

Total Dominator Colorings in Paths

A.Vijayalekshmi

(S.T.Hindu College, Nagercoil, Tamil Nadu, India)

E-mail: vijimath.a@gmail.com

Abstract: Let G be a graph without isolated vertices. A total dominator coloring of a graph G is a proper coloring of the graph G with the extra property that every vertex in the graph G properly dominates a color class. The smallest number of colors for which there exists a total dominator coloring of G is called the total dominator chromatic number of G and is denoted by $\chi_{td}(G)$. In this paper we determine the total dominator chromatic number in paths. Unless otherwise specified, n denotes an integer greater than or equal to 2.

Key Words: Total domination number, chromatic number and total dominator chromatic number, Smarandachely k -domination coloring, Smarandachely k -dominator chromatic number.

AMS(2010): 05C15, 05C69

§1. Introduction

All graphs considered in this paper are finite, undirected graphs and we follow standard definitions of graph theory as found in [2].

Let $G = (V, E)$ be a graph of order n with minimum degree at least one. The open neighborhood $N(v)$ of a vertex $v \in V(G)$ consists of the set of all vertices adjacent to v . The closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood $N(S)$ is defined to be $\cup_{v \in S} N(v)$, and the closed neighborhood of S is $N[S] = N(S) \cup S$. A subset S of V is called a dominating (total dominating) set if every vertex in $V - S$ (V) is adjacent to some vertex in S . A dominating (total dominating) set is minimal dominating (total dominating) set if no proper subset of S is a dominating (total dominating) set of G . The domination number γ (total domination number γ_t) is the minimum cardinality taken over all minimal dominating (total dominating) sets of G . A γ -set (γ_t -set) is any minimal dominating (total dominating) set with cardinality γ (γ_t).

A proper coloring of G is an assignment of colors to the vertices of G , such that adjacent vertices have different colors. The smallest number of colors for which there exists a proper coloring of G is called chromatic number of G and is denoted by $\chi(G)$. Let $V = \{u_1, u_2, u_3, \dots, u_p\}$ and $\mathcal{C} = \{C_1, C_2, C_3, \dots, C_n\}$ be a collection of subsets $C_i \subset V$. A color represented in a vertex u is called a non-repeated color if there exists one color class $C_i \in \mathcal{C}$ such that $C_i = \{u\}$.

Let G be a graph without isolated vertices. A *total dominator coloring* of a graph G is

¹Received September 19, 2011. Accepted June 22, 2012.

a proper coloring of the graph G with the extra property that every vertex in the graph G properly dominates a color class. The smallest number of colors for which there exists a total dominator coloring of G is called the *total dominator chromatic number* of G and is denoted by $\chi_{td}(G)$. Generally, for an integer $k \geq 1$, a *Smarandachely k -dominator coloring* of G is a proper coloring on G such that every vertex in the graph G properly dominates a k color classes and the smallest number of colors for which there exists a Smarandachely k -dominator coloring of G is called the *Smarandachely k -dominator chromatic number* of G , denoted by $\chi_{td}^S(G)$. Clearly, if $k = 1$, such a Smarandachely 1-dominator coloring and Smarandachely 1-dominator chromatic number are nothing but the total dominator coloring and total dominator chromatic number of G .

In this paper we determine total dominator chromatic number in paths.

Throughout this paper, we use the following notations.

Notation 1.1 Usually, the vertices of P_n are denoted by $u_1, u_2, \dots, u_n$ in order. We also denote a vertex $u_i \in V(P_n)$ with $i > \lceil \frac{n}{2} \rceil$ by $u_{i-(n+1)}$. For example, u_{n-1} by u_{-2} . This helps us to visualize the position of the vertex more clearly.

Notation 1.2 For $i < j$, we use the notation $\langle [i, j] \rangle$ for the subpath induced by $\langle u_i, u_{i+1}, \dots, u_j \rangle$. For a given coloring C of P_n , $C|_{\langle [i, j] \rangle}$ refers to the coloring C restricted to $\langle [i, j] \rangle$.

We have the following theorem from [1].

Theorem 1.3 For any graph G with $\delta(G) \geq 1$, $\max\{\chi(G), \gamma_t(G)\} \leq \chi_{td}(G) \leq \chi(G) + \gamma_t(G)$.

Definition 1.4 We know from Theorem 1.3 that $\chi_{td}(P_n) \in \{\gamma_t(P_n), \gamma_t(P_n) + 1, \gamma_t(P_n) + 2\}$. We call the integer n , good (respectively bad, very bad) if $\chi_{td}(P_n) = \gamma_t(P_n) + 2$ (if respectively $\chi_{td}(P_n) = \gamma_t(P_n) + 1, \chi_{td}(P_n) = \gamma_t(P_n)$).

§2. Determination of $\chi_{td}(P_n)$

First, we note the values of $\chi_{td}(P_n)$ for small n . Some of these values are computed in Theorems 2.7, 2.8 and the remaining can be computed similarly.

n	$\gamma_t(P_n)$	$\chi_{td}(P_n)$
2	2	2
3	2	2
4	2	3
5	3	4
6	4	4

n	$\gamma_t(P_n)$	$\chi_{td}(P_n)$
7	4	5
8	4	6
9	5	6
10	6	7

Thus $n = 2, 3, 6$ are very bad integers and we shall show that these are the only bad integers. First, we prove a result which shows that for large values of n , the behavior of $\chi_{td}(P_n)$ depends only on the residue class of $n \bmod 4$ [More precisely, if n is good, $m > n$ and $m \equiv n \pmod{4}$ then m is also good]. We then show that $n = 8, 13, 15, 22$ are the least good integers in their respective residue classes. This therefore classifies the good integers.

Fact 2.1 Let $1 < i < n$ and let C be a td-coloring of P_n . Then, if either u_i has a repeated color or u_{i+2} has a non-repeated color, $C| \langle [i+1, n] \rangle$ is also a td-coloring. This fact is used extensively in this paper.

Lemma 2.2 $\chi_{td}(P_{n+4}) \geq \chi_{td}(P_n) + 2$.

Proof For $2 \leq n \leq 5$, this is directly verified from the table. We may assume $n \geq 6$. Let $u_1, u_2, u_3, \dots, u_{n+4}$ be the vertices of P_{n+4} in order. Let C be a minimal td-coloring of P_{n+4} . Clearly, u_2 and u_{-2} are non-repeated colors. First suppose u_4 is a repeated color. Then $C| \langle [5, n+4] \rangle$ is a td-coloring of P_n . Further, $C| \langle [1, 4] \rangle$ contains at least two color classes of C . Thus $\chi_{td}(P_n + 4) \geq \chi_{td}(P_n) + 2$. Similarly the result follows if u_{-4} is a repeated color. Thus we may assume u_4 and u_{-4} are non-repeated colors. But the $C| \langle [3, n+2] \rangle$ is a td-coloring and since u_2 and u_{-2} are non-repeated colors, we have in this case also $\chi_{td}(P_{n+4}) \geq \chi_{td}(P_n) + 2$. $\square$

Corollary 2.3 If for any n , $\chi_{td}(P_n) = \gamma_t(P_n) + 2$, $\chi_{td}(P_m) = \gamma_t(P_m) + 2$, for all $m > n$ with $m \equiv n \pmod{4}$.

Proof By Lemma 2.2, $\chi_{td}(P_{n+4}) \geq \chi_{td}(P_n) + 2 = \gamma_t(P_n) + 2 + 2 = \gamma_t(P_{n+4}) + 2$. $\square$

Corollary 2.4 No integer $n \geq 7$ is a very bad integer.

Proof For $n = 7, 8, 9, 10$, this is verified from the table. The result then follows from the Lemma 2.2. $\square$

Corollary 2.5 The integers 2, 3, 6 are the only very bad integers.

Next, we show that $n = 8, 13, 15, 22$ are good integers. In fact, we determine $\chi_{td}(P_n)$ for small integers and also all possible minimum td-colorings for such paths. These ideas are used more strongly in determination of $\chi_{td}(P_n)$ for $n = 8, 13, 15, 22$.

Definition 2.6 Two td-colorings C_1 and C_2 of a given graph G are said to be equivalent if there exists an automorphism $f : G \rightarrow G$ such that $C_2(v) = C_1(f(v))$ for all vertices v of G . This is clearly an equivalence relation on the set of td-colorings of G .

Theorem 2.7 Let $V(P_n) = \{u_1, u_2, \dots, u_n\}$ as usual. Then

- (1) $\chi_{td}(P_2) = 2$. The only minimum td-coloring is (given by the color classes) $\{\{u_1\}, \{u_2\}\}$
- (2) $\chi_{td}(P_3) = 2$. The only minimum td-coloring is $\{\{u_1, u_3\}, \{u_2\}\}$.
- (3) $\chi_{td}(P_4) = 3$ with unique minimum coloring $\{\{u_1, u_4\}, \{u_2\}, \{u_3\}\}$.
- (4) $\chi_{td}(P_5) = 4$. Any minimum coloring is equivalent to one of $\{\{u_1, u_3\}, \{u_2\}, \{u_4\}, \{u_5\}\}$ or $\{\{u_1, u_5\}, \{u_2\}, \{u_3\}, \{u_4\}\}$ or $\{\{u_1\}, \{u_2\}, \{u_4\}, \{u_3, u_5\}\}$.
- (5) $\chi_{td}(P_6) = 4$ with unique minimum coloring $\{\{u_1, u_3\}, \{u_4, u_6\}, \{u_2\}, \{u_5\}\}$.
- (6) $\chi_{td}(P_7) = 5$. Any minimum coloring is equivalent to one of $\{\{u_1, u_3\}, \{u_2\}, \{u_4, u_7\}, \{u_5\}, \{u_6\}\}$ or $\{\{u_1, u_4\}, \{u_2\}, \{u_3\}, \{u_5, u_7\}, \{u_6\}\}$ or $\{\{u_1, u_4, u_7\}, \{u_2\}, \{u_3\}, \{u_5\}, \{u_6\}\}$.

Proof We prove only (vi). The rest are easy to prove. Now, $\gamma_t(P_7) = \lceil \frac{7}{2} \rceil = 4$. Clearly $\chi_{td}(P_7) \geq 4$. We first show that $\chi_{td}(P_7) \neq 4$. Let C be a td-coloring of P_7 with 4 colors. The vertices u_2 and $u_{-2} = u_6$ must have non-repeated colors. Suppose now that u_3 has a repeated color. Then $\{u_1, u_2, u_3\}$ must contain two color classes and $C|_{\langle [4, 7] \rangle}$ must be a td-coloring which will require at least 3 new colors (by (3)). Hence u_3 and similarly u_{-3} must be non-repeated colors. But, then we require more than 4 colors. Thus $\chi_{td}(P_7) = 5$. Let C be a minimal td-coloring of P_7 . Let u_2 and u_{-2} have colors 1 and 2 respectively. Suppose that both u_3 and u_{-3} are non-repeated colors. Then, we have the coloring $\{\{u_1, u_4, u_7\}, \{u_2\}, \{u_3\}, \{u_5\}, \{u_6\}\}$. If either u_3 or u_{-3} is a repeated color, then the coloring C can be verified to be equivalent to the coloring given by $\{\{u_1, u_3\}, \{u_2\}, \{u_4, u_7\}, \{u_5\}, \{u_6\}\}$, or by $\{\{u_1, u_4\}, \{u_2\}, \{u_3\}, \{u_5, u_7\}, \{u_6\}\}$. $\square$

We next show that $n = 8, 13, 15, 22$ are good integers.

Theorem 2.8 $\chi_{td}(P_n) = \gamma_t(P_n) + 2$ if $n = 8, 13, 15, 22$.

Proof As usual, we always adopt the convention $V(P_n) = \{u_1, u_2, \dots, u_n\}; u_{-i} = u_{n+1-i}$ for $i \geq \lceil \frac{n}{2} \rceil$; C denotes a minimum td-coloring of P_n .

We have only to prove $|C| > \gamma_t(P_n) + 1$. We consider the following four cases.

Case 1 $n = 8$

Let $|C| = 5$. Then, as before u_2 , being the only vertex dominated by u_1 has a non-repeated color. The same argument is true for u_{-2} also. If now u_3 has a repeated color, $\{u_1, u_2, u_3\}$ contains 2-color classes. As $C|_{\langle [4, 8] \rangle}$ is a td-coloring, we require at least 4 more colors. Hence, u_3 and similarly u_{-3} must have non-repeated colors. Thus, there are 4 singleton color classes and $\{u_2\}, \{u_3\}, \{u_{-2}\}$ and $\{u_{-3}\}$. The two adjacent vertices u_4 and u_{-4} contribute two more colors. Thus $|C|$ has to be 6.

Case 2 $n = 13$

Let $|C| = 8 = \gamma_t(P_{13}) + 1$. As before u_2 and u_{-2} are non-repeated colors. Since $\chi_{td}(P_{10}) = 7 + 2 = 9$, u_3 can not be a repeated color, arguing as in case (i). Thus, u_3 and u_{-3} are also non-repeated colors. Now, if u_1 and u_{-1} have different colors, a diagonal of the color classes chosen

as $\{u_1, u_{-1}, u_2, u_{-2}, u_3, u_{-3}, \dots\}$ form a totally dominating set of cardinality $8 = \gamma_t(P_{13}) + 1$. However, clearly u_1 and u_{-1} can be omitted from this set without affecting total dominating set giving $\gamma_t(P_{13}) \leq 6$, a contradiction. Thus, u_1 and $u_{-1} = u_{13}$ have the same color say 1. Thus, $\langle [4, -4] \rangle = \langle [4, 10] \rangle$ is colored with 4 colors including the repeated color 1. Now, each of the pair of vertices $\{u_4, u_6\}, \{u_5, u_7\}, \{u_8, u_{10}\}$ contains a color classes. Thus $u_9 = u_{-5}$ must be colored with 1. Similarly, u_5 . Now, if $\{u_4, u_6\}$ is not a color class, the vertex with repeated color must be colored with 1 which is not possible, since an adjacent vertex u_5 which also has color 1. Therefore $\{u_4, u_6\}$ is a color class. Similarly $\{u_8, u_{10}\}$ is also a color class. But then, u_7 will not dominate any color class. Thus $|C| = 9$.

Case 3 $n = 15$

Let $|C| = 9$. Arguing as before, u_2, u_{-2}, u_3 and u_{-3} have non-repeated colors $[\chi_{td}(P_{12}) = 8]$; u_1 and u_{-1} have the same color, say 1. The section $\langle [4, -4] \rangle = \langle [4, 12] \rangle$ consisting of 9 vertices is colored by 5 colors including the color 1. An argument similar to the one used in Case (2), gives u_4 (and u_{-4}) must have color 1. Thus, $C| \langle [5, -5] \rangle$ is a td-coloring with 4 colors including 1. Now, the possible minimum td-coloring of P_7 are given by Theorem 2.7. We can check that 1 can not occur in any color class in any of the minimum colorings given. e.g. take the coloring given by $\{u_5, u_8\}, \{u_6\}, \{u_7\}, \{u_9, u_{11}\}, \{u_{10}\}$. If u_6 has color 1, u_5 can not dominate a color class. Since u_4 has color 1, $\{u_5, u_8\}$ can not be color class 1 and so on. Thus $\chi_{td}(P_{15}) = 10$.

Case 4 $n = 22$

Let $|C| = \gamma_t(P_{22}) + 1 = 13$. We note that $\chi_{td}(P_{19}) = \gamma_t(P_{19}) + 2 = 12$. Then, arguing as in previous cases, we get the following facts.

Fact 1 u_2, u_{-2}, u_3, u_{-3} have non-repeated colors.

Fact 2 u_1 and u_{-1} have the same color, say 1.

Fact 3 u_7 is a non-repeated color.

This follows from the facts, otherwise $C| \langle [8, 22] \rangle$ will be a td-coloring; The section $\langle [1, 7] \rangle$ contain 4 color classes which together imply $\chi_{td}(P_{22}) \geq 4 + \chi_{td}(P_{15}) = 4 + 10 = 14$. In particular $\{u_5, u_7\}$ is not a color class.

Fact 4 The Facts 1 and 2, it follows that $C| \langle [4, -4] \rangle = C| \langle [4, 19] \rangle$ is colored with 9 colors including 1. Since each of the pair $\{\{u_4, u_6\}, \{u_5, u_7\}, \{u_8, u_{10}\}, \{u_9, u_{11}\}, \{u_{12}, u_{14}\}, \{u_{13}, u_{15}\}, \{u_{16}, u_{18}\}, \{u_{17}, u_{19}\}\}$ contain a color class, if any of these pairs is not a color class, one of the vertices must have a non-repeated color and the other colored with 1. From Fact 3, it then follows that the vertex u_5 must be colored with 1. It follows that $\{u_4, u_6\}$ must be a color class, since otherwise either u_4 or u_6 must be colored with 1.

Since $\{u_4, u_6\}$ is a color class, u_7 must dominate the color class $\{u_8\}$.

We summarize:

- u_2, u_3, u_7, u_8 have non-repeated colors.
- $\{u_4, u_6\}$ is a color class

- u_1 and u_5 are colored with color 1.

Similarly,

- $u_{-2}, u_{-3}, u_{-7}, u_{-8}$ have non-repeated colors.
- $\{u_{-4}, u_{-6}\}$ is a color class.
- u_{-1} and u_{-5} are colored with color 1.

Thus the section $\langle [9, -9] \rangle = \langle [9, 14] \rangle$ must be colored with 3 colors including 1. This is easily seen to be not possible, since for instance this will imply both u_{13} and u_{14} must be colored with color 1. Thus, we arrive at a contradiction. Thus $\chi_{td}(P_{22}) = 14$. $\square$

Theorem 2.9 *Let n be an integer. Then,*

- (1) *any integer of the form $4k$, $k \geq 2$ is good;*
- (2) *any integer of the form $4k + 1$, $k \geq 3$ is good;*
- (3) *any integer of the form $4k + 2$, $k \geq 5$ is good;*
- (4) *any integer of the form $4k + 3$, $k \geq 3$ is good.*

Proof The integers $n = 2, 3, 6$ are very bad and $n = 4, 5, 7, 9, 10, 11, 14, 18$ are bad. $\square$

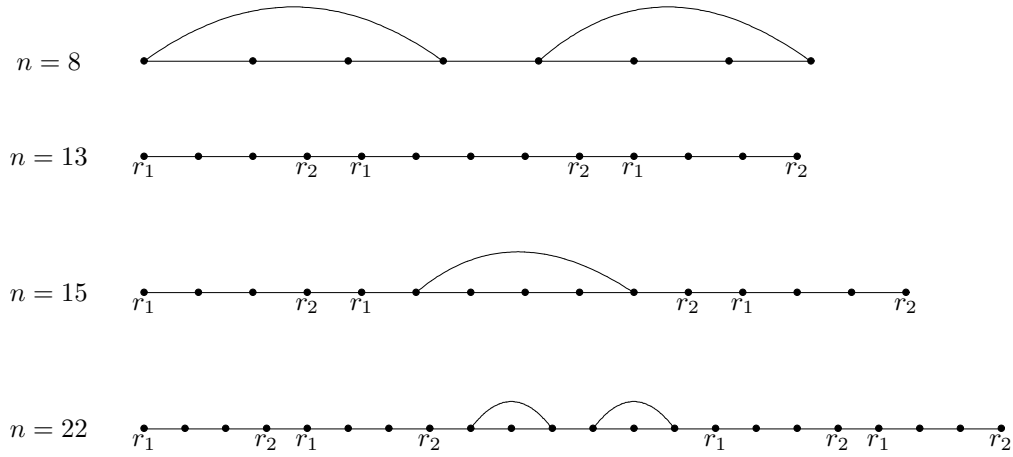
Remark 2.10 Let C be a minimal td-coloring of G . We call a color class in C , a non-dominated color class (n-d color class) if it is not dominated by any vertex of G . These color classes are useful because we can add vertices to these color classes without affecting td-coloring.

Lemma 2.11 *Suppose n is a good number and P_n has a minimal td-coloring in which there are two non-dominated color class. Then the same is true for $n + 4$ also.*

Proof Let $C_1, C_2, \dots, C_r$ be the color classes for P_n where C_1 and C_2 are non-dominated color classes. Suppose u_n does not have color C_1 . Then $C_1 \cup \{u_{n+1}\}, C_2 \cup \{u_{n+4}\}, \{u_{n+2}\}, \{u_{n+3}\}, C_3, C_4, \dots, C_r$ are required color classes for P_{n+4} . i.e. we add a section of 4 vertices with middle vertices having non-repeated colors and end vertices having C_1 and C_2 with the coloring being proper. Further, suppose the minimum coloring for P_n , the end vertices have different colors. Then the same is true for the coloring of P_{n+4} also. If the vertex u_1 of P_n does not have the color C_2 , the new coloring for P_{n+4} has this property. If u_1 has color C_2 , then u_n does not have the color C_2 . Therefore, we can take the first two color classes of P_{n+4} as $C_1 \cup \{u_{n+4}\}$ and $C_2 \cup \{u_{n+1}\}$. $\square$

Corollary 2.12 *Let n be a good number. Then P_n has a minimal td-coloring in which the end vertices have different colors. [It can be verified that the conclusion of the corollary is true for all $n \neq 3, 4, 11$ and 18].*

Proof We claim that P_n has a minimum td-coloring in which: (1) there are two non-dominated color classes; (2) the end vertices have different colors.

**Fig.1**

Now, it follows from the Lemma 2.11 that (1) and (2) are true for every good integer. $\square$

Corollary 2.13 *Let n be a good integer. Then, there exists a minimum td-coloring for P_n with two n -d color classes.*

References

- [1] M.I.Jinnah and A.Vijayalekshmi, *Total Dominator Colorings in Graphs*, Ph.D Thesis, University of Kerala, 2010.
- [2] F.Harry, *Graph Theory*, Addison - Wesley Reading Mass, 1969.
- [3] Terasa W.Haynes, Stephen T.Hedetniemi, Peter J.Slater, *Domination in Graphs*, Marcel Dekker, New York, 1998.
- [4] Terasa W.Haynes, Stephen T.Hedetniemi, Peter J.Slater, *Domination in Graphs - Advanced Topics*, Marcel Dekker, New York, 1998.