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A Note on Odd Graceful Labeling of a Class of Trees

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Abstract: A connected graph with n vertices and q edges is called odd graceful if it is possible to label the vertices x with pairwise distinct integers $f(x)$ in $\{0, 1, 2, 3, \cdots, 2q-1\}$ so that when each edge, xy is labeled $|f(x)-f(y)|$, the resulting edge labels are pairwise distinct and thus form the entire set $\{1, 3, 5, \cdots, 2q-1\}$. In this paper we study the odd graceful labeling of class of T_n trees.

Key Words: Labeling, Odd graceful graph, Tree.

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§1. Introduction

Unless mentioned otherwise, a graph in this paper shall mean a simple finite graph without isolated vertices.

For all terminology and notations in graph theory, we follow Harary [1] and for all terminology regarding odd graceful labeling, we follow [2]. A connected graph with n vertices and q edges is called odd graceful if it is possible to label the vertices x with pairwise distinct integers $f(x)$ in $\{0, 1, 2, 3, \cdots, 2q-1\}$ so that each edge, xy, is labeled $|f(x)-f(y)|$, the resulting edge labels are pairwise distinct. (and thus form the entire set $\{1, 3, 5, \cdots, 2q-1\}$). In this article we study the odd graceful labeling of typical class of T_n trees.

§2. On T_n-Class of Trees

Definition 2.1([3]) *Let T be a tree and x and y be two adjacent vertices in T. Let there be two end vertices (non-adjacent vertices of degree one) $x_1, y_1 \in T$ such that the length of the path $x - x_1$ is equal to the length of the path $y - y_1$. If the edge xy is deleted from T and x_1, y_1 are joined by an edge x_1y_1; then such a transformations of the edge from xy to x_1y_1 is called an elementary parallel transformation (or an EPT of T) and the edge xy is called a transformable edge.*

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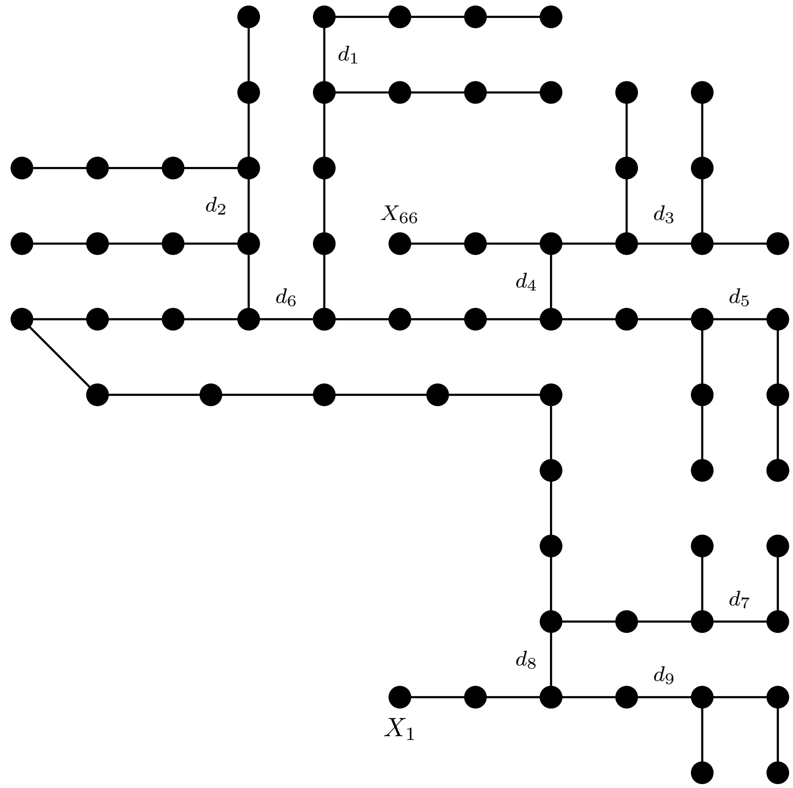

Fig.1 A T_{66}-tree T

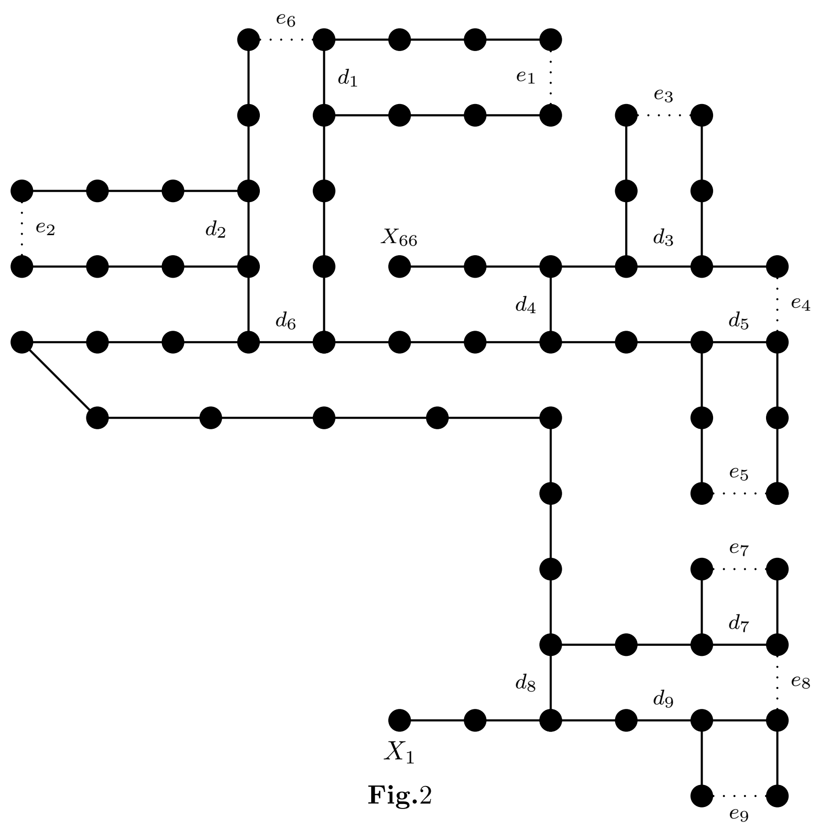

Fig.2

Definition 2.2 *If by a sequence of EPT's, the tree, T can be reduced to a Hamiltonian path, then T is called a T_n-tree (transformed tree) and such a Hamiltonian path is denoted as $P^H(T)$. Any such sequence regarded as a composition mapping (EPT's) denoted by P is called parallel transformation of T*[3].

A T_n-tree and a sequence of nine EPT's reducing it to a hamiltonian path are illustrated in Fig.1 to Fig.3.

In Fig.2, let $d_1, d_2, \cdots d_9$ are the deleted edges and $e_1, e_2, \cdots, e_9$ are the corresponding added edges (Given in broken lines).

An EPT $P_i^H(T)$; for $i = 1, 2, \cdots, 9$. The hamiltonian path $P^H(T)$ for the tree in Fig. 1 is given in Fig.3.

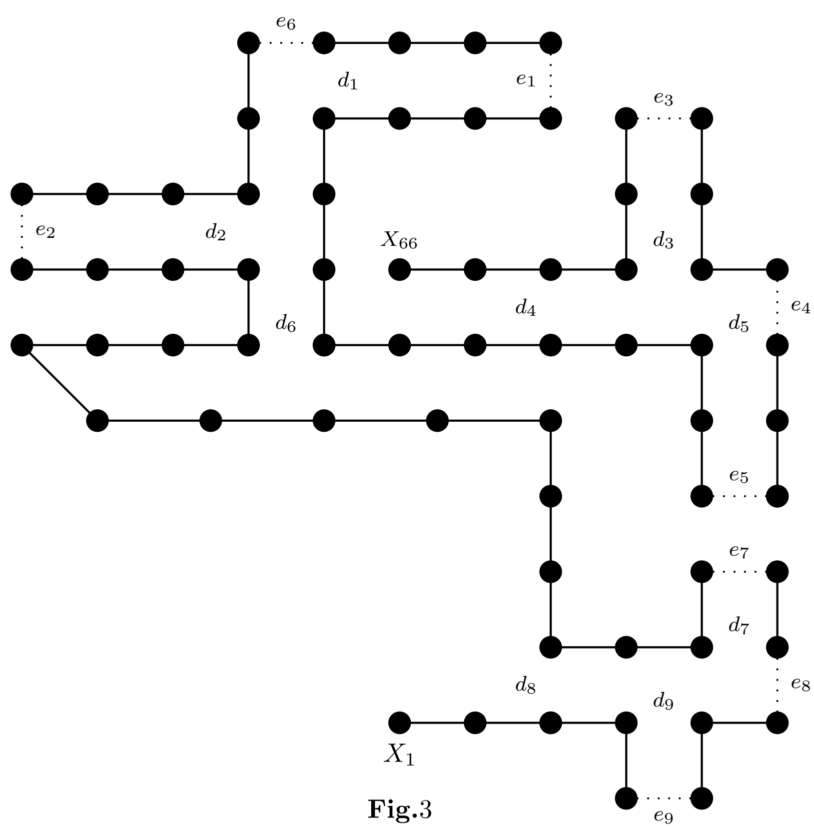

Fig.3

Theorem 2.3 *Every T_n tree is odd graceful.*

Proof Let T be a T_n tree with $(n+1)$ vertices. By definition there exist a path $P^H(T)$ corresponding to T_n. Let $E_d = \{d_1, d_2, \cdots, d_r\}$ be the set of edges deleted from tree T and E_p is the set of edges newly added through the sequence $\{e_1, e_2, \cdots, e_r\}$ of the EPT's used to arrive at the path (Hamiltonian path) $P^H(T)$. Clearly E_d and E_p have the same number of edges. Now we have $V(P^H(T)) = V(T)$ and $E(P^H(T)) = \{E\{T\} - E_d\} \cup E_p$: Now denote the vertices of $P^H(T)$ successively as $v_1, v_2, \cdots, v_{n+1}$ starting from one pendant vertex of $P^H(T)$ right up to other. Define the vertex numbering of f from $V(P^H(T)) \to \{0, 1, 2, \cdots, 2q-1\}$ as follows.

$$\begin{aligned} f(v_i) &= 2\Big[\frac{i-1}{2}\Big] \text{ if } i \text{ is odd, } 1 \leqslant i \leqslant n+1 \\ &= (2q-1) - 2\Big[\frac{i-2}{2}\Big] \text{ if } i \text{ is even, } 2 \leqslant i \leqslant n+1 \end{aligned}$$

where, q is the number of edges of T and [.] denote the integer part.

Now it can be easily seen that f is injective. Let g_f^* be the induced mapping defined from the edge set of $P^H(T)$ in to the set $\{1, 3, 5, \cdots, 2q-1\}$ as follows:
$g_f^*(uv) = |f(u) - f(v)|$ whenever $uv \in E(PH(T))$.
Since $P^H(T)$ is a path, every edge in $P^H(T)$ is of the form v_iv_{i+1} for $i = 1, 2, \cdots, n$.

Case 1 When i is even, then

$$\begin{aligned} g_f^*(v_iv_{i+1}) &= |f(v_i) - f(v_{i+1})| \\ &= \left|(2q-1) - 2\left[\frac{i-2}{2}\right] - 2\left[\frac{i+1-1}{2}\right]\right| \\ &= \left|(2q-1) - 2\left\{\left[\frac{i-2}{2}\right] + \left[\frac{i}{2}\right]\right\}\right| \\ &= \left|(2q-1) - 2\left[\frac{i-2+i}{2}\right]\right| \\ &= \left|(2q-1) - 2\left[\frac{2i-2}{2}\right]\right| \\ &= \left|(2q-1) - 4\left[\frac{i-1}{2}\right]\right| \qquad (1) \end{aligned}$$

Case 2 When i is odd, then

$$\begin{aligned} g_f^*(v_iv_{i+1}) &= |f(v_i) - f(v_{i+1})| \\ &= \left[2\left[\frac{i-1}{2}\right] - \left((2q-1) - 2\left[\frac{i+1-2}{2}\right]\right)\right] \\ &= \left|2\left[\frac{i-1}{2}\right] - (2q-1) + 2\left[\frac{i-1}{2}\right]\right| \\ &= \left|(2q-1) - 4\left[\frac{i-1}{2}\right]\right| \qquad (2) \end{aligned}$$

From (1) and (2), we get for all i,

$$g_f^*(v_iv_{i+1}) = \left|(2q-1) - 4\left[\frac{i-1}{2}\right]\right| \qquad (3)$$

From (3), it is clear that g_f^* is injective and its range is $\{1, 3, 5, \cdots, 2q-1\}$. Then f is odd graceful on $P^H(T)$.

In order to prove that f is also odd graceful on T_n, it is enough to show that $g_f^*(d_s) = g_f^*(e_s)$. Let $d_s = v_iv_j$ be an edge of T for same indices i and j, $1 \leqslant i \leqslant n+1; 1 \leqslant j \leqslant n+1$ and d_s be deleted and e_s be the corresponding edge joined to obtain $P^H(T)$ at a distance k from u_i and u_j. Then $e_s = v_{i+k}v_{j-k}$. Since e_s is an edge in $P^H(T)$, it must be of the form $e_s = v_{i+k}v_{i+k+1}$.

We have $(v_{i+k}, v_{j-k}) = (v_{i+k}, v_{i+k+1}) \Longrightarrow j - k = i + k + 1 \Longrightarrow j = i + 2k + 1$. Therefore i and j are of opposite parity $\Longrightarrow$ one of i, j is odd and other is even.

Case a When i is odd, $1 \leqslant i \leqslant n$. The value of the edge $e_s = v_iv_j$ is given by

$$\begin{aligned}
g_f^*(d_s) &= g_f^*(v_iv_j) \\
&= g_f^*(v_iv_{i+2k+1}) \\
&= |f(v_i) - f(v_{i+2k+1}| \qquad (4) \\
&= \left|(2q-1) - 2\left[\frac{i-2}{2}\right] - 2\left[\frac{i+2k+1-1}{2}\right]\right| \\
&= \left|(2q-1) - 2\left\{\left[\frac{i-2}{2}\right] + 2\left[\frac{i+2k}{2}\right]\right\}\right| \\
&= |(2q-1) - (2i+2k-2)| \\
&= |(2q-1) - 2(i+k-1)| \qquad (5)
\end{aligned}$$

Case b When i is even, $2 \leqslant i \leqslant n$.

$$\begin{aligned}
g_f^*(d_s) &= |f(v_i) - f(v_{i+2k+1})| \\
&= \left|2\left[\frac{i-2}{2}\right] - \left((2q-1) - 2\left[\frac{i+2k+1-2}{2}\right]\right)\right| \\
&= \left|2\left[\frac{i-2}{2}\right] + 2\left[\frac{i+2k-1}{2}\right] - (2q-1)\right| \\
&= |(2i+2k-2) - 2 - (2q-1)| \\
&= |(2q-1) - 2(i+k-1)| \qquad (6)
\end{aligned}$$

From (4), (5) and (6) it follows that

$$g_f^*(d_s) = g_f^*(v_iv_j) = |(2q-1) - 2(i+k-1)|, 1 \leqslant i \leqslant n \qquad (7)$$

Now again,

$$\begin{aligned}
g_f^*(e_s) &= g_f^*(v_{i+k}v_{j-k}) = g_f^*(v_kv_{i+k+1}) \\
&= |f(v_{i+k}) - f(v_{i+k+1})| \\
&= \left|(2q-1) - 2\left[\frac{i+k-2}{2}\right] - 2\left[\frac{i+k+1-1}{2}\right]\right. \\
&= |(2q-1) - (2i+2k-2)| \\
&= |(2q-1) - 2(i+k-1)|, 1 \leqslant i \leqslant n \qquad (8)
\end{aligned}$$

From (7) and (8), it follows that

$$g_f^*(e_s) = g_f^*(d_s).$$

Then f is odd graceful on T_n also. Hence the graph T_n-tree is odd graceful. The proof is complete. □

For example, an odd graceful labelling of a T_n-tree using 2.3, is shown in Fig.4.

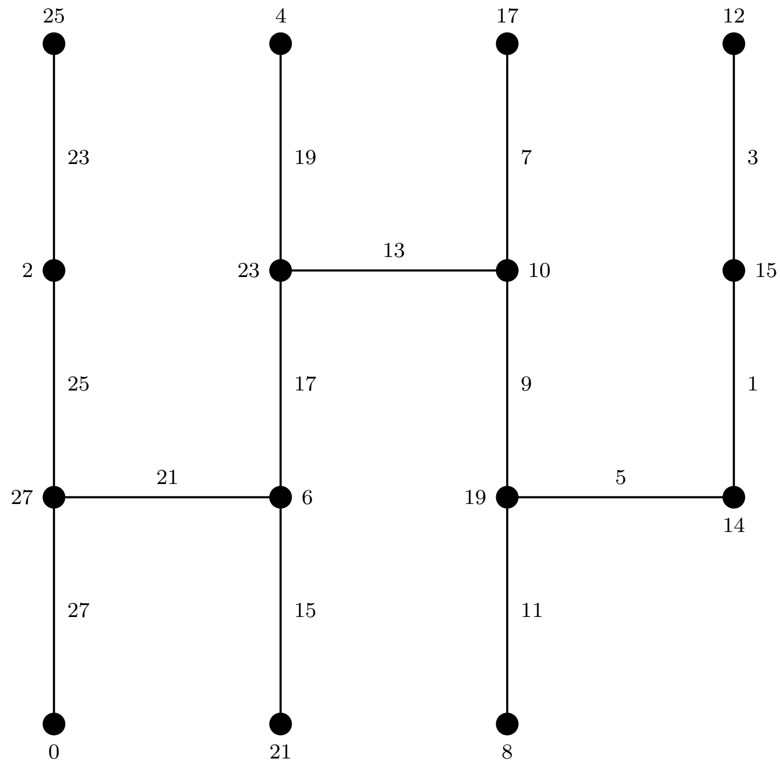

Fig.4

An odd graceful labeling of a T_n-tree using Theorem 2.3.

References

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