International J.Math. Combin. Vol.2(2012), 76-80

Perfect Domination Excellent Trees

Sharada B.

(Department of Studies in Computer Science, University of Mysore, Manasagangothri, Mysore-570006, India)

E-mail: sharadab21@gmail.com

Abstract: A set D of vertices of a graph G is a *perfect dominating set* if every vertex in $V \setminus D$ is adjacent to exactly one vertex in D. In this paper we introduce the concept of *perfect domination excellent graph* as a graph in which every vertex belongs to some perfect dominating set of minimum cardinality. We also provide a constructive characterization of perfect domination excellent trees.

Key Words: Tree, perfect domination, Smarandachely k-dominating set, Smarandachely k-domination number.

AMS(2010): 05C69

§1. Introduction

Let $G = (V, E)$ be a graph. A set D of vertices is a perfect dominating set if every vertex in $V \setminus D$ is adjacent to exactly one vertex in D. The perfect domination number of G, denoted $\gamma_p(G)$, is the minimum cardinality of a perfect dominating set of G. A perfect dominating set of cardinality $\gamma_p(G)$ is called a $\gamma_p(G)$-set. Generally, a set of vertices S in a graph G is said to be a *Smarandachely k-dominating set* if each vertex of G is dominated by at least k vertices of S and the *Smarandachely k-domination number* $\gamma_k(G)$ of G is the minimum cardinality of a Smarandachely k-dominating set of G. Particularly, if $k = 1$, such a set is called a dominating set of G and the Smarandachely 1-domination number of G is nothing but the domination number of G and denoted by $\gamma(G)$. Domination and its parameters are well studied in graph theory. For a survey on this subject one can go through the two books by Haynes et al [3,4].

Sumner [7] defined a graph to be $\gamma-excellent$ if every vertex is in some minimum dominating set. Also, he has characterized $\gamma - excellent$ trees. Similar to this concept, Fricke et al [2] defined a graph to be $i-excellent$ if every vertex is in some minimum independent dominating set. The i-excellent trees have been characterized by Haynes et al [5]. Fricke et al [2] defined $\gamma_t - excellent$ graph as a graph in which every vertex is in some minimum total dominating set. The γ_t-excellent trees have been characterized by Henning [6].

In this paper we introduce the concept of γ_p-excellent graph. Also, we provide a constructive characterization of perfect domination excellent trees.

We define the perfect domination number of G relative to the vertex u, denoted $\gamma_p^u(G)$, as the minimum cardinality of a perfect dominating set of G that contains u. We call a perfect dominating set of cardinality $\gamma_p^u(G)$ containing u to be a $\gamma_p^u(G)$-set. We define a graph G to be

[1]Received February 1, 2012. Accepted June 18, 2012.

$\gamma_p - excellent$ if $\gamma_p^u(G) = \gamma_p(G)$ for every vertex u of G.

All graphs considered in this paper are finite and simple. For definitions and notations not given here see [4]. A tree is an acyclic connected graph. A *leaf* of a tree is a vertex of degree 1. A *support vertex* is a vertex adjacent to a leaf. A *strong support vertex* is a support vertex that is adjacent to more than one leaf.

§2. Perfect Domination Excellent Graph

Proposition 2.1 *A path P_n is $\gamma_p - excellent$ if and only if $n = 2$ or $n \equiv 1(mod3)$.*

Proof It is easy to see that the paths P_2 and P_n for $n \equiv 1(mod3)$ are γ_p-excellent. Let $P_n, n \geq 3$, be a γ_p-excellent path and suppose that $n \equiv 0, 2(mod3)$. If $n \equiv 0(mod3)$, then P_n has a unique γ_p-set, which does not include all the vertices. If $n \equiv 2(mod3)$, then no γ_p-set of P_n contains the third vertex on the path. □

Proposition 2.2 *Every graph is an induced subgraph of a γ_p-excellent graph.*

Proof Consider a graph H and let $G = HoK_1$, the 1-corona of a graph H. Every vertex in $V(H)$ is now a support vertex in G. Therefore, $V(H)$ is a γ_p-set of G. As well, the set of end vertices in G is a γ_p-set. Hence every vertex in $V(G)$ is in some γ_p-set and G is γ_p-excellent. Since H is an induced subgraph of G, the result follows. □

§3. Characterization of Trees

We now provide a constructive characterization of perfect domination excellent trees. We accomplish this by defining a family of labelled trees as defined in [1].

Let $\mathcal{F} = \{T_n\}_{n \geq 1}$ be the family of trees constructed inductively such that T_1 is a path P_4 and $T_n = T$, a tree. If $n \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the two operations $\mathcal{F}_1$, $\mathcal{F}_2$ for $i = 1, 2, \cdots, n-1$. Then we say that T has length n in $\mathcal{F}$.

We define the status of a vertex v, denoted sta(v) to be A or B. Initially if $T_1 = P_4$, then sta(v) $= A$ if v is a support vertex and sta(v) $= B$, if v is a leaf. Once a vertex is assigned a status, this status remains unchanged as the tree is constructed.

Operation $\mathcal{F}_1$ Assume $y \in T_n$ and sta(y) $= A$. The tree T_{n+1} is obtained from T_n by adding a path x, w and the edge xy. Let sta(x) $= A$ and sta(w) $= B$.

Operation $\mathcal{F}_2$ Assume $y \in T_n$ and sta(y) $= B$. The tree T_{n+1} is obtained from T_n by adding a path x, w, v and the edge xy. Let sta(x) $=$ sta(w) $= A$ and sta(v) $= B$.

$\mathcal{F}$ is closed under the two operations $\mathcal{F}_1$ and $\mathcal{F}_2$. For $T \in \mathcal{F}$, let $A(T)$ and $B(T)$ be the sets of vertices of status A and B respectively. We have the following observation, which follow from the construction of $\mathcal{F}$.

Observation 3.1 Let $T \in \mathcal{F}$ and $v \in V(T)$

1. If sta$(v) = A$, then v is adjacent to exactly one vertex of $B(T)$ and at least one vertex of $A(T)$.

2. If sta$(v) = B$, then $N(v)$ is a subset of $A(T)$.

3. If v is a support vertex, then sta$(v) = A$.

4. If v is a leaf, then sta$(v) = B$.

5. $|A(T)| \geq |B(T)|$

6. Distance between any two vertices in $B(T)$ is at least three.

Lemma 3.2 *If $T \in \mathcal{F}$, then $B(T)$ is a $\gamma_p(T)$-set. Moreover if T is obtained from $T' \in \mathcal{F}$ using operation $\mathcal{F}_1$ or $\mathcal{F}_2$, then $\gamma_p(T) = \gamma_p(T') + 1$.*

Proof By Observation 3.1, it is clear that $B(T)$ is a perfect dominating set. Now we prove that, $B(T)$ is a $\gamma_p(T)$-set. We proceed by induction on the length n of the sequence of trees needed to construct the tree T. Suppose $n = 1$, then $T = P_4$, belongs to $\mathcal{F}$. Let the vertices of P_4 be labeled as a, b, c, d. Then, $B(P_4) = \{a, d\}$ and is a $\gamma_p(P_4)$-set. This establishes the base case. Assume then that the result holds for all trees in $\mathcal{F}$ that can be constructed from a sequence of fewer than n trees where $n \geq 2$. Let $T \in \mathcal{F}$ be obtained from a sequence $T_1, T_2, \cdots, T_n$ of n trees, where $T' = T_{n-1}$ and $T = T_n$. By our inductive hypothesis $B(T')$ is a $\gamma_p(T')$-set.

We now consider two possibilities depending on whether T is obtained from T' by operation $\mathcal{F}_1$ or $\mathcal{F}_2$.

Case 1 T is obtained from T' by operation $\mathcal{F}_1$.

Suppose T is obtained from T' by adding a path y, x, w of length 2 to the attacher vertex $y \in V(T')$. Any $\gamma_p(T')$-set can be extended to a $\gamma_p(T)$-set by adding to it the vertex w, which is of status B. Hence $B(T) = B(T') \cup \{w\}$ is a $\gamma_p(T)$-set.

Case 2 T is obtained from T' by operation $\mathcal{F}_2$.

The proof is very similar to Case 1.

If T is obtained from $T' \in \mathcal{F}$ using operation $\mathcal{F}_1$ or $\mathcal{F}_2$, then T can have exactly one more vertex with status B than T'. Since $\gamma_p(T) = |B(T)|$ and $\gamma_p(T') = |B(T')|$, it follows that $\gamma_p(T) = \gamma_p(T') + 1$. □

Lemma 3.3 *If $T \in \mathcal{F}$ have length n, then T is a $\gamma_p -$ excellent tree.*

Proof Since T has length n in $\mathcal{F}$, T can be obtained from a sequence $T_1, T_2, \cdots, T_n$ of trees such that T_1 is a path P_4 and $T_n = T$, a tree. If $n \geq 2$, T_{i+1} can be obtained from T_i by one of the two operations $\mathcal{F}_1$, $\mathcal{F}_2$ for $i = 1, 2, \cdots, n-1$. To prove the desired result, we proceed by induction on the length n of the sequence of trees needed to construct the tree T.

If $n = 1$, then $T = P_4$ and therefore, T is γ_p-excellent. Hence the lemma is true for the base case.

Assume that the result holds for all trees in $\mathcal{F}$ of length less than n, where $n \geq 2$. Let $T \in \mathcal{F}$ be obtained from a sequence $T_1, T_2, \cdots, T_n$ of n trees. For notational convenience, we denote T_{n-1} by T'. We now consider two possibilities depending on whether T is obtained from T' by operation $\mathcal{F}_1$ or $\mathcal{F}_2$.

Case 1 T is obtained from T' by operation $\mathcal{F}_1$.

By Lemma 3.2, $\gamma_p(T) = \gamma_p(T') + 1$. Let u be an arbitrary element of $V(T)$.

Subcase 1.1 $u \in V(T')$.

Since T' is γ_p-excellent, $\gamma_p^u(T') = \gamma_p(T')$. Now any $\gamma_p^u(T')$-set can be extended to a perfect dominating set of T by adding either x or w and so $\gamma_p^u(T) \leq \gamma_p^u(T') + 1 = \gamma_p(T') + 1 = \gamma_p(T)$.

Subcase 1.2 $u \in V(T) \setminus V(T')$.

Any $\gamma_p^y(T')$-set can be extended to a perfect dominating set of T by adding the vertex w and so $\gamma_p^u(T) \leq \gamma_p^y(T') + 1 = \gamma_p(T') + 1 = \gamma_p(T)$.

Consequently, we have $\gamma_p^u(T) = \gamma_p(T)$ for any arbitrary vertex u of T. Hence T is γ_p-excellent.

Case 2 T is obtained from T' by operation $\mathcal{F}_2$.

The proof is very similar to Case 1. □

Proposition 3.4 *If T is a tree obtained from a tree T' by adding a path x, w or a path x, w, v and an edge joining x to the vertex y of T', then $\gamma_p(T) = \gamma_p(T') + 1$.*

Proof Suppose T is a tree obtained from a tree T' by adding a path x, w and an edge joining x to the vertex y of T', then any $\gamma_p(T')$-set can be extended to a perfect dominating set of T by adding x or w and so $\gamma_p(T) \leq \gamma_p(T') + 1$. Now let S be a $\gamma_p(T)$-set and let $S' = S \cap V(T')$. Then S' is a perfect dominating set of T'. Hence, $\gamma_p(T') \leq |S'| \leq |S| - 1 = \gamma_p(T) - 1$. Thus, $\gamma_p(T) \geq \gamma_p(T') + 1$. Hence $\gamma_p(T) = \gamma_p(T') + 1$. The other case can be proved on the same lines. □

Theorem 3.5 *A tree T of order $n \geq 4$ is $\gamma_p - excellent$ if and only if $T \in \mathcal{F}$.*

Proof By Lemma 3.3, it is sufficient to prove that the condition is necessary. We proceed by induction on the order n of a γ_p-excellent tree T. For $n = 4$, $T = P_4$ is γ_p-excellent and also it belongs to the family $\mathcal{F}$. Assume that $n \geq 5$ and all γ_p-excellent trees with order less than n belong to $\mathcal{F}$. Let T be a γ_p-excellent tree of order n. Let $P : v_1, v_2, \cdots, v_k$ be a longest path in T. Obviously $\deg(v_1) = \deg(v_k) = 1$ and $\deg(v_2) = \deg(v_{k-1}) = 2$ and $k \geq 5$. We consider two possibilities.

Case 1 v_3 is a support vertex.

Let $T' = T \setminus \{v_1, v_2\}$. We prove that T' is γ_p-excellent, that is for any $u \in T', \gamma_p^u(T') = \gamma_p(T')$. Since $u \in T' \subset T$ and T is γ_p-excellent, there exists a $\gamma_p^u(T)$-set such that $\gamma_p^u(T) = \gamma_p(T)$. Let S be a $\gamma_p^u(T) - set$ and $S' = S \cap V(T')$. Then S' is a perfect dominating set of T'. Also, $|S'| \leq |S| - 1 = \gamma_p(T) - 1 = \gamma_p(T')$, by Proposition 3.4. Since $u \in S', S'$ is a $\gamma_p^u(T')$-set such that $\gamma_p^u(T') = \gamma_p(T')$. Thus T' is γ_p-excellent. Hence by the inductive hypothesis $T' \in \mathcal{F}$, since $|V(T')| < |V(T)|$. The $sta(v_3) = A$ in T', because v_3 is a support vertex. Thus, T is obtained from $T' \in \mathcal{F}$ by the operation $\mathcal{F}_1$. Hence $T \in \mathcal{F}$ as desired.

Case 2 v_3 is not a support vertex.

Let $T' = T \setminus \{v_1, v_2, v_3\}$. As in Case 1, we can prove that T' is γ_p-excellent. Since $|V(T')| < |V(T)|$, $T' \in \mathcal{F}$ by the inductive hypothesis.

If v_4 is a support vertex or has a neighbor which is a support vertex then v_3 is present in none of the γ_p-sets. So, T cannot be γ_p-excellent. Hence either $deg(v_4) = 2$ and v_4 is a leaf of T' so that $v_4 \in B(T')$ by Observation 3.1 or $deg(v_4) \geq 3$ and all the neighbors of v_4 in $T' \setminus \{v_5\}$ are at distance exactly 2 from a leaf of T'. Hence all the neighbors of v_4 in $T' \setminus \{v_5\}$ are in $A(T')$ by Observation 3.1, and have no neighbors in $B(T')$ except v_4. Hence $v_4 \in B(T')$, again by Observation 3.1. Thus, T can be obtained from T' by the operation $\mathcal{F}_2$. Hence $T \in \mathcal{F}$. □

References

[1] H.Aram, S.M.Sheikholeslami and O.Favaron, Domination subdivision numbers of trees, *Discrete Math.*, 309 (2009), 622-628.

[2] G.H.Fricke, T.W.Haynes, S.S.Hedetniemi and R.C.Laskar, Excellent trees, *Bulletin of ICA*, 34(2002) 27-38.

[3] T.W.Haynes, S.T.Hedetniemi and P.J.Slater (Eds.), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.

[4] T.W.Haynes, S.T.Hedetniemi and P.J.Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[5] T.W.Haynes, M.A.Henning, A characterization of i-excellent trees, *Discrete Math.*, 248 (2002), 69-77.

[6] M.A.Henning, Total domination excellent trees, *Discrete Math.*, 263(2003), 93-104.

[7] D.Sumner, personal communication, May 2000.