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On Set-Semigraceful Graphs

Ullas Thomas

Department of Basic Sciences, Amal Jyothi College of Engineering

Koovappally P.O. - 686 518, Kottayam, Kerala, India

Sunil C. Mathew

Department of Mathematics, St.Thomas College Palai, Arunapuram P.O. - 686 574, Kottayam, Kerala, India

E-mail: ullasmanickathu@rediffmail.com, sunilcmathew@gmail.com

Abstract: This paper studies certain properties of set-semigraceful graphs and obtain certain bounds for the order and size of such graphs. More set-semigraceful graphs from given ones are also obtained through various graph theoretic methods.

Key Words: Set-indexer, set-graceful, set-semigraceful.

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§1. Introduction

In 1986, B. D. Acharya introduced the concept of set-indexer of a graph G which is an assignment of distinct subsets of a finite set X to the vertices of G subject to certain conditions. Based on this, the notions of set-graceful and set-semigraceful graphs were derived. Later many authors have studied about set-graceful graphs and obtained many significant results.

This paper sheds more light on set-semigraceful graphs. Apart from many classes of set-semigraceful graphs, several properties of them are also investigated. Certain bounds for the order and size of set-semigraceful graphs are derived. More set-semigraceful graphs from given ones are also obtained through various techniques of graph theory.

§2. Preliminaries

In this section we include certain definitions and known results needed for the subsequent development of the study. Throughout this paper, l, m and n stand for natural numbers without restrictions unless and otherwise mentioned. For a nonempty set X, the set of all subsets of X is denoted by 2^X. We always denote a graph under consideration by G and its vertex and edge sets by V and E respectively and G' being a subgraph of a graph G is denoted by $G' \subseteq G$. When G' is a proper subgraph of G we denote it by $G' \subset G$. By the term graph we mean a simple graph and the basic notations and definitions of graph theory are assumed to be familiar to the readers.

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Definition 2.1([1]) *Let $G = (V, E)$ be a given graph and X be a nonempty set. Then a mapping $f : V \to 2^X$, or $f : E \to 2^X$, or $f : V \cup E \to 2^X$ is called a set-assignment or set-valuation of the vertices or edges or both.*

Definition 2.2([1]) *Let G be a given graph and X be a nonempty set. Then a set-valuation $f : V \cup E \to 2^X$ is a set-indexer of G if*

1. *$f(uv) = f(u) \oplus f(v), \forall uv \in E$, where '$\oplus$' denotes the binary operation of taking the symmetric difference of the sets in 2^X*
2. *the restriction maps $f|_V$ and $f|_E$ are both injective.*

In this case, X is called an indexing set of G. Clearly a graph can have many indexing sets and the minimum of the cardinalities of the indexing sets is said to be the set-indexing number of G, denoted by $\gamma(G)$. The set-indexing number of trivial graph K_1 is defined to be zero.

Theorem 2.3([1]) *Every graph has a set-indexer.*

Theorem 2.4([1]) *If X is an indexing set of $G = (V, E)$. Then*

(*i*) $|E| \leq 2^{|X|} - 1$ *and*
(*ii*) $\lceil \log_2(|E| + 1) \rceil \leq \gamma(G) \leq |V| - 1$, *where $\lceil \ \rceil$ is the ceiling function.*

Theorem 2.5([1]) *If G' is subgraph of G, then $\gamma(G') \leq \gamma(G)$.*

Theorem 2.6([3]) *The set-indexing number of the Heawood Graph is 5.*

Theorem 2.7([2]) *The set-indexing number of the Peterson graph is 5.*

Theorem 2.8([13]) *If G is a graph of order n, then $\gamma(G) \geq \gamma(K_{1,n-1})$.*

Theorem 2.9([13]) *$\gamma(K_{1,m}) = n + 1$ if and only if $2^n \leq m \leq 2^{n+1} - 1$.*

Theorem 2.10([15]) *$\gamma(P_m) = n + 1$, where $2^n \leq m \leq 2^{n+1} - 1$.*

Definition 2.11([17]) *The double star graph $ST(m, n)$ is the graph formed by two stars $K_{1,m}$ and $K_{1,n}$ by joining their centers by an edge.*

Theorem 2.12([16]) *For a double star graph $ST(m, n)$ with $|V| = 2^l$, $l \geq 2$,*

$$\gamma(ST(m,n)) = \begin{cases} l & \text{if } m \text{ is even,} \\ l+1 & \text{if } m \text{ is odd.} \end{cases}$$

Theorem 2.13([16]) *$\gamma(ST(m, n)) = l + 1$ if $2^l + 1 \leq |V| \leq 2^{l+1} - 1$; $l \geq 2$.*

Theorem 2.14([1]) *$\gamma(C_5) = 4$.*

Theorem 2.15([1]) *$\gamma(C_6) = 4$.*

Definition 2.16([11]) *The join $K_1 \vee P_{n-1}$ of K_1 and P_{n-1} is called a fan graph and is denoted by F_n.*

Theorem 2.17([15]) $\gamma(F_n) = m + 1$, *where* $m = \lceil log_2 n \rceil$ *and* $n \geq 4$.

Definition 2.18([6]) *The double fan graph is obtained by joining P_n and K_2.*

Theorem 2.19([1]) $\gamma(K_n) = \begin{cases} n-1 & \text{if } 1 \leq n \leq 5 \\ n-2 & \text{if } 6 \leq n \leq 7 \end{cases}$

Theorem 2.20([1]) $\gamma(K_n) = \begin{cases} 6 & \text{if } 8 \leq n \leq 9 \\ 7 & \text{if } 10 \leq n \leq 12 \\ 8 & \text{if } 13 \leq n \leq 15 \end{cases}$

Definition 2.21([10]) *For a graph G, the splitting graph $S'(G)$ is obtained from G by adding for each vertex v of G, a new vertex say v' so that v' is adjacent to every vertex that is adjacent to v.*

Definition 2.22([4]) *An n-sun is a graph that consists of a cycle C_n and an edge terminating in a vertex of degree one attached to each vertex of C_n.*

Definition 2.23([8]) *The wheel graph with n spokes, W_n, is the graph that consists of an n-cycle and one additional vertex, say u, that is adjacent to all the vertices of the cycle.*

Definition 2.24([11]) *The helm graph H_n is the graph obtained from a wheel $W_n = C_n \vee K_1$ by attaching a pendant edge at each vertex of C_n.*

Definition 2.25([10]) *The twing is a graph obtained from a path by attaching exactly two pendant edges to each internal vertex of the path.*

Definition 2.26([3]) *The triangular book is the graph $K_2 \vee N_m$, where N_m is the null graph of order m.*

Definition 2.27([6]) *The Gear graph is obtained from the wheel by adding a vertex between every pair of adjacent vertices of the cycle.*

Definition 2.28([10]) *Embedding is a mapping ζ of the vertices of G into the set of vertices of a graph H such that the subgraph induced by the set $\{\zeta(u) : u \in V(G)\}$ is isomorphic to G; for all practical purposes, we shall assume then that G is indeed a subgraph of H.*

Definition 2.29([1]) *A graph G is said to be set-graceful if $\gamma(G) = log_2(|E| + 1)$ and the corresponding optimal set-indexer is called a set-graceful labeling of G.*

Theorem 2.30([9]) *Every cycle C_{2^n-1}, $n \geq 2$ is set-graceful.*

Theorem 2.31([10]) *$K_{3,5}$ is not set-graceful.*

Theorem 2.32([15]) $\gamma(H_{2^n-1}) = n + 2$ *for* $n \geq 2$.

Theorem 2.33([1]) *The star $K_{1,2^n-1}$ is set-graceful.*

§3. Certain Properties of Set-Semigraceful Graphs

In this section we derive certain properties of set-semigraceful graphs and obtain certain bounds for the order and size of such graphs.

Definition 3.1([1]) *A graph G is said to be set-semigraceful if $\gamma(G) = \lceil log_2(|E|+1)\rceil$.*

Remark 3.2 (1) The Heawood graph is set-semigraceful by Theorem 2.6.

(2) The stars are set-semigraceful by Theorem 2.9.

(3) The Paths $P_n;\ n \neq 2^m,\ m \geq 2$ are set-semigraceful by Theorem 2.10.

(4) K_n is set-semigraceful if $n \in \{1, \cdots, 7, 9, 12\}$ by Theorems 2.19 and 2.20.

(5) The Double Stars $ST(m,n);\ m+n \neq 2^l$, m is odd are set-semigraceful by Theorems 2.12 and 2.13.

(6) The helm graph H_{2^n-1} is set-semigraceful by Theorem 2.32.

(7) All set graceful graphs are set-semigraceful.

(8) The path P_7 is set-semigraceful but it is not set-graceful.

(9) A graph G of size 2^m-1 is set-semigraceful if and only if it is set-graceful.

(10) Not all graphs of size 2^m-1 is set-semigraceful. For example $K_{3,5}$ is not set-semigraceful by Theorem 2.31.

The following theorem is an immediate consequence of the above definition.

Theorem 3.3 *Let G be a (p,q)-graph with $\gamma(G) = m$. Then G is set-semigraceful if and only if $2^{m-1} \leq q \leq 2^m - 1$.*

Corollary 3.4 *Let G' be a (p,q') spanning subgraph of a set-semigraceful (p,q)-graph G with $q' \geq 2^{\gamma(G)-1}$. Then G' is set-semigraceful.*

Proof The proof follows from Theorems 2.4 and 3.3. □

Remark 3.5 (1) The Peterson graph is not set-semigraceful by Theorem 2.7.

(2) The paths $P_{2^m};\ m \geq 2$ are not set-semigraceful by Theorem 2.10.

(3) The double stars $ST(m,n);\ m+n = 2^l$, m is odd are not set-semigraceful by Theorem 2.12.

(4) K_n is not set-semigraceful if $n \in \{8, 10, 11, 13, 14, 15\}$ by Theorem 2.20.

Corollary 3.6 *Let T be a set-semigraceful tree of order p, then $2^{\gamma(T)-1} + 1 \leq p \leq 2^{\gamma(T)}$.*

While Theorem 3.3 gives bounds for the size of a set-semigraceful graph in terms of the set-indexing number, the following one gives the same in terms of its order.

Theorem 3.7 *Let G be a set-semigraceful (p,q)-graph. Then*

$$2^{\lceil log_2 p\rceil -1} \leq q \leq 2^{\left\lceil log_2(\frac{p(p-1)}{2}+1)\right\rceil} - 1.$$

Proof By Theorems 2.4, 2.5 and 2.8, we have

$$\lceil log_2 p\rceil \leq \gamma(K_{1,p-1}) \leq \gamma(G) = \lceil log_2(q+1)\rceil \leq \left\lceil log_2(\frac{p(p-1)}{2}+1)\right\rceil.$$

Now by Theorem 3.3 we have

$$2^{\lceil log_2 p\rceil -1} \leq q \leq 2^{\left\lceil log_2(\frac{p(p-1)}{2}+1)\right\rceil} - 1. \qquad \square$$

Remark 3.8 The converse of Theorem 3.7 is not always true. By Theorem 2.14 we have, $\lceil log_2(|E(C_5)|+1)\rceil = \lceil log_2 6\rceil = 3 < \gamma(C_5) = 4$. But C_5 is not set-semigraceful even if $2^2 \leq |E(C_5)| \leq 2^4 - 1$ holds. Further as a consequence of the above theorem we have the graphs $C_6 \cup 3K_1$, $C_5 \cup 4K_1$ and $C_5 \cup 2K_2$ are not set-semigraceful.

Remark 3.9 By Theorem 2.5, for any subgraph G' of G, $\gamma(G') \leq \gamma(G)$. But subgraphs of a set-semigraceful graph need not be set-semigraceful. For example K_6 is set-semigraceful but the spanning subgraph C_6 of K_6 is not set-semigraceful, by Theorem 2.19.

In fact the result given by Theorem 3.3 holds for any set-semigraceful graph as we see in the following.

Theorem 3.10 *Every connected set-semigraceful $(p, p-1)$-graph is a tree such that*

$$2^{m-1} + 1 \leq p \leq 2^m$$

and for every m, such a tree always exists.

Proof Clearly every connected $(p, p-1)$ graph T is a tree and by Theorem 3.3 we have $2^{m-1} + 1 \leq p \leq 2^m$ if T is set-semigraceful. On the other hand, for a given m, the star graph $K_{1,2^m-1}$ is set-semigraceful. $\square$

Theorem 3.11 *If the complete graph K_n; $n \geq 2$ is set-semigraceful then*

$$2m - 1 \leq \gamma(K_n) \leq 2m + 1, \quad where, \ m = \lfloor log_2 n\rfloor.$$

Proof If K_n is set-semigraceful then

$$\begin{aligned}\left\lceil log_2 \frac{n(n-1)}{2} + 1\right\rceil &= \gamma(K_n) \geq \left\lceil log_2 \frac{n(n-1)}{2}\right\rceil \\ &= \lceil log_2 n + log_2(n-1) - log_2 2\rceil \\ &= \lceil log_2 n + log_2(n-1) - 1\rceil.\end{aligned}$$

For any n, there exists m such that $2^m \leq n \leq 2^{m+1} - 1$ so that from above $\gamma(K_n) \geq 2m - 1$. But we have

$$\gamma(K_n) = \left\lceil log_2 \frac{n(n-1)}{2} + 1\right\rceil = \left\lceil log_2 \frac{n(n-1)+2}{2}\right\rceil \leq 2m + 1.$$

Thus we have $2m-1 \le \gamma(K_n) \le 2m+1$; $m = \lfloor log_2 n \rfloor$. □

Remark 3.12 The converse of Theorem 3.11 is not always true. For example, by Theorem 2.20 we have

$$2\lfloor log_2 n \rfloor - 1 \le \gamma(K_8) \le 2\lfloor log_2 n \rfloor + 1.$$

But the complete graph K_8 is not set-semigraceful. Also by Theorems 2.20 and 3.11 we have K_{13}, K_{14} and K_{15} are not set-semigraceful.

Theorem 3.13 *If a (p,q)-graph G has a set-semigraceful labeling with respect to a set X of cardinality $m \ge 2$, there exists a partition of the vertex set $V(G)$ into two nonempty sets V_1 and V_2 such that the number of edges joining the vertices of V_1 with those of V_2 is at most 2^{m-1}.*

Proof Let $f : V \cup E \to 2^X$ be a set-semigraceful labeling of G with indexing set X of cardinality m. Let $V_1 = \{u \in V : |f(u)| \text{ is odd}\}$ and $V_2 = V - V_1$. We have $|A \oplus B| = |A| + |B| - 2|A \cap B|$ for any two subsets A, B of X and hence $|A \oplus B| = 1(\text{mod } 2) \Leftrightarrow A$ and B does not belong to the same set V_i; $i = 1, 2$. Therefore all odd cardinality subsets of X in $f(E)$ must appear on edges joining V_1 and V_2. Consequently there exists at most 2^{m-1} edges between V_1 and V_2. □

Remark 3.14 In 1986, B.D.Acharya [1] conjectured that the cycle C_{2^n-1}; $n \ge 2$ is set-graceful and in 1989, Mollard and Payan [9] settled this in the affirmative. The idea of their proof is the following:

Consider the field $GF(2^n)$ constructed by a binary primitive polynomial say $p(x)$ of degree n. Let α be a root of $p(x)$ in $GF(2^n)$. Then $GF(2^n) = \{0, 1, \alpha, \alpha^2, \ldots, \alpha^{2^n-2}\}$. Now by assigning $\alpha^{i-1} \text{mod } p(\alpha)$, $1 \le i \le 2^n-1$, to the vertices v_i of the cycle $C_{2^n-1} = (v_1, \ldots, v_{2^n-1}, v_1)$ we get a set-graceful labeling of C_{2^n-1} with indexing set $X = \{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}$. Note that here $\alpha^j \text{mod } p(\alpha) = a_0\alpha^0 + a_1\alpha^1 + \ldots + a_{n-1}\alpha^{n-1}$; $a_i = 0$ or 1 for $0 \le i \le n-1$ with $\alpha^0 = 1$ and we identify it as $\{a_i\alpha^i /\ a_i = 1;\ 0 \le i \le n-1\}$ which is a subset of X.

Theorem 3.15 *All cycles C_k with $2^n - 1 \le k \le 2^n + 2^{n-1} - 2$, $n \ge 3$ are set-semigraceful.*

Proof The cycle $C_{2^n-1} = (v_1, v_2, \cdots, v_{2^n-1}, v_1)$ has a set-graceful labeling f as described in the above Remark 3.14. Take $l = 2^{n-1} - 1$ new vertices $u_1, \cdots, u_l$ and form the cycle $C_{2^n+l-1} = (v_1, u_1, v_2, v_3, u_2, v_4, v_5, u_3, v_6, v_7, \cdots, v_{2^n-3}, u_l, v_{2^n-2}, v_{2^n-1}, v_1)$. Now define a set-indexer g of C_{2^n+l-1} with indexing set $Y = X \cup \{x\}$ as follows: $g(v_i) = f(v_i)$; $1 \le i \le 2^n - 1$ and $g(u_j) = f(v_{2j-1}, v_{2j}) \cup \{x\}$; $1 \le j \le l$. Then by Theorem 2.4 we have $\gamma(C_{2^n+l-1}) = n+1$. Now by removing the vertices u_j; $2 \le j \le l$ and joining $v_{2j-1}v_{2j}$ in succession we get the cycles C_{2^n+l-2}, $C_{2^n+l-3}, \ldots, C_{2^n}$. Clearly g induces optimal set-indexers for these cycles by Theorem 2.4 and $\gamma(C_k) = n+1$; $2^n \le k \le 2^n + l - 2$ so that these cycles are set-semigraceful. □

Theorem 3.16 *The set-indexing number of the twing graph obtained from P_{2^n-1} is $n+2$; $n \ge 3$ and hence it is set-semigraceful.*

Proof Let $P_{2^n-1} = (v_1, \ldots, v_{2^n-1})$. By Theorem 2.10 we have $\gamma(P_{2^n-1}) = n$. Let f be an optimal set-indexer of P_{2^n-1} with indexing set X. Let T be a twing graph obtained from P_{2^n-1} by joining each vertex v_i; $i \in \{2, 3, \cdots, 2^n-2\}$ of P_{2^n-1} to two new vertices say u_i and w_i by pendant edges. Consider the set-indexer g of T with indexing set $Y = X \cup \{x, y\}$ defined as follows: $g(v) = f(v)$ for all $v \in V(P_{2^n-1})$, $g(u_i) = f(u_{i-1}) \cup \{x\}$ and $g(w_i) = f(u_{i-1}) \cup \{y\}$; $2 \leq i \leq 2^n - 2$. Consequently $\gamma(T) \leq n + 2$. But by Theorem 2.4 we have

$$\lceil log_2(|E(T)| + 1)\rceil = \lceil log_2(2^n - 2 + 2^n - 3 + 2^n - 3 + 1)\rceil = n + 2 \leq \gamma(T).$$

Thus T is set-semigraceful. □

§4. Construction of Set-Semigraceful Graphs

In this section we construct more set-semigraceful graphs from given ones through various graph theoretic methods.

Theorem 4.1 *Every set-semigraceful (p, q)-graph G with $\gamma(G) = m$ can be embedded in a set-semigraceful $(2^m, q)$-graph.*

Proof Let f be a set-semigraceful labeling of G with indexing set X of cardinality $m = \gamma(G)$. Now add $2^m - p$ isolated vertices to G and assign the unassigned subsets of X under f to these vertices in a one to one manner. Clearly the resulting graph is set-semigraceful. □

Theorem 4.2 *A graph G is set-semigraceful with $\gamma(G) = m$, then every subgraph H of G with $2^{m-1} \leq |E(H)| \leq 2^m - 1$ is also set-semigraceful.*

Proof Since every set-indexer of G is a set-indexer of H, the result follows from Theorem 2.4. □

Corollary 4.3 *All subgraphs G of the star $K_{1,n}$ is set-semigraceful with the same set-indexing number m if and only if $2^{m-1} \leq |E(G)| \leq 2^m - 1$.*

Proof The proof follows from Theorems 4.2 and 2.9. □

Theorem 4.4 *If a $(p, p-1)$-graph G is set-semigraceful, then $G \vee N_{2^n-1}$ is set-semigraceful.*

Proof Let G be set-semigraceful with set-indexing number m. By Theorem 2.4 we have

$$\begin{aligned}\gamma(G \vee N_{2^n-1}) &\geq \lceil log_2(|E(G \vee N_{2^n-1})| + 1)\rceil \\ &= \lceil log_2(p - 1 + p(2^n - 1) + 1)\rceil \\ &= \lceil log_2 p(2^n)\rceil = \lceil log_2(2^n) + log_2(p)\rceil \geq n + m.\end{aligned}$$

Let f be a set-semigraceful labeling of G with indexing set X of cardinality m. Consider the set $Y = \{y_1, \ldots, y_n\}$ and let $V(N_{2^n-1}) = \{v_1, \ldots, v_n\}$. We can find a set-semigraceful labeling say g of $G \vee N_{2^n-1}$ with indexing set $X \cup Y$ as follows: $g(u) = f(u)$ for all $u \in V(G)$ and assign the distinct nonempty subsets of Y to the vertices $v_1, \ldots, v_n$ in any order. Thus $G \vee N_{2^n-1}$ is set-semigraceful. □

Remark 4.5 The converse of Theorem 4.4 is not true in general. For example consider the wheel graph $W_5 = C_5 \vee K_1 = (u_1, u_2, \ldots u_5, u_1) \vee \{u\}$. Now assigning the subsets $\{a\}$, $\{a,b\}$, $\{a,b,c\}$, $\{a,d\}$, $\{a,b,c,d\}$ and ϕ of the set $X = \{a,b,c,d\}$ to the vertices $u_1,\ldots,u_5$ and u in that order we get W_5 as set-semigraceful whereas C_5 is not set-semigraceful by Theorem 2.14.

Corollary 4.6 *The triangular book $K_2 \vee N_{2^n-1}$ is set-semigraceful.*

Proof The proof follows from Theorems 4.4 and 2.10. □

Theorem 4.7 *The fan graph F_n is set-semigraceful if and only if $n \neq 2^m + 1$; $n \geq 4$.*

Proof If $n-1 \neq 2^m$, by Theorem 2.10 we have P_{n-1} is set-semigraceful so that by Theorem 4.4, $F_n = P_{n-1} \vee K_1$ is set-semigraceful.

Conversely if F_n is set-semigraceful, then by Theorem 3.3, we have

$$\gamma(F_n) = \lceil log_2(n-1+n-2+1)\rceil = \lceil log_2(2n-2)\rceil = \lceil log_2(n-1)\rceil + 1.$$

But by Theorem 2.17 we already have $\gamma(F_n) = \lceil log_2 n\rceil + 1$. Consequently we must have $n \neq 2^m + 1$. □

Theorem 4.8 *Every graph can be embedded as an induced subgraph of a connected set-semigraceful graph.*

Proof Let $\{v_1, \ldots, v_n\}$ be the vertex set of the given graph G. Now take a new vertex say u and join it with all the vertices of G. Consider the set $X = \{x_1, \ldots, x_n\}$. Let $m = 2^n - (|E| + n) - 1$. Take m new vertices $u_1,\ldots,u_m$ and join them with u. A set-indexer of the resulting graph G' can be defined as follows: Assign ϕ to u and $\{x_i\}$ to v_i; $1 \leq i \leq n$. Let $S = \{f(e) : e \in E\} \cup \{\{x_i\} : 1 \leq i \leq n\}$. Note that $|S| = |E| + n$. Now by assigning the m elements of $2^X - (S \cup \phi)$ to the vertices $u_1,\ldots,u_m$ in any order we get a set-indexer of G' with X as the indexing set, making G' set-semigraceful. □

Theorem 4.9 *The splitting graph $S'(G)$ of a set-semigraceful bipartite (p,q)-graph G with $\gamma(G) = m$ and $3q \geq 2^{m+1}$, is set-semigraceful.*

Proof Let f be an optimal set-indexer of G with indexing set X of cardinality m. Let $V_1 = \{v_1, \ldots, v_n\}$ and $V_2 = \{u_1, \ldots, u_l\}$ be the partition of $V(G)$, where $n = p - k$. Since G is set-semigraceful with $\gamma(G) = m$, by Theorem 3.3 we have $2^{m-1} \leq q \leq 2^m - 1$. To form the splitting graph $S'(G)$ of G, for each v_i or u_j in G, add a new vertex v'_i or u'_j and add edges joining v'_i or u'_j to all neighbours of v_i or u_j in G respectively. Since $S'(G)$ has $3q$ edges, by Theorem 2.4 we have

$$\gamma(S'(G)) \geq \lceil log_2(|E(S'(G))| + 1)\rceil = \lceil log_2(3q+1)\rceil \geq \left\lceil log_2(2^{m+1}+1)\right\rceil = m + 2.$$

We can define a set-indexer g of $S'(G)$ with indexing set $Y = X \cup \{x, y\}$ as follows: $g(v) = f(v)$ for all $v \in V(G)$, $g(v'_i) = f(v_i) \cup \{x\}$; $1 \leq i \leq n$ and $g(u'_j) = f(u_j) \cup \{y\}$; $1 \leq j \leq l$. Consequently

$$\gamma(S'(G)) = m + 2 = \lceil log_2(|E(S'(G))| + 1)\rceil$$

and hence $S'(G)$ is set-semigraceful. □

Remark 4.10 (1) Even though C_3 is not bipartite, both C_3 and its splitting graph are set-semigraceful.

(2) Splitting graph of a path P_4 is set-semigraceful. But P_4 is not set-semigraceful.

Theorem 4.11 *For any set-graceful graph G, the graph H; $G \cup K_1 \subset H \subseteq G \vee K_1$ is set-semigraceful.*

Proof Let $m = \gamma(G) = log_2(|E(G)| + 1)$. Then by Theorem 2.4 we have

$$\gamma(H) \geq \lceil log_2(|E(H)| + 1) \rceil \geq \lceil log_2(2^m + 1) \rceil = m + 1.$$

Let f be a set-graceful labeling of G with indexing set X. Now we can extend f to a set-indexer g of $G \vee K_1$ with indexing set $Y = X \cup \{x\}$ of cardinality $m + 1$ as follows: $g(u) = f(u)$ for all $u \in V(G)$ and $g(v) = \{x\}$ where $\{v\} = V(G)$. Clearly $g(e) = f(e)$ for all $e \in E(G)$ and $g(uv) = g(u) \cup \{x\}$ are all distinct. Then by Theorem 2.5 we have

$$\gamma(H) = m + 1 = \lceil log_2(|E(H)| + 1) \rceil .$$ □

Corollary 4.12 *The wheel W_{2^n-1} is set-semigraceful.*

Proof The proof follows from Theorems 4.11 and 2.30. □

Theorem 4.13 *Let G be a set-graceful $(p, p-1)$-graph, then $G \vee N_m$ is set-semigraceful.*

Proof Let G be set-graceful graph with set-indexing number n. For every m, there exists l such that $2^l \leq m \leq 2^{l+1} - 1$. By Theorem 2.4 we have

$$\begin{aligned} \gamma(G \vee N_m) &\geq \lceil log_2(|E(G \vee N_m)| + 1) \rceil \\ &= \lceil log_2(p - 1 + pm + 1) \rceil = \lceil log_2 p(m + 1) \rceil \\ &= \lceil log_2(2^n)(m + 1) \rceil = \lceil log_2(2^n) + log_2(m + 1) \rceil \geq n + l + 1. \end{aligned}$$

Let f be a set-semigraceful labeling of G with X; $|X| = n$ as the indexing set. Consider the set $Y = \{y_1, \ldots, y_{l+1}\}$ and $V(N_m) = \{v_1, \ldots, v_m\}$. Now we can extend f to $G \vee N_m$ by assigning the distinct nonempty subsets of Y to the vertices $v_1, \ldots, v_m$ in that order to get a set-indexer of $G \vee N_m$ with indexing set $X \cup Y$. Hence $G \vee N_m$ is set-semigraceful. □

Corollary 4.14 *$K_{1,2^n-1,m}$ is set-semigraceful.*

Proof The proof follows from Theorems 2.33 and 4.13. □

Theorem 4.15 *Let G be a $(p, p-1)$ set-graceful graph, then $G \vee K_2$ and $G \vee K_3$ are set-semigraceful.*

Proof Let f be a set-graceful labeling of G with indexing set X of cardinality n. By

Theorem 2.4 we have

$$\begin{aligned}\gamma(G \vee K_2) &\geq \lceil log_2(|E(G \vee K_2)|+1)\rceil = \lceil log_2(p-1+2p+1+1)\rceil \\ &= \lceil log_2(3p+1)\rceil = \left\lceil log_2 3 + log_2(p+\frac{1}{3})\right\rceil \\ &= \left\lceil log_2 3 + log_2(2^n+\frac{1}{3})\right\rceil \geq n+2.\end{aligned}$$

Consider the set $Y = X \cup \{x_{n+1}, x_{n+2}\}$ and the set-indexer g of $G \vee K_2$ defined by $g(u) = f(u)$ for all $u \in V(G)$, $g(v_1) = \{x_{n+1}\}$ and $g(v_2) = \{x_{n+2}\}$; $v_1,\ v_2 \in V(K_2)$. Consequently $\gamma(G \vee K_2) = n+1$ so that it is set-semigraceful.

Let $K_3 = (v_1, v_2, v_3, v_1)$. By Theorem 2.4 we have $\gamma(G \vee K_3) \geq \lceil log_2(|E(G \vee K_3)|+1)\rceil$ $=\lceil log_2(p-1+3p+3+1)\rceil = \lceil log_2(4p+3)\rceil = \lceil log_2(4.2^n+3)\rceil = \left\lceil log_2(2^{n+2}+3)\right\rceil \geq n+3$. We can find a set-indexer h of $G \vee K_3$ with indexing set $Z = Y \cup \{x_{n+3}\}$ as follows: Assign $\{x_{n+1}\}$, $\{x_{n+2}\}$ and $\{x_{n+3}\}$ to the vertices of K_3 and $h(u) = f(u)$ for all $u \in V(G)$. Clearly h is a set-semigraceful labeling of $G \vee K_3$. □

Corollary 4.16 *All double fans $P_n \vee K_2$; $n \neq 2^m$, $m \geq 2$ are set-semigraceful.*

Proof The proof follows from Theorems 4.15 and 2.10. □

Corollary 4.17 *The graph $K_{1,2^n-1} \vee K_2$ is set-semigraceful.*

Proof The proof follows from Theorems 4.15 and 2.13. □

Theorem 4.18 *If C_n is set-semigraceful, then the graph $C_n \vee K_2$ is set-semigraceful. Moreover $\gamma(C_n \vee K_2) = m+2$, where $2^m \leq n \leq 2^m + 2^{m-1} - 2$, $n \geq 7$.*

Proof The proof follows from Theorems 3.15 and 4.15. □

Theorem 4.19 *Let G be a set-semigraceful (p,q)-graph with $\gamma(G) = m$. If $p \geq 2^{m-1}$, then $G \vee K_1$ set-semigraceful.*

Proof By Theorem 3.3 we have $2^{m-1} \leq |E(G)| \leq 2^m - 1$. Since $|V| \geq 2^{m-1}$, by Theorem 2.4 we have $\gamma(G \vee K_1) \geq \lceil log_2(|E|+1)\rceil = m+1$. Let f be a set-indexer of G with indexing set X of cardinality $m = \gamma(G)$. Now we can define a set-indexer g of $G \vee K_1$; $V(K_1) = \{v\}$ with indexing set $Y = X \cup \{x\}$ as follows: $g(u) = f(u)$ for all $u \in V(G)$ and $g(v) = Y$. This shows that $G \vee K_1$ is set-semigraceful. □

Corollary 4.20 *If C_m is set-semigraceful, then W_m is also set-semigraceful.*

Proof The proof follows from Theorem 4.19. □

Corollary 4.21 *W_n is set-semigraceful, where $2^m + 1 \leq n \leq 2^m + 2^{m-1} - 1$.*

proof The proof follows from Theorem 3.15 and Corollary 4.20. □

Theorem 4.22 *The gear graph of order $2n+1$ with $2^m - 1 \leq n \leq 2^{m-1} + 2^{m-3}$, $m \geq 3$ is set-semigraceful.*

Proof The proof follows from Theorem 2.5 and Corollary 4.21. □

Theorem 4.23 *Let G be a set-semigraceful hamiltonian (p,q)-graph with $\gamma(G) = m$ and $p \geq 2^{m-1}$. If G' is a graph obtained from G by joining a pendand vertex to each vertex of G, then G' is set-semigraceful.*

Proof Let $C = (v_1, v_2, \cdots, v_n, v_1)$ be a hamiltonian cycle in G. Let f be a set-indexer of G with $\gamma(G) = m$ and X be the corresponding indexing set. Now take n new vertices v_i'; $1 \leq i \leq n$ and let $G' = G \cup \{v_i v_i' / \ 1 \leq i \leq n\}$. By Theorem 2.4 we have $\gamma(G') \geq \lceil log_2(|E(G')| + 1) \rceil = m+1$. We can define a set-indexer g of G' with indexing set $Y = X \cup \{x\}$ as follows: $g(u) = f(u)$ for all $u \in V(G)$, $g(v_i') = f(v_i v_{i+1}) \cup \{x\}$; $1 \leq i \leq n$ with $v_{n+1} = v_1$. Clearly G' is set-semigraceful. □

Corollary 4.24 *If C_n is set-semigraceful, then the sun-graph obtained from C_n is set-semigraceful.*

Proof The proof follows from Theorem 4.23. □

Corollary 4.25 *The sun-graph of order $2n$; $2^m \leq n \leq 2^m + 2^{m-1} - 2$; $m \geq 3$ is set-semigraceful.*

Proof The proof follows from Theorem 3.15 and Corollary 4.24. □

References

[1] B.D.Acharya, Set valuations of graphs and their applications, *Proc. Sympos. on Optimization, Design of Experiments and Graph Theory*, Indian Institute of Technology Bombay, 1986, 231-238.

[2] B.D.Acharya, Set indexers of a graph and set graceful graphs, *Bull. Allahabad Math. Soc.*, 16 (2001), 1-23.

[3] B.D.Acharya, S.Arumugam and A.Rosa, *Labelings of Discrete Structures and Applications*, Narosa Publishing House, New Delhi, 2008.

[4] R.Anitha and R.S.Lekshmi, N-sun decomposition of complete, complete-bipartite and some Harary graphs, *International Journal of Computational and Mathematical Sciences*, 2(2008), 33-38.

[5] G.Chartrand and P.Zhang, *Introduction to Graph Theory*, Tata Mcgraw Hill, New Delhi, 2005.

[6] J.A.Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, 13 (2010).

[7] S.M.Hegde, On set valuations of graphs, *Nat. Acad. Sci. Letteres*, 14 (1991), 181-182.

[8] A.Kirlangic, The repture degree and gear graphs, *Bull. Malays. Math. Sci. Soc.*, 32 (2009), 31-36.

[9] M.Mollard and C.Payan, On two conjectures about set-graceful graphs, *European J. Com-*

binatorics 10 (1989), 185-187.

[10] K.L.Princy, *Some Studies on Set Valuations of Graphs-Embedding and NP Completeness*, Ph.D. Thesis, Kannur University, 2007.

[11] W.C.Shiu and P.C.B.Lam, Super-edge-graceful labelings of multi-level wheel graphs, fan graphs and actinia graphs, *Congr. Numerantium*, 174 (2005), 49-63.

[12] U.Thomas and S.C.Mathew, On set indexers of multi-partite graphs, *STARS: Int. Journal (Sciences)*, 3 (2009), 1-10.

[13] U.Thomas and S.C.Mathew, On set indexer number of complete k-partite graphs, *Int. J. Math. Computation*, 5 (2009), 18-28.

[14] U.Thomas and S.C.Mathew, On topological set indexers of graphs, *Advances and Applications in Discrete Mathematics*, 5 (2010), 115-130.

[15] U. Thomas and S. C. Mathew, On set indexers of paths, cycles and certain related graphs, *Discrete Mathematics, Algorithms and Applications* (Accepted).

[16] U.Thomas and S.C.Mathew, Topologically set graceful stars, paths and related graphs, *South Asian Journal of Mathematics* (Accepted).

[17] T.M.Wang and C.C.Hsiao, New constructions of antimagic graph labeling, *Proc. 24th Workshop on Combinatorial Mathematics and Computation Theory*, National Chi Nan University Taiwan, 2007, 267-272.