

## Characterizations of the Quaternionic Mannheim Curves In Euclidean space $\mathbb{E}^4$

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**Abstract:** In [5], Matsuda and Yorozu obtained that Mannheim curves in 4-dimensional Euclidean space. In this study, we define *quaternionic Mannheim curves* and we give some characterizations of them in Euclidean 3-space and 4-space.

**Key Words:** Quaternion algebra, Mannheim curve, Euclidean space.

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### §1. Introduction

The geometry of curves in a Euclidean space have been developed a long time ago and we have a deep knowledge about it. In the theory of curves in Euclidean space, one of the important and interesting problem is characterizations of a regular curve. We can characterize some curves via their relations between the Frenet vectors of them. For instance Mannheim curve is a special curve and it is characterized using the Frenet vectors of its Mannheim curve couple.

In 2007, the definition of Mannheim curves in Euclidean 3-space is given by H.Liu and F.Wang [4] with the following:

**Definition 1.1** *Let  $\alpha$  and  $\beta$  be two curves in Euclidean 3-space. If there exists a corresponding relationship between the space curves  $\alpha$  and  $\beta$  such that, at the corresponding points of the curves, the principal normal lines of  $\alpha$  coincides with the binormal lines of  $\beta$ , then  $\alpha$  is called a Mannheim curve, and  $\beta$  is called a Mannheim partner curve of  $\alpha$ .*

In their paper, they proved that a given curve is a Mannheim curve if and only if then for  $\lambda \in \mathbb{R}$ , it has  $\lambda(\kappa^2 + \tau^2) = \varkappa$ , where  $\varkappa$  and  $\tau$  are curvature functions of curve. Also in 2009, Matsuda and Yorozu, in [5], defined generalized Mannheim curves in Euclidean 4-space. If the first normal line at each point of  $\alpha$  is included in the plane generated by the second normal line and the third normal line of  $\beta$  at corresponding point under a bijection, which is from  $\alpha$  to  $\beta$ . Then the curve  $\alpha$  is called generalized Mannheim curve and the curve  $\beta$  is called generalized Mannheim mate curve of  $\alpha$ . And they gave a theorem such that if the curve  $\alpha$  is a generalized Mannheim curve in Euclidean 4-space, then the first curvature function  $k_1$  and

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second curvature functions  $k_2$  of the curve  $\alpha$  satisfy the equality:

$$k_1(s) = \mu \left\{ (k_1(s))^2 + (k_2(s))^2 \right\},$$

where  $\mu$  is a positive constant number. The quaternion was introduced by Hamilton. His initial attempt to generalize the complex numbers by introducing a three-dimensional object failed in the sense that the algebra he constructed for these three-dimensional object did not have the desired properties. On the 16th October 1843 Hamilton discovered that the appropriate generalization is one in which the scalar(real) axis is left unchanged whereas the vector(imaginary) axis is supplemented by adding two further vectors axes.

In 1987, The Serret-Frenet formulas for a quaternionic curve in  $\mathbb{E}^3$  and  $\mathbb{E}^4$  was defined by Bharathi and Nagaraj, in [7], and then in 2004, Serret-Frenet formulas for quaternionic curves and quaternionic inclined curves have been defined in Semi-Euclidean space by Çöken and Tuna in 2004, [1].

In 2011 Güngör and Tosun studied quaternionic rectifying curves, [8]. Also, Gök et.al and Kahraman et.al, in [6,3], defined a new kind of slant helix in Euclidean space  $\mathbb{E}^4$  and semi-Euclidean space  $\mathbb{E}_2^4$ . It called quaternionic  $B_2$ -slant helix in Euclidean space  $\mathbb{E}^4$  and semi-real quaternionic  $B_2$ -slant helix in semi-Euclidean space  $\mathbb{E}_2^4$ , respectively. Recently, Sağlam, in [2], has studied on the osculating spheres of a real quaternionic curve in Euclidean 4-space. In this paper, we define quaternionic Mannheim curves and we give some characterizations of them in Euclidean 3 and 4 space.

## §2. Preliminaries

Let  $Q_H$  denotes a four dimensional vector space over the field  $H$  of characteristic grater than 2. Let  $e_i$  ( $1 \leq i \leq 4$ ) denote a basis for the vector space. Let the rule of multiplication on  $Q_H$  be defined on  $e_i$  ( $1 \leq i \leq 4$ ) and extended to the whole of the vector space by distributivity as follows:

A real quaternion is defined with  $q = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3 + de_4$  where  $a, b, c, d$  are ordinary numbers. Such that

$$\begin{aligned} e_4 &= 1, & e_1^2 &= e_2^2 = e_3^2 = -1, \\ e_1e_2 &= e_3, & e_2e_3 &= e_1, \quad e_3e_1 = e_2, \\ e_2e_1 &= -e_3, & e_3e_2 &= -e_1, \quad e_1e_3 = -e_2. \end{aligned} \tag{2.1}$$

If we denote  $S_q = d$  and  $\vec{V}_q = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3$ , we can rewrite real quaternions the basic algebraic form  $q = S_q + \vec{V}_q$  where  $S_q$  is scalar part of  $q$  and  $\vec{V}_q$  is vectorial part. Using these basic products we can now expand the product of two quaternions to give

$$p \times q = S_p S_q - \langle \vec{V}_p, \vec{V}_q \rangle + S_p \vec{V}_q + S_q \vec{V}_p + \vec{V}_p \wedge \vec{V}_q \text{ for every } p, q \in Q_H, \tag{2.2}$$

where we have use the inner and cross products in Euclidean space  $\mathbb{E}^3$ , [7]. There is a unique involutory antiautomorphism of the quaternion algebra, denoted by the symbol  $\gamma$  and defined as follows:

$$\gamma q = -a\vec{e}_1 - b\vec{e}_2 - c\vec{e}_3 + de_4 \text{ for every } q = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3 + de_4 \in Q_H,$$

which is called the “Hamiltonian conjugation”. This defines the symmetric, real valued, non-degenerate, bilinear form  $h$  are follows:

$$h(p, q) = \frac{1}{2} [p \times \gamma q + q \times \gamma p] \text{ for } p, q \in Q_H.$$

And then, the norm of any  $q$  real quaternion denotes

$$\|q\|^2 = h(q, q) = q \times \gamma q. \quad (2.3)$$

The concept of a spatial quaternion will be used of throughout our work.  $q$  is called a spatial quaternion whenever  $q + \gamma q = 0$ , [2].

The Serret-Frenet formulae for quaternionic curves in  $\mathbb{E}^3$  and  $\mathbb{E}^4$  are follows:

**Theorem 2.1**([7]) *The three-dimensional Euclidean space  $\mathbb{E}^3$  is identified with the space of spatial quaternions  $\{p \in Q_H \mid p + \gamma p = 0\}$  in an obvious manner. Let  $I = [0, 1]$  denotes the unit interval of the real line  $\mathbb{R}$ . Let*

$$\begin{aligned} \alpha : I \subset \mathbb{R} &\longrightarrow Q_H \\ s &\longrightarrow \alpha(s) = \sum_{i=1}^3 \alpha_i(s) \vec{e}_i, \quad 1 \leq i \leq 3, \end{aligned}$$

be an arc-lengthed curve with nonzero curvatures  $\{k, r\}$  and  $\{t(s), n(s), b(s)\}$  denotes the Frenet frame of the curve  $\alpha$ . Then Frenet formulas are given by

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & r \\ 0 & -r & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}, \quad (2.4)$$

where  $k$  is the principal curvature,  $r$  is torsion of  $\alpha$ .

**Theorem 2.2**([7]) *The four-dimensional Euclidean spaces  $\mathbb{E}^4$  is identified with the space of quaternions. Let  $I = [0, 1]$  denotes the unit interval of the real line  $\mathbb{R}$ . Let*

$$\begin{aligned} \alpha^{(4)} : I \subset \mathbb{R} &\longrightarrow Q_H \\ s &\longrightarrow \alpha^{(4)}(s) = \sum_{i=1}^4 \alpha_i(s) \vec{e}_i, \quad 1 \leq i \leq 4, \quad \vec{e}_4 = 1, \end{aligned}$$

be a smooth curve in  $\mathbb{E}^4$  with nonzero curvatures  $\{K, k, r - K\}$  and  $\{T(s), N(s), B_1(s), B_2(s)\}$  denotes the Frenet frame of the curve  $\alpha$ . Then Frenet formulas are given by

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & K & 0 & 0 \\ -K & 0 & k & 0 \\ 0 & -k & 0 & (r - K) \\ 0 & 0 & -(r - K) & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} \quad (2.5)$$

where  $K$  is the principal curvature,  $k$  is the torsion and  $(r - K)$  is bitorsion of  $\alpha^{(4)}$ .

### §3. Characterizations of the Quaternionic Mannheim Curve

In this section, we define quaternionic Mannheim curves and we give some characterizations of them in Euclidean 3 and 4 space.

**Definition 3.1** Let  $\alpha(s)$  and  $\beta(s^*)$  be two spatial quaternionic curves in  $\mathbb{E}^3$ . Let  $\{t(s), n(s), b(s)\}$  and  $\{t^*(s^*), n^*(s^*), b^*(s^*)\}$  be Frenet frames, respectively, on these curves.  $\alpha(s)$  and  $\beta(s^*)$  are called spatial quaternionic Mannheim curve couple if  $n(s)$  and  $b^*(s^*)$  are linearly dependent.

**Theorem 3.2** Let  $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$  be a spatial quaternionic Mannheim curve with the arc lenght parameter  $s$  and  $\beta : \bar{I} \subset \mathbb{R} \rightarrow \mathbb{E}^3$  be spatial quaternionic Mannheim partner curve of  $\alpha$  with the arc lenght parameter  $s^*$ . Then

$$d(\alpha(s), \beta(s^*)) = \text{constant}, \quad \text{for all } s \in I.$$

*Proof* From Definition 3.1, we can write

$$\alpha(s) = \beta(s^*) + \lambda^*(s^*) b^*(s^*) \quad (3.1)$$

Differentiating the Eq.(3.1) with respect to  $s^*$  and by using the Frenet equations, we get

$$\frac{d\alpha(s)}{ds} \frac{ds}{ds^*} = t^*(s^*) + \lambda^{*'}(s^*) b^*(s^*) - \lambda^*(s^*) r^*(s^*) n^*(s^*)$$

If we denote  $\frac{d\alpha(s)}{ds} = t(s)$

$$t(s) = \frac{ds^*}{ds} \left[ t^*(s^*) + \lambda^{*'}(s^*) b^*(s^*) - \lambda^*(s^*) r^*(s^*) n^*(s^*) \right]$$

and

$$h(t(s), n(s)) = \frac{ds^*}{ds} \left[ h(t^*(s^*), n(s)) + \lambda^{*'}(s^*) h(b^*(s^*), n(s)) - \lambda^*(s^*) r^*(s^*) h(n^*(s^*), n(s)) \right].$$

Since  $\{n(s), b^*(s^*)\}$  is a linearly dependent set, we get

$$\lambda^{*'}(s^*) = 0$$

that is,  $\lambda^*$  is constant function on  $\bar{I}$ . This completes the proof.  $\square$

**Theorem 3.3** Let  $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$  be spatial quaternionic curves with the arc-length parameter  $s$ . Then  $\alpha$  is spatial quaternionic Mannheim curve if and only if

$$k(s) = \lambda(k^2(s) + r^2(s)), \quad (3.2)$$

where  $\lambda$  is constant.

*Proof* If  $\alpha$  is spatial quaternionic Mannheim curve, we can write

$$\beta(s) = \alpha(s) + \lambda(s)n(s)$$

Differentiating the above equality and by using the Frenet equations, we get

$$\frac{d\beta(s)}{ds} = [(1 - \lambda(s)k(s))t(s) + \lambda'(s)n(s) + \lambda(s)r(s)b(s)].$$

As  $\{n(s), b^*(s^*)\}$  is a linearly dependent set, we get

$$\lambda'(s) = 0.$$

This means that  $\lambda$  is constant. Thus we have

$$\frac{d\beta(s)}{ds} = (1 - \lambda k(s))t(s) + \lambda r(s)b(s).$$

On the other hand, we have

$$t^* = \frac{d\beta}{ds} \frac{ds}{ds^*} = [(1 - \lambda k(s))t(s) + \lambda r(s)b(s)] \frac{ds}{ds^*}.$$

By taking the derivative of this equation with respect to  $s^*$  and applying the Frenet formulas we obtain

$$\begin{aligned} \frac{dt^*}{ds^*} \frac{ds}{ds^*} &= [-\lambda k'(s)t(s) + (k(s) - \lambda k^2(s) - \lambda r^2(s))n(s) + \lambda r'(s)b(s)] \left(\frac{ds}{ds^*}\right)^2 \\ &\quad + [(1 - \lambda k(s))t(s) + \lambda r(s)b(s)] \frac{d^2s}{ds^{*2}}. \end{aligned}$$

From this equation, we get

$$k(s) = \lambda(k^2(s) + r^2(s)).$$

Conversely, if  $k(s) = \lambda(k^2(s) + r^2(s))$ , then we can easily see that  $\alpha$  is a Mannheim curve.  $\square$

**Theorem 3.4** *Let  $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$  be spatial quaternionic Mannheim curve with arc-length parameter  $s$ . Then  $\beta$  is the spatial quaternionic Mannheim partner curve of  $\alpha$  if and only if the curvature functions  $k^*(s^*)$  and  $r^*(s^*)$  of  $\beta$  satisfy the following equation*

$$\frac{dr^*}{ds^*} = \frac{k^*}{\mu} (1 + \mu^2 r^{*2}),$$

where  $\mu$  is constant.

*Proof* Let  $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$  be spatial quaternionic Mannheim curve. Then, we can write

$$\alpha(s^*) = \beta(s^*) + \mu(s^*)b^*(s^*) \quad (3.3)$$

for some function  $\mu(s^*)$ . By taking the derivative of Eq.(3.3) with respect to  $s^*$  and using the Frenet equations, we obtain

$$t \frac{ds}{ds^*} = t^*(s^*) + \mu'(s^*)b^*(s^*) - \mu(s^*)r^*(s^*)n^*(s^*).$$

And then, we know that  $\{n(s), b^*(s^*)\}$  is a linearly dependent set, so we have

$$\mu'(s^*) = 0.$$

This means that  $\mu(s^*)$  is a constant function. Thus we have

$$t \frac{ds}{ds^*} = t^*(s^*) - \mu r^*(s^*) n^*(s^*). \quad (3.4)$$

On the other hand, we have

$$t = t^* \cos \theta + n^* \sin \theta, \quad (3.5)$$

where  $\theta$  is the angle between  $t$  and  $t^*$  at the corresponding points of the curves  $\alpha$  and  $\beta$ . By taking the derivative of Eq.(3.5) with respect to  $s^*$  and using the Frenet equations, we obtain

$$kn \frac{ds}{ds^*} = - \left( k^* + \frac{d\theta}{ds^*} \right) \sin \theta t^* + \left( k^* + \frac{d\theta}{ds^*} \right) \cos \theta n^* + r^* \sin \theta b^*.$$

From this equation and the fact that the  $\{n(s), b^*(s^*)\}$  is a linearly dependent set, we get

$$\begin{cases} \left( k^* + \frac{d\theta}{ds^*} \right) \sin \theta = 0 \\ \left( k^* + \frac{d\theta}{ds^*} \right) \cos \theta = 0. \end{cases}$$

For this reason, we have

$$\frac{d\theta}{ds^*} = -k^*. \quad (3.6)$$

From the Eq.(3.4), Eq.(3.5) and notice that  $t^*$  is orthogonal to  $b^*$ , we find that

$$\frac{ds}{ds^*} = \frac{1}{\cos \theta} = -\frac{\mu r^*}{\sin \theta}.$$

Then we have

$$\mu r^* = -\tan \theta.$$

By taking the derivative of this equation and applying Eq.(3-6), we get

$$\mu \frac{dr^*}{ds^*} = k^* (1 + \mu^2 r^{*2})$$

that is

$$\frac{dr^*}{ds^*} = \frac{k^*}{\mu} (1 + \mu^2 r^{*2}).$$

Conversely, if the curvature  $k^*$  and torsion  $r^*$  of the curve  $\beta$  satisfy the equality:

$$\frac{dr^*}{ds^*} = \frac{k^*}{\mu} (1 + \mu^2 r^{*2})$$

for constant  $\mu$ , then we define a curve by

$$\alpha(s^*) = \beta(s^*) + \mu b^*(s^*) \quad (3.7)$$

and we will show that  $\alpha$  is a spatial quaternionic Mannheim curve and  $\beta$  is the spatial quaternionic Mannheim partner curve of  $\alpha$ . By taking the derivative of Eq.(3.7) with respect to  $s^*$  twice, we get

$$t \frac{ds}{ds^*} = t^* - \mu r^* n^*, \quad (3.8)$$

$$kn \left( \frac{ds}{ds^*} \right)^2 + t \frac{d^2s}{ds^{*2}} = \mu k^* r^* t^* + \left( k^* - \mu \frac{dr^*}{ds^*} \right) n^* - \mu r^{*2} b^*, \quad (3.9)$$

respectively. Taking the cross product of Eq.(3.8) with Eq.(3.9) and noticing that

$$k^* - \mu \frac{dr^*}{ds^*} + \mu^2 k^* r^{*2} = 0,$$

we have

$$kb \left( \frac{ds}{ds^*} \right)^3 = \mu^2 r^{*3} t^* + \mu r^{*2} n^*. \quad (3.10)$$

By taking the cross product of Eq.(3.8) with Eq.(3.10), we get

$$kn \left( \frac{ds}{ds^*} \right)^4 = -\mu r^{*2} (1 + \mu^2 r^{*2}) b^*.$$

This means that the principal normal vector field of the spatial quaternionic curve  $\alpha$  and binormal vector field of the spatial quaternionic curve  $\beta$  are linearly dependent set. And so  $\alpha$  is a spatial quaternionic Mannheim curve and  $\beta$  is spatial quaternionic Mannheim partner curve of  $\alpha$ .  $\square$

**Theorem 3.5** *Let  $\{\alpha, \beta\}$  be a spatial quaternionic Mannheim curve couple in  $\mathbb{E}^3$ . Then measure of the angle  $\theta$  between the tangent vector fields of spatial quaternionic curves  $\alpha(s)$  and  $\beta(s^*)$  is constant if and only if the spatial quaternionic curve  $\beta(s^*)$  is a geodesic.*

*Proof* From Eq.(3-6), we know that  $\frac{d\theta}{ds^*} = -k^*$ , where  $\theta$  is the angle between  $t$  and  $t^*$  at the corresponding points of the curves  $\alpha$  and  $\beta$ .

If  $\theta$  is a constant angle, the curvature of the curve  $\beta$ ,

$$k^* = 0,$$

that is the curve  $\beta$  is a geodesic.

Conversely, if the curve  $\beta$  is a geodesic, the angel  $\theta$  between  $t$  and  $t^*$  at the corresponding points of the curves  $\alpha$  and  $\beta$  satisfy the following equality:

$$\frac{d\theta}{ds^*} = 0,$$

that is  $\theta$  is a constant angle.  $\square$

**Definition 3.6** *A quaternionic curve  $\alpha^{(4)} : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$  is a quaternionic Mannheim curve if there exists a quaternionic curve  $\beta^{(4)} : \bar{I} \subset \mathbb{R} \rightarrow \mathbb{E}^4$  such that the second Frenet vector at each point of  $\alpha^{(4)}$  is included the plane generated by the third Frenet vector and the fourth Frenet vector of  $\beta^{(4)}$  at corresponding point under  $\varphi$ , where  $\varphi$  is a bijection from  $\alpha^{(4)}$  to  $\beta^{(4)}$ . The curve  $\beta^{(4)}$  is called the quaternionic Mannheim partner curve of  $\alpha^{(4)}$ .*

**Theorem 3.7** *Let  $\alpha^{(4)} : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$  and  $\beta^{(4)} : \bar{I} \subset \mathbb{R} \rightarrow \mathbb{E}^4$  be quaternionic Mannheim curve couple with arc-length  $s$  and  $\bar{s}$ , respectively. Then*

$$d \left( \alpha^{(4)}(s), \beta^{(4)}(\bar{s}) \right) = \lambda(s) = \text{constant}, \text{ for all } s \in I \quad (3.11)$$

*Proof* From the Definition 3.6, quaternionic Mannheim partner curve  $\beta^{(4)}$  of  $\alpha^{(4)}$  is given by the following equation

$$\beta^{(4)}(s) = \alpha^{(4)}(s) + \lambda(s)N(s),$$

where  $\lambda(s)$  is a smooth function. A smooth function  $\psi : s \in I \rightarrow \psi(s) = \bar{s} \in \bar{I}$  is defined by

$$\psi(s) = \int_0^s \left\| \frac{d\alpha^{(4)}(s)}{ds} \right\| ds = \bar{s}.$$

The bijection  $\varphi: \alpha^{(4)} \rightarrow \beta^{(4)}$  is defined by  $\varphi(\alpha^{(4)}(s)) = \beta^{(4)}(\psi(s))$ . Since the second Frenet vector at each point of  $\alpha^{(4)}$  is included in the plane generated by the third Frenet vector and the fourth Frenet vector of  $\beta^{(4)}$  at corresponding point under  $\varphi$ , for each  $s \in I$ , the Frenet vector  $N(s)$  is given by the linear combination of Frenet vectors  $\bar{B}_1(\psi(s))$  and  $\bar{B}_2(\psi(s))$  of  $\beta^{(4)}$ , that is, we can write

$$N(s) = g(s)\bar{B}_1(\psi(s)) + h(s)\bar{B}_2(\psi(s)),$$

where  $g(s)$  and  $h(s)$  are smooth functions on  $I$ . So we can write

$$\beta^{(4)}(\psi(s)) = \alpha^{(4)}(s) + \lambda(s)N(s). \quad (3.12)$$

Differentiating the Eq.(3.12) with respect to  $s$  and by using the Frenet equations, we get

$$\bar{T}(\psi(s))\psi'(s) = [(1 - \lambda K(s))T(s) + \lambda'(s)N(s) + \lambda(s)k(s)B_1(s)].$$

By the fact that:

$$h(\bar{T}(\varphi(s)), g(s)\bar{B}_1(\psi(s)) + h(s)\bar{B}_2(\psi(s))) = 0,$$

we have

$$\lambda'(s) = 0$$

that is,  $\lambda(s)$  is constant function on  $I$ . This completes the proof.  $\square$

**Theorem 3.78** *If the quaternionic curve  $\alpha^{(4)} : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$  is a quaternionic Mannheim curve, then the first and second curvature functions of  $\alpha^{(4)}$  satisfy the equality:*

$$K(s) = \lambda \{ K^2(s) + k^2(s) \},$$

where  $\lambda$  is constant.

*Proof* Let  $\beta^{(4)}$  be a quaternionic Mannheim partner curve of  $\alpha^{(4)}$ . Then we can write

$$\beta^{(4)}(\psi(s)) = \alpha^{(4)}(s) + \lambda N(s).$$

Differentiating the above equation, we get

$$\bar{T}(\psi(s))\psi'(s) = [(1 - \lambda K(s))T(s) + \lambda k(s)B_1(s)],$$

that is,

$$\bar{T}(\psi(s)) = \frac{1 - \lambda K(s)}{\psi'(s)}T(s) + \frac{\lambda k(s)}{\psi'(s)}B_1(s),$$



where  $\psi'(s) = \sqrt{(1 - \lambda K(s))^2 + (\lambda k(s))^2}$  for  $s \in I$ . By differentiation of both sides of the above equality with respect to  $s$ , we have

$$\begin{aligned} \psi'(s) \overline{K}(\overline{s}) \overline{N}(\overline{s}) &= \left( \frac{1 - \lambda K(s)}{\psi'(s)} \right)' T(s) \\ &+ \left( \frac{(1 - \lambda K(s)) K(s) - \lambda k(s)^2}{\psi'(s)} \right) N(s) \\ &+ \left( \frac{\lambda k(s)}{\psi'(s)} \right)' B_1(s) - \frac{\lambda k(s) (r(s) - K(s))}{\psi'(s)} B_2(s). \end{aligned}$$

By the fact:

$$h(\overline{N}(\varphi(s)), g(s) \overline{B}_1(\psi(s)) + h(s) \overline{B}_2(\psi(s))) = 0,$$

we have that coefficient of  $N$  in the above equation is zero, that is,

$$(1 - \lambda K(s)) K(s) - \lambda k(s)^2 = 0.$$

Thus, we have

$$K(s) = \lambda \{K^2(s) + k^2(s)\}$$

for  $s \in I$ . This completes the proof.  $\square$

**Theorem 3.9** Let  $\alpha^{(4)} : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$  be a quaternionic curve with arc-length  $s$ , whose curvature functions  $K(s)$  and  $k(s)$  are non-constant functions and satisfy the equality:  $K(s) = \lambda \{K^2(s) + k^2(s)\}$ , where  $\lambda$  is positive constant number. If the quaternionic curve  $\beta^{(4)}$  is given by  $\beta^{(4)}(\overline{s}) = \alpha^{(4)}(s) + \lambda N(s)$ , then  $\alpha^{(4)}$  is a quaternionic Mannheim curve and  $\beta^{(4)}$  is the quaternionic Mannheim partner curve of  $\alpha^{(4)}$ .

*Proof* Let  $\overline{s}$  be the arc-length of the quaternionic curve  $\beta^{(4)}$ . That is,  $\overline{s}$  is defined by

$$\overline{s} = \int_0^s \left\| \frac{d\alpha^{(4)}(s)}{ds} \right\| ds$$

for  $s \in I$ . We can write a smooth function  $\psi : s \in I \rightarrow \psi(s) = \overline{s} \in \overline{I}$ . By the assumption of the curvature functions  $K(s)$  and  $k(s)$ , we have

$$\begin{aligned} \psi'(s) &= \sqrt{(1 - \lambda K(s))^2 + (\lambda k(s))^2}, \\ \psi'(s) &= \sqrt{1 - \lambda K(s)} \end{aligned}$$

for  $s \in I$ . Then we can easily write

$$\begin{aligned} \beta^{(4)}(\overline{s}) &= \beta^{(4)}(\psi(s)) \\ &= \alpha^{(4)}(s) + \lambda N(s) \end{aligned}$$

for the quaternionic curve  $\beta^{(4)}$ . If we differentiate both sides of the above equality with respect to  $s$ , we get

$$\psi'(s) \overline{T}(\psi(s)) = T(s) + \lambda \{-K(s)T(s) + k(s)B_1(s)\}.$$

And so we have,

$$\overline{T}(\psi(s)) = \sqrt{1 - \lambda K(s)} T(s) + \frac{\lambda k(s)}{\sqrt{1 - \lambda K(s)}} B_1(s). \quad (3.13)$$

Differentiating the above equality with respect to  $s$  and by using the Frenet equations, we get

$$\begin{aligned} \psi'(s) \overline{K}(\psi(s)) \overline{N}(\psi(s)) &= \left( \sqrt{1 - \lambda K(s)} \right)' T(s) + \left( \frac{K(s)(1 - \lambda K(s)) - \lambda k^2(s)}{\sqrt{1 - \lambda K(s)}} \right) N(s) \\ &+ \left( \frac{\lambda k(s)}{\sqrt{1 - \lambda K(s)}} \right)' B_1(s) + \frac{\lambda k(s)(r(s) - K(s))}{\sqrt{1 - \lambda K(s)}} B_2(s) \end{aligned}$$

From our assumption, it holds

$$\frac{K(s)(1 - \lambda K(s)) - \lambda k^2(s)}{\sqrt{1 - \lambda K(s)}} = 0.$$

We find the coefficient of  $N(s)$  in the above equality vanishes. Thus the vector  $\overline{N}(\psi(s))$  is given by linear combination of  $T(s)$ ,  $B_1(s)$  and  $B_2(s)$  for each  $s \in I$ . And the vector  $\overline{T}(\psi(s))$  is given by linear combination of  $T(s)$  and  $B_1(s)$  for each  $s \in I$  in the Eq.(3.13). As the curve  $\beta^{(4)}$  is quaternionic curve in  $\mathbb{E}^4$ , the vector  $N(s)$  is given by linear combination of  $\overline{B}_1(\overline{s})$  and  $\overline{B}_2(\overline{s})$ . For this reason, the second Frenet curve at each point of  $\alpha^{(4)}$  is included in the plane generated the third Frenet vector and the fourth Frenet vector of  $\beta^{(4)}$  at corresponding point under  $\varphi$ . Here the bijection  $\varphi : \alpha^{(4)} \rightarrow \beta^{(4)}$  is defined by  $\varphi(\alpha^{(4)}(s)) = \beta^{(4)}(\psi(s))$ . This completes the proof.  $\square$

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