_1,s_1)(r,s)},\\
&\quad e_{(r_1,s)(r,s_1)}, \cdots, e_{(1,s)(r_1+1,s)}, \cdots, e_{(r_1,s)(r,s)}, e_{(1,r+1)(2,s_1+1)}, e_{(1,s_1+1)(2,r+1)}, \cdots,\\
&\quad e_{(r_1-1,r+1)(r_1,s_1+1)}, e_{(r_1-1,s_1+1)(r_1,r+1)}, \cdots, e_{(1,r+1)(2,s)}, e_{(1,s)(2,r+1)}, \cdots,\\
&\quad e_{(r_1-1,r+1)(r_1,s)}, e_{(r_1-1,s)(r_1,r+1)}, \cdots, e_{(1,s_1)(2,s)}, e_{(1,s)(2,s_1)}, \cdots, e_{(r_1-1,s_1)(r_1,s)},\\
&\quad e_{(r_1-1,s)(r_1,s_1)}, e_{(1,s_1+1)(2,s_1+1)}, \cdots, e_{(r_1-1,s_1+1)(r_1,s_1+1)}, \cdots, e_{(1,s_1+1)(2,s)},\\
&\quad e_{(1,s)(2,s_1+1)}, \cdots, e_{(r_1-1,s_1+1)(r_1,s)}, e_{(r_1-1,s)(r_1,s_1+1)}, \cdots, e_{(1,s)(2,s)}, \cdots, e_{(r_1-1,s)(r_1,s)}\big)^T
\end{aligned}
$$

It is clear that if $I(G_1)$ has order $m_1 \times n_1$ and $I(G_2)$ has order $m_2 \times n_2$, then $I(G_1[G_2])$ has order $(n_1 m_2 + n_2 m_1) \times n_1 n_2$.

Example 4.2 Let G_1 and G_2 be two graphs such that $V(G_1) = \{v_1, v_2, v_3, v_4\}$, $E(G_1) = \{e_{12}, e_{13}, e_{14}, e_{23}, e_{34}\}$, $V(G_2) = \{v_5, v_6, v_7\}$, $E(G_2) = \{e_{56}, e_{57}\}$, and $f : G_1 \to G'_1$, $g : G_2 \to G'_2$ be graph foldings, see Fig.3.

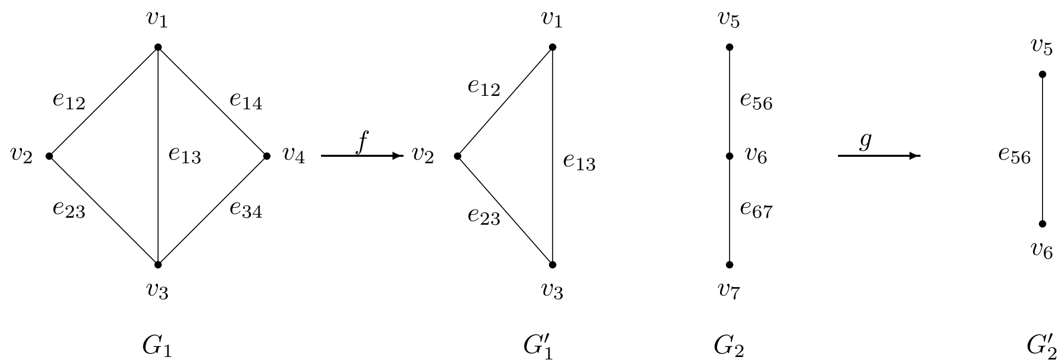


Fig.3

Their incidence matrixes are shown in the following.

$$I(G_1) = \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ e_{12} \\ e_{13} \\ e_{23} \\ e_{14} \\ e_{34} \end{array} \qquad I(G_2) = \begin{array}{c} \begin{array}{ccc} v_5 & v_6 & v_7 \end{array} \\ \left[\begin{array}{cc|c} 1 & 1 & 0 \\ \hline 0 & 1 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ e_{56} \\ e_{67} \end{array}$$

Then we know that $f \vee g$ is a graph folding, see Fig.4.

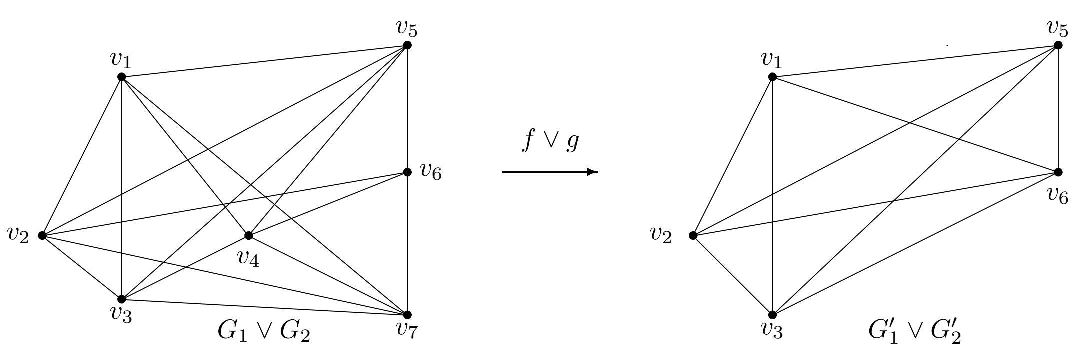
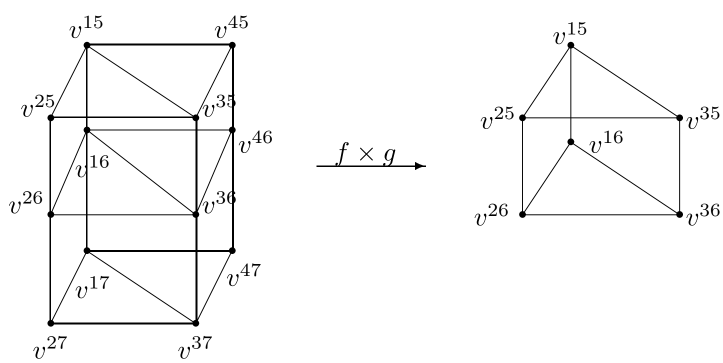


Fig.4

$$I(G_1 \vee G_2) = \begin{array}{c} \begin{array}{ccccccc} v_1 & v_2 & v_3 & v_5 & v_6 & v_4 & v_7 \end{array} \\ \left[\begin{array}{ccccc|cc} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \end{array} \begin{array}{l} \\ e_{12} \\ e_{13} \\ e_{23} \\ e_{56} \\ e_{15} \\ e_{16} \\ e_{25} \\ e_{26} \\ e_{35} \\ e_{36} \\ e_{14} \\ e_{34} \\ e_{67} \\ e_{17} \\ e_{27} \\ e_{37} \\ e_{45} \\ e_{46} \\ e_{47} \end{array}$$

We also know that $f \times g$ is a graph folding, seeing Fig.5,



$G_1 \times G_2$ $\qquad$ $G'_1 \times G'_2$

Fig.5

where $v^{15} = (v_1, v_5), v^{16} = (v_1, v_6), v^{25} = (v_2, v_5)$, $v^{16} = (v_2, v_6), v^{35} = (v_3, v_5), v^{36} = (v_3, v_6), v^{17} = (v_1, v_7), v^{27} = (v_2, v_7), v^{37} = (v_3, v_7), v^{45} = (v_4, v_5), v^{46} = (v_4, v_6), v^{47} = (v_4, v_7)$.

$$I(G_1 \times G_2) = \begin{array}{cc}
\begin{matrix} v^{15} & v^{16} & v^{25} & v^{26} & v^{35} & v^{36} & v^{17} & v^{27} & v^{37} & v^{45} & v^{46} & v^{47} \end{matrix} & \\
\left[\begin{array}{cccccc|cccccc}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}\right] &
\begin{array}{c}
(e_{12}, v_5) \\ (e_{12}, v_6) \\ (e_{13}, v_5) \\ (e_{13}, v_6) \\ (e_{23}, v_5) \\ (e_{23}, v_6) \\ (v_1, e_{56}) \\ (v_2, e_{56}) \\ (v_3, e_{56}) \\
(e_{14}, v_5) \\ (e_{14}, v_6) \\ (e_{34}, v_5) \\ (e_{34}, v_6) \\ (v_1, e_{67}) \\ (v_2, e_{67}) \\ (v_3, e_{67}) \\ (e_{12}, v_7) \\ (e_{13}, v_7), \\ (e_{23}, v_7) \\ (v_4, e_{56}) \\ (e_{14}, v_7) \\ (e_{34}, v_7) \\ (v_4, e_{67})
\end{array}
\end{array}$$

Similarly, we know that $f \otimes g$, $f \circ g$ and $f[g]$ are also graph foldings, seeing Fig 6- Fig.8, where $v^{ij} = (v_i, v_j)$ for integers $1 \leq i, j \leq 7$.

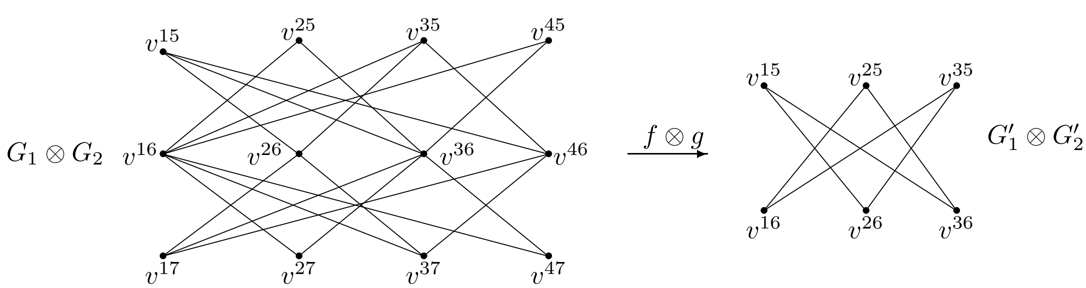
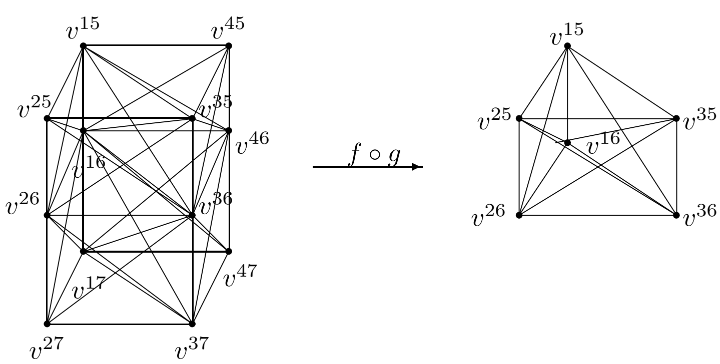


Fig.6

$$I(G_1 \otimes G_2) = \begin{array}{c} \begin{array}{cccccccccccc} v^{15} & v^{16} & v^{25} & v^{26} & v^{35} & v^{36} & v^{17} & v^{27} & v^{37} & v^{45} & v^{46} & v^{47} \end{array} \\ \left[\begin{array}{cccccc|cccccc}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0
\end{array}\right] \end{array}
\begin{array}{l} \\ e_{(1,5)(2,6)} \\ e_{(1,6)(2,5)} \\ e_{(1,5)(3,6)} \\ e_{(1,6)(3,5)} \\ e_{(2,5)(3,6)} \\ e_{(2,6)(3,5)} \\ e_{(1,6)(3,7)} \\ e_{(1,7)(2,6)} \\ e_{(1,6)(3,7)} \\ e_{(1,7)(3,6)} \\ e_{(2,6)(3,7)} \\ e_{(2,7)(3,6)} \\ e_{(1,5)(4,6)} \\ e_{(1,6)(4,5)} \\ e_{(3,5)(4,6)} \\ e_{(3,6)(4,5)} \\ e_{(1,6)(4,7)} \\ e_{(1,7)(4,6)} \\ e_{(3,6)(4,7)} \\ e_{(3,7)(4,6)} \end{array}$$



$G_1 \circ G_2$ $\qquad$ $G'_1 \circ G'_2$

Fig.7

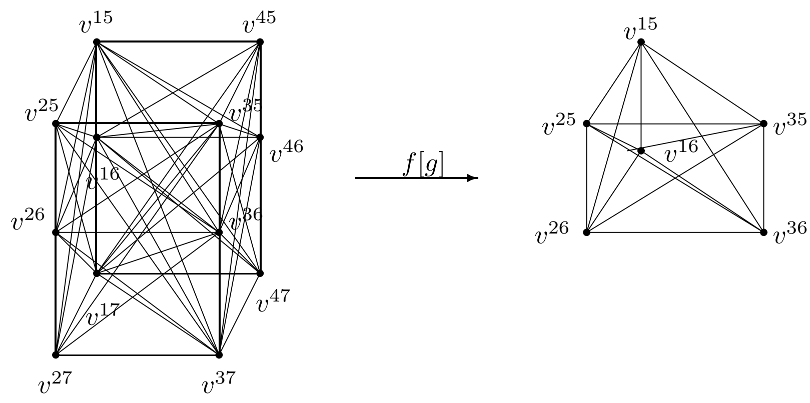
$$I(G_1 \circ G_2) = \left[\begin{array}{c|c} I(G'_1 \circ G'_2) & O \\ \hline Q & R \end{array}\right]$$

where,

$$I(G'_1 \circ G'_2) = \begin{array}{c} \begin{array}{cccccc} v^{15} & v^{16} & v^{25} & v^{26} & v^{35} & v^{36} \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ (e_{12}, v_5) \\ (e_{12}, v_6) \\ (e_{13}, v_5) \\ (e_{13}, v_6) \\ (e_{23}, v_5) \\ (e_{23}, v_6) \\ (v_1, e_{56}) \\ (v_2, e_{56}) \\ (v_3, e_{56}) \\ e_{(1,5)(2,6)} \\ e_{(1,6)(2,5)} \\ e_{(1,5)(3,6)} \\ e_{(1,6)(3,5)} \\ e_{(2,5)(3,6)} \\ e_{(2,6)(3,5)} \end{array}$$

$$O = (0)_{15\times 6}$$

$$
Q = \begin{array}{cc}
\begin{array}{cccccc} v^{15} & v^{16} & v^{25} & v^{26} & v^{35} & v^{36} \end{array} & \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} &
\begin{array}{c}
(e_{14}, v_5) \\
(e_{14}, v_6) \\
(e_{34}, v_5) \\
(e_{34}, v_6) \\
(v_1, e_{67}) \\
(v_2, e_{67}) \\
(v_3, e_{67}) \\
(e_{12}, v_7) \\
(e_{13}, v_7), \\
(e_{23}, v_7) \\
(v_4, e_{56}) \\
(e_{14}, v_7) \\
(e_{34}, v_7) \\
(v_4, e_{67}) \\
e_{(1,6)(2,7)} \\
e_{(1,7)(2,6)} \\
e_{(1,6)(3,7)} \\
e_{(1,7)(3,6)} \\
e_{(2,6)(3,7)} \\
e_{(2,7)(3,6)} \\
e_{(1,5)(4,6)} \\
e_{(1,6)(4,5)} \\
e_{(3,5)(4,6)} \\
e_{(3,6)(4,5)} \\
e_{(1,6)(4,7)} \\
e_{(1,7)(4,6)} \\
e_{(3,6)(4,7)} \\
e_{(3,7)(4,6)}
\end{array}
\end{array}
\qquad
R = \begin{array}{cc}
\begin{array}{cccccc} v^{15} & v^{16} & v^{25} & v^{26} & v^{35} & v^{36} \end{array} & \\
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0
\end{bmatrix} &
\begin{array}{c}
(e_{14}, v_5) \\
(e_{14}, v_6) \\
(e_{34}, v_5) \\
(e_{34}, v_6) \\
(v_1, e_{67}) \\
(v_2, e_{67}) \\
(v_3, e_{67}) \\
(e_{12}, v_7) \\
(e_{13}, v_7), \\
(e_{23}, v_7) \\
(v_4, e_{56}) \\
(e_{14}, v_7) \\
(e_{34}, v_7) \\
(v_4, e_{67}) \\
e_{(1,6)(2,7)} \\
e_{(1,7)(2,6)} \\
e_{(1,6)(3,7)} \\
e_{(1,7)(3,6)} \\
e_{(2,6)(3,7)} \\
e_{(2,7)(3,6)} \\
e_{(1,5)(4,6)} \\
e_{(1,6)(4,5)} \\
e_{(3,5)(4,6)} \\
e_{(3,6)(4,5)} \\
e_{(1,6)(4,7)} \\
e_{(1,7)(4,6)} \\
e_{(3,6)(4,7)} \\
e_{(3,7)(4,6)}
\end{array}
\end{array}
$$



$G_1[G_2]$ $\qquad$ $G_1'[G_2']$

Fig.8

$$I(G_1[G_2]) = \left[\begin{array}{c|c} I(G_1'[G_2']) & O \\ \hline Q & R \end{array}\right], \quad I(G_1'[G_2']) = \begin{array}{c} \begin{array}{cccccc} v^{15} & v^{16} & v^{25} & v^{26} & v^{35} & v^{36} \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array} , \quad \begin{array}{c} (v_1, e_{56}) \\ (v_2, e_{56}) \\ (v_3, e_{56}) \\ e_{(1,5)(2,5)} \\ e_{(1,5)(3,5)} \\ e_{(2,5)(3,5)} \\ e_{(1,5)(2,6)} \\ e_{(1,6)(2,5)} \\ e_{(1,5)(3,6)} \\ e_{(1,6)(3,5)} \\ e_{(2,5)(3,6)} \\ e_{(3,5)(2,6)} \\ e_{(1,6)(2,6)} \\ e_{(1,6)(3,6)} \\ e_{(2,6)(3,6)} \end{array}$$

$O = (0)_{15\times 6}$ and

$$Q = \begin{array}{cc}
\begin{array}{cccccc} v^{15} & v^{16} & v^{25} & v^{26} & v^{35} & v^{36} \end{array} & \\
\left[\begin{array}{cccccc}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 1 \\
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0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] &
\begin{array}{c}
(v_1, e_{67}) \\ (v_2, e_{67}) \\ (v_3, e_{67}) \\ (v_4, e_{56}) \\ (v_4, e_{67}) \\
e_{(1,5)(4,5)} \\ e_{(3,5)(4,5)} \\ e_{(1,5)(4,6)} \\ e_{(1,6)(4,5)} \\ e_{(3,5)(4,6)} \\ e_{(3,6)(4,5)} \\ e_{(1,5)(4,7)} \\ e_{(1,7)(4,5)} \\ e_{(3,5)(4,7)} \\ e_{(3,7)(4,5)} \\ e_{(1,6)(4,6)} \\ e_{(3,6)(4,6)} \\ e_{(1,6)(4,7)} \\ e_{(1,7)(4,6)} \\ e_{(3,6)(4,7)} \\ e_{(3,7)(4,6)} \\ e_{(1,7)(4,7)} \\ e_{(3,7)(4,7)} \\ e_{(1,5)(2,7)} \\ e_{(1,7)(2,5)} \\ e_{(1,5)(3,7)} \\ e_{(1,7)(3,5)} \\ e_{(2,5)(3,7)} \\ e_{(2,7)(3,7)} \\ e_{(2,7)(3,5)} \\ e_{(1,6)(2,7)} \\ e_{(1,7)(2,6)} \\ e_{(1,6)(3,7)} \\ e_{(1,7)(3,6)} \\ e_{(2,6)(3,7)} \\ e_{(1,7)(2,7)} \\ e_{(1,7)(3,7)} \\ e_{(2,7)(3,7)}
\end{array}
\end{array}$$

$$R = \begin{array}{cc}
\begin{array}{cccccc} v^{15} & v^{16} & v^{25} & v^{26} & v^{35} & v^{36} \end{array} & \\
\left[\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
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1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
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0 & 0 & 1 & 0 & 1 & 0 \\
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1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1
\end{array}\right] &
\begin{array}{c}
(v_1, e_{67}) \\ (v_2, e_{67}) \\ (v_3, e_{67}) \\ (v_4, e_{56}) \\ (v_4, e_{67}) \\
e_{(1,5)(4,5)} \\ e_{(3,5)(4,5)} \\ e_{(1,5)(4,6)} \\ e_{(1,6)(4,5)} \\ e_{(3,5)(4,6)} \\ e_{(3,6)(4,5)} \\ e_{(1,5)(4,7)} \\ e_{(1,7)(4,5)} \\ e_{(3,5)(4,7)} \\ e_{(3,7)(4,5)} \\ e_{(1,6)(4,6)} \\ e_{(3,6)(4,6)} \\ e_{(1,6)(4,7)} \\ e_{(1,7)(4,6)} \\ e_{(3,6)(4,7)} \\ e_{(3,7)(4,6)} \\ e_{(1,7)(4,7)} \\ e_{(3,7)(4,7)} \\ e_{(1,5)(2,7)} \\ e_{(1,7)(2,5)} \\ e_{(1,5)(3,7)} \\ e_{(1,7)(3,5)} \\ e_{(2,5)(3,7)} \\ e_{(2,7)(3,7)} \\ e_{(2,7)(3,5)} \\ e_{(1,6)(2,7)} \\ e_{(1,7)(2,6)} \\ e_{(1,6)(3,7)} \\ e_{(1,7)(3,6)} \\ e_{(2,6)(3,7)} \\ e_{(1,7)(2,7)} \\ e_{(1,7)(3,7)} \\ e_{(2,7)(3,7)}
\end{array}
\end{array}$$

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[1] E.M.El-Kholy and A.El-Esawy, Graph folding of some special graphs, *J. Math. Statistics* Nal., 1(2005), pp. 66-70, Sci. pub. U.S.A.

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