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Roman Domination in Complementary Prism Graphs

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Abstract: A Roman domination function on a complementary prism graph GG^c is a function $f : V \cup V^c \rightarrow \{0, 1, 2\}$ such that every vertex with label 0 has a neighbor with label 2. The Roman domination number $\gamma_R(GG^c)$ of a graph $G = (V, E)$ is the minimum of $\sum_{x \in V \cup V^c} f(x)$ over such functions, where the complementary prism GG^c of G is graph obtained from disjoint union of G and its complement G^c by adding edges of a perfect matching between corresponding vertices of G and G^c. In this paper, we have investigated few properties of $\gamma_R(GG^c)$ and its relation with other parameters are obtained.

Key Words: Graph, domination number, Roman domination number, Smarandachely Roman s-domination function, complementary prism, Roman domination of complementary prism.

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§1. Introduction

In this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$. Let $n = |V|$ and $m = |E|$ denote the number of vertices and edges of a graph G, respectively. For any vertex v of G, let $N(v)$ and $N[v]$ denote its open and closed neighborhoods respectively. $\alpha_0(G)(\alpha_1(G))$, is the minimum number of vertices (edges) in a vertex (edge) cover of G. $\beta_0(G)(\beta_1(G))$, is the minimum number of vertices (edges) in a maximal independent set of vertex (edge) of G. Let $deg(v)$ be the degree of vertex v in G. Then $\triangle(G)$ and $\delta(G)$ be maximum and minimum degree of G, respectively. If M is a matching in a graph G with the property that every vertex of G is incident with an edge of M, then M is a perfect matching in G. The complement G^c of a graph G is the graph having the same set of vertices as G denoted by V^c and in which two vertices are adjacent, if and only if they are not adjacent in G. Refer to [5] for additional graph theory terminology.

A dominating set $D \subseteq V$ for a graph G is such that each $v \in V$ is either in D or adjacent to a vertex of D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. Further, a dominating set D is a minimal dominating set of G, if and only if for each vertex $v \in D$, $D - v$ is not a dominating set of G. For complete review on theory of domination

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and its related parameters, we refer [1], [6]and [7].

For a graph $G = (V, E)$, let $f : V \to \{0, 1, 2\}$ and let (V_0, V_1, V_2) be the ordered partition of V induced by f, where $V_i = \{v \in V/f(v) = i\}$ and $|V_i| = n_i$ for $i = 0, 1, 2$. There exist 1-1 correspondence between the functions $f : V \to \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) of V. Thus we write $f = (V_0, V_1, V_2)$.

A function $f = (V_0, V_1, V_2)$ is a Roman dominating function (RDF) if $V_2 \succ V_0$, where $\succ$ signifies that the set V_2 dominates the set V_0. The weight of a Roman dominating function is the value $f(V) = \sum_{v \epsilon V} f(v) = 2|V_2| + |V_1|$. Roman dominating number $\gamma_R(G)$, equals the minimum weight of an RDF of G, we say that a function $f = (V_0, V_1, V_2)$ is a γ_R-function if it is an RDF and $f(V) = \gamma_R(G)$. Generally, let $I \subset \{0, 1, 2, \cdots, n\}$. A *Smarandachely Roman s-dominating function* for an integer s, $2 \leq s \leq n$ on a graph G = (V,E) is a function $f : V \to \{0, 1, 2, \cdots, n\}$ satisfying the condition that $|f(u) - f(v)| \geq s$ for each edge $uv \in E$ with $f(u)$ or $f(v) \in I$. Particularly, if we choose $n = s = 2$ and $I = \{0\}$, such a Smarandachely Roman s-dominating function is nothing but the Roman domination function. For more details on Roman dominations and its related parameters we refer [3]-[4] and [9]-[11].

In [8], Haynes etal., introduced the concept of domination and total domination in complementary prisms. Analogously, we initiate the Roman domination in complementary prism as follows:

A Roman domination function on a complementary prism graph GG^c is a function $f : V \cup V^c \to \{0, 1, 2\}$ such that every vertex with label 0 has a neighbor with label 2. The Roman domination number $\gamma_R(GG^c)$ of a graph $G = (V, E)$ is the minimum of $\sum_{x \in V \cup V^c} f(x)$ over such functions, where the complementary prism GG^c of G is graph obtained from disjoint union of G and its complement G^c by adding edges of a perfect matching between corresponding vertices of G and G^c.

§2. Results

We begin by making a couple of observations.

Observation 2.1 *For any graph G with order n and size m,*

$$m(GG^c) = n(n + 1)/2.$$

Observation 2.2 *For any graph G,*

(i) $\beta_1(GG^c) = n$.

(ii) $\alpha_1(GG^c) + \beta_1(GG^c) = 2n$.

Proof Let G be a graph and GG^c be its complementary prism graph with perfect matching M. If one to one correspondence between vertices of a graph G and its complement G^c in GG^c, then GG^c has even order and M is a 1-regular spanning sub graph of GG^c, thus (i) follows and due to the fact of $\alpha_1(G) + \beta_1(G) = n$,(ii) follows. □

Observation 2.3 *For any graph G,*

$$\gamma(GG^c) = n$$

if and only if G or G^c is totally disconnected graph.

Proof Let there be n vertices of degree 1 in GG^c. Let D be a dominating set of GG^c and v be a vertex of G of degree $n-1$, $v \in D$. In GG^c, v dominates n vertices and remaining $n-1$ vertices are pendent vertices which has to dominate itself. Hence $\gamma(GG^c) = n$. Conversely, if $\gamma(GG^c) = n$, then there are n vertices in minimal dominating set D. □

Theorem 2.1 *For any graph G,*

$$\gamma_R(GG^c) = \alpha_1(GG^c) + \beta_1(GG^c)$$

if and only if G being an isolated vertex.

Proof If G is an isolated vertex, then GG^c is K_2 and $\gamma_R(GG^c) = 2, \alpha_1(GG^c) = 1$ and $\beta_1(GG^c) = 1$. Conversely, if $\gamma_R(GG^c) = \alpha_1(GG^c) + \beta_1(GG^c)$. By above observation, then we have $\gamma_R(GG^c) = 2|V_2| + |V_1|$. Thus we consider the following cases:

Case 1 If $V_2 = \phi, |V_1| = 2$, then $V_0 = \phi$ and $GG^c \cong K_2$.

Case 2 If $|V_2| = 1, |V_1| = \phi$, then GG^c is a complete graph.

Hence the result follows. □

Theorem 2.2 *Let G and G^c be two complete graphs then GG^c is also complete if and only if $G \cong K_1$.*

Proof If $G \cong K_1$ then $G^c \cong K_1$ and $GG^c \cong K_2$ which is a complete graph. Conversely, if GG^c is complete graph then any vertex v of G is adjacent to $n-1$ vertices of G and n vertices of G^c. According to definition of complementary prism this is not possible for graph other than K_1. □

Theorem 2.3 *For any graph G,*

$$\gamma(GG^c) < \gamma_R(GG^c) \leq 2\gamma(GG^c).$$

Further,the upper bound is attained if $V_1(GG^c) = \phi$.

Proof Let $f = (V_0, V_1, V_2)$ be γ_R-function. If $V_2 \succ V_0$ and $(V_1 \cup V_2)$ dominates GG^c, then $\gamma(GG^c) < |V_1 \cup V_2| = |V_1| + 2|V_2| = \gamma_R(GG^c)$. Thus the result follows.

Let $f = (V_0, V_1, V_2)$ be an RDF of GG^c with $|D| = \gamma(GG^c)$. Let $V_2 = D, V_1 = \phi$ and $V_0 = V - D$. Since f is an RDF and $\gamma_R(GG^c)$ denotes minimum weight of $f(V)$. It follows $\gamma_R(GG^c) \leq f(V) = |V_1| + 2|V_2| = 2|S| = 2\gamma(GG^c)$. Hence the upper bound follows. For graph GG^c, let v be vertex not in V_1, implies that either $v \in V_2$ or $v \in V_0$. If $v \in V_2$ then $v \in D$, $\gamma_R(GG^c) = 2|V_2| + |V_1| = 2|D| = 2\gamma(GG^c)$. If $v \in V_0$ then $N(v) \subseteq V_2$ or $N(v) \subseteq V_0$ as v does not belong to V_1. Hence the result. □

Theorem 2.4 *For any graph G,*

$$2 \leq \gamma_R(GG^c) \leq (n+1).$$

Further, the lower bound is attained if and only if $G \cong K_1$ and the upper bound is attained if G or G^c is totally disconnected graph.

Proof Let G be a graph with $n \geq 1$. If $f = \{V_0, V_1, V_2\}$ be a RDF of GG^c, then $\gamma_R(GG^c) \geq 2$. Thus the lower bound follows.

Upper bound is proved by using mathematical induction on number of vertices of G. For $n = 1$, $GG^c \cong K_2, \gamma_R(GG^c) = n+1$. For $n = 2$, $GG^c \cong P_4, \gamma_R(GG^c) = n+1$. Assume the result to be true for some graph H with $n-1$ vertices, $\gamma_R(HH^c) \leq n$. Let G be a graph obtained by adding a vertex v to H. If v is adjacent to a vertex w in H which belongs to V_2, then $v \in V_0, \gamma_R(GG^c) = n < n+1$. If v is adjacent to a vertex either in V_0 or V_1, then $\gamma_R(GG^c) = n+1$. If v is adjacent to all vertices of H then $\gamma_R(GG^c) < n < n+1$. Hence upper bound follows for any number of vertices of G.

Now, we prove the second part. If $G \cong K_1$, then $\gamma_R(GG^c) = 2$. On the other hand, if $\gamma_R(GG^c) = 2 = 2|V_2| + |V_1|$ then we have following cases:

Case 1 If $|V_2| = 1, |V_1| = 0$, then there exist a vertex $v \in V(GG^c)$ such that degree of $v = (n-1)$, thus one and only graph with this property is $GG^c \cong K_2$. Hence $G = K_1$.

Case 2 If $|V_2| = 0, |V_1| = 2$, then there are only two vertices in the GG^c which are connected by an edge. Hence the result.

If G is totally disconnected then G^c is a complete graph. Any vertex v^c in G^c dominates n vertices in GG^c. Remaining $n-1$ vertices of GG^c are in V_1. Hence $\gamma_R(GG^c) = n+1$. □

Proposition 2.1([3]) *For any path P_n and cycle C_n with $n \geq 3$ vertices,*

$$\gamma_R(P_n) = \gamma_R(C_n) = \lceil 2n/3 \rceil,$$

where $\lceil x \rceil$ is the smallest integer not less than x.

Theorem 2.5 *For any graph G,*

(*i*) *if $G = P_n$ with $n \geq 3$ vertices, then*

$$\gamma_R(GG^c) = 4 + \lceil 2(n-3)/3 \rceil;$$

(*ii*) *if $G = C_n$ with $n \geq 4$ vertices, then*

$$\gamma_R(GG^c) = 4 + \lceil 2(n-2)/3 \rceil.$$

Proof (*i*) Let $G = P_n$ be a path with with $n \geq 3$ vertices. Then we have the following cases:

Case 1 Let $f = (V_0, V_1, V_2)$ be an RDF and a pendent vertex v is adjacent to a vertex u in G. The vertex v^c is not adjacent to a vertex u^c in V^c. But the vertex of v^c in V^c is adjacent to n vertices of GG^c. Let $v^c \in V_2$ and $N(v^c) \subseteq V_0$. There are n vertices left and $u^c \in N[u]$ but $\{N(u^c) - u\} \subseteq V_0$. Hence $u \in V_2, N(u) \subseteq V_0$. There are $(n-3)$ vertices left, whose induced subgraph H forms a path with $\gamma_R(H) = \lceil 2(n-3)/3 \rceil$, this implies that $\gamma_R(G) = 4 + \lceil 2(n-3)/3 \rceil$.

Case 2 If v is not a pendent vertex, let it be adjacent to vertices u and w in G. Repeating same procedure as above case , $\gamma_R(GG^c) = 6 + \lceil 2(n-3)/3 \rceil$, which is a contradiction to fact of RDF.

(ii) Let $G = C_n$ be a cycle with $n \geq 4$ vertices. Let $f = (V_0, V_1, V_2)$ be an RDF and w be a vertex adjacent to vertex u and v in G, and w^c is not adjacent to u^c and v^c in V^c. But w^c is adjacent to $(n-2)$ vertices of GG^c. Let $w^c \in V_2$ and $N(w^c) \subseteq V_0$. There are $(n+1)$-vertices left with u^c or $v^c \in V_2$. With out loss of generality, let $u^c \in V_2$, $N(u^c) \subseteq V_0$. There are $(n-2)$ vertices left, whose induced subgraph H forms a path with $\gamma_R(H) = \lceil 2(n-2)/3 \rceil$ and $V_2 = \{w, u^c\}$, this implies that $\gamma_R(G) = 4 + \lceil 2(n-2)/3 \rceil$. □

Theorem 2.6 *For any graph G,*

$$max\{\gamma_R(G), \gamma_R(G^c)\} < \gamma_R(GG^c) \leq (\gamma_R(G) + \gamma_R(G^c)).$$

Further, the upper bound is attained if and only if the graph G is isomorphic with K_1.

Proof Let G be a graph and let $f : V \to \{0, 1, 2\}$ and $f = (V_0, V_1, V_2)$ be RDF. Since GG^c has $2n$ vertices when G has n vertices, hence $max\{\gamma_R(G), \gamma_R(G^c)\} < \gamma_R(GG^c)$ follows.

For any graph G with $n \geq 1$ vertices. By Theorem 2.4, we have $\gamma_R(GG^c) \leq (n+1)$ and $(\gamma_R(G) + \gamma_R(G^c)) \leq (n+2) = (n+1)+1$. Hence the upper bound follows.

Let $G \cong K_1$. Then $GG^c = K_2$, thus the upper bound is attained. Conversely, suppose $G \ncong K_1$. Let u and v be two adjacent vertices in G and u is adjacent to v and u^c in GG^c. The set $\{u, v^c\}$ is a dominating set out of which $u \in V_2, v^c \in V_1$. $\gamma_R(G) = 2, \gamma_R(G^c) = 0$ and $\gamma_R(GG^c) = 3$ which is a contradiction. Hence no two vertices are adjacent in G. □

Theorem 2.7 *If degree of every vertex of a graph G is one less than number of vertices of G, then*

$$\gamma_R(GG^c) = \gamma(GG^c) + 1.$$

Proof Let $f = (V_0, V_1, V_2)$ be an RDF and let v be a vertex of G of degree $n-1$. In GG^c, v is adjacent to n vertices. If D is a minimum dominating set of GG^c then $v \in D$, $v \in V_2$ also $N(v) \subseteq V_0$. Remaining $n-1$ belongs to V_1 and D. $|D| = \gamma(GG^c) = n$ and $\gamma_R(GG^c) = n+1 = \gamma(GG^c) + 1$. □

Theorem 2.8 *For any graph G with $n \geq 1$ vertices,*

$$\gamma_R(GG^c) \leq [2n - (\Delta(GG^c) + 1)].$$

Further, the bound is attained if G is a complete graph.

Proof Let G be any graph with $n \geq 1$ vertices. Then GG^c has $2n$- vertices. Let $f = (V_0, V_1, V_2)$ be an RDF and v be any vertex of GG^c such that $deg(v) = \Delta(GG^c)$. Then v

dominates $\Delta(GG^c)+1$ vertices. Let $v \in V_2$ and$N(v) \subseteq V_0$. There are $(2n-(\Delta(GG^c)+1)$ vertices left in GG^c, which belongs to one of V_0, V_1 or V_2. If all these vertices $\in V_1$, then $\gamma_R(GG^c) = 2|V_2| + |V_1| = 2 + (2n - \Delta(GG^c) + 1) = 2n - \Delta(GG^c) + 1$. Hence lower bound is attained when $G \cong K_n$, where v is a vertex of G. If not all remaining vertices belong to V_1, then there may be vertices belonging to V_2 and which implies there neighbors belong to V_0. Hence the result follows. □

Theorem 2.9 *For any graph G,*

$$\gamma_R(GG^c)^c \leq \gamma_R(GG^c).$$

Further, the bound is attained for one of the following conditions:

(i) $GG^c \cong (GG^c)^c$;

(ii) GG^c *is a complete graph.*

Proof Let G be a graph, GG^c be its complementary graph and $(GG^c)^c$ be complement of complementary prism. According to definition of GG^c there should be one to one matching between vertices of G and G^c, where as in $(GG^c)^c$ there will be one to $(n-1)$ matching between vertices of G and G^c implies that adjacency of vertices will be more in $(GG^c)^c$. Hence the result. If $GG^c \cong (GG^c)^c$, domination and Roman domination of these two graphs are same. The only complete graph GG^c can be is K_2. $(GG^c)^c$ will be two isolated vertices, $\gamma_R(GG^c) = 2$ and $\gamma_R(GG^c)^c = 2$. Hence bound is attained. □

To prove our next results, we make use of following definitions:

A *rooted tree* is a tree with a countable number of vertices, in which a particular vertex is distinguished from the others and called the root. In a rooted tree, the parent of a vertex is the vertex connected to it on the path to the root; every vertex except the root has a unique parent. A child of a vertex v is a vertex of which v is the parent. A leaf is a vertex without children.

A graph with exactly one induced cycle is called *unicyclic.*

Theorem 2.10 *For any rooted tree T,*

$$\gamma_R(TT^c) = 2|S_2| + |S_1|,$$

where $S_1 \subseteq V_1$ and $S_2 \subseteq V_2$.

Proof Let T be a rooted tree and $f = (V_0, V_1, V_2)$ be RDF of a complementary prism TT^c. We label all parent vertices of T as $P_1, P_2, ...P_k$ where P_k is root of a tree T. Let S_p be set of all parent vertices of T, S_l be set of all leaf vertices of T and $v \in S_l$ be a vertex farthest from P_k. The vertex v^c is adjacent to $(n-1)-$vertices in TT^c. Let $v^c \in S_2$, and $N(v^c) \subseteq V_0$. Let P_1 be parent vertex of $v \in T$. For i=1 to k if P_i is not assigned weight then $P_i \in S_2$ and $N(P_i) \subseteq V_0$. If P_i is assigned weight and check its leaf vertices in T, then we consider the following cases:

Case 1 If P_i has at least 2 leaf vertices, then $P_i \in S_2$ and $N(P_i) \subseteq V_0$.

Case 2 If P_i has at most 1 leaf vertex, then all such leaf vertices belong to S_1. Thus $\gamma_R(GG^c) = 2|S_2| + |S_1|$ follows. □

Theorem 2.11 *Let G^c be a complement of a graph G. Then the complementary prism GG^c is*

(i) isomorphic with a tree T if and only if G or G^c has at most two vertices.

(ii) $(n+1)/2$-regular graph if and only if G is $(n-1)/2$-regular.

(iii) unicyclic graph if and only if G has exactly 3 vertices.

Proof (i) Suppose GG^c is a tree T with the graph G having minimum three vertices. Then we have the following cases:

Case 1 Let u, v and w be vertices of G with v is adjacent to both u and w. In GG^c, u^c is connected to u and w^c also v^c is connected to v. Hence there is a closed path u-v-w-w^c-u^c-u, which is a contradicting to our assumption.

Case 2 If vertices u, v and w are totally disconnected in G, then G^c is a complete graph. Since every complete graph G with $n \geq 3$ has cycle. Hence GG^c is not a tree.

Case 3 If u and v are adjacent but which is not adjacent to w in G, then in GG^c there is a closed path u-u^c-w^c-v^c-v^c-u, again which is a contradicting to assumption.

On the other hand, if G has one vertex, then $GG^c \cong K_2$ and if G have two vertices, then $GG^c \cong P_4$. In both the cases GG^c is a tree.

(ii) Let G be r-regular graph, where $r = (n-1)/2$, then G^c is $n-r-1$ regular. In GG^c, degree of every vertex in G is $r+1 = (n+1)/2$ and degree of every vertex in G^c is $n-r = (n+1)/2$, which implies GG^c is $(n+1)/2$-regular. Conversely, suppose GG^c is $s = (n+1)/2$-regular. Let E be set of all edges making perfect match between G and G^c. In $GG^c - E$, G is $s-1$-regular and G^c is $(n-s-1)$-regular. Hence the graph G is $(n-1)/2$-regular.

(iii) If GG^c has at most two vertices, then from (i), GG^c is a tree. Minimum vertices required for a graph to be unicyclic is 3. Because of perfect matching in complementary prism and G and G^c are connected if there are more than 3 vertices there will be more than 1 cycle.□

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