

On Pathos Total Semitotal and Entire Total Block Graph of a Tree

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Abstract: In this communication, the concept of pathos total semitotal and entire total block graph of a tree is introduced. Its study is concentrated only on trees. We present a characterization of graphs whose pathos total semitotal block graphs are planar, maximal outerplanar, minimally nonouterplanar, nonminimally nonouterplanar, noneulerian and hamiltonian. Also, we present a characterization of those graphs whose pathos entire total block graphs are planar, maximal outerplanar, minimally nonouterplanar, nonminimally nonouterplanar, noneulerian, hamiltonian and graphs with crossing number one.

Key Words: Pathos, path number, Smarandachely block graph, semitotal block graph, total block graph, pathos total semitotal block graph, pathos entire total block graph, pathos length, pathos point, inner point number.

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§1. Introduction

The concept of pathos of a graph G was introduced by Harary [2], as a collection of minimum number of line disjoint open paths whose union is G . The path number of a graph G is the number of paths in a pathos. A new concept of a graph valued functions called the semitotal and total block graph of a graph was introduced by Kulli [5]. For a graph $G(p, q)$ if $B = u_1, u_2, u_3, \dots, u_r; r \geq 2$ is a block of G . Then we say that point u_1 and block B are incident with each other, as are u_2 and B and soon. If two distinct blocks B_1 and B_2 are incident with a common cut point, then they are called adjacent blocks. The points and blocks of a graph are called its members. A *Smarandachely block graph* $T_S^V(G)$ for a subset $V \subset V(G)$ is such a graph with vertices $V \cup \mathcal{B}$ in which two points are adjacent if and only if the corresponding members of G are adjacent in $\langle V \rangle_G$ or incident in G , where $\mathcal{B}$ is the set of blocks of G . The semitotal block graph of a graph G denoted $T_b(G)$ is defined as the graph whose point set is the union of set of points, set of blocks of G in which two points are adjacent if and only if

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members of G are incident. The total block graph of a graph G denoted by $T_B(G)$ is defined as the graph whose point set is the union of set of points, set of blocks of G in which two points are adjacent if and only if the corresponding members of G are adjacent or incident. Also, a new concept called pathos semitotal and total block graph of a tree has been introduced by Muddebihal [10]. The pathos semitotal graph of a tree T denoted by $P_{T_b}(T)$ is defined as the graph whose point set is the union of set of points, set of blocks and the set of path of pathos of T in which two points are adjacent if and only if the corresponding members of G are incident and the lines lie on the corresponding path P_i of pathos. The pathos total block graph of a tree T denoted by $P_{T_B}(T)$ is defined as the graph whose point set is the union of set of points, set of blocks and the set of path of pathos of T in which two points are adjacent if and only if the corresponding members of G are adjacent or incident and the lines lie on the corresponding path P_i of pathos. Stanton [11] and Harary [3] have calculated the path number for certain classes of graphs like trees and complete graphs.

All undefined terminology will conform with that in Harary [1]. All graphs considered here are finite, undirected and without loops or multiple lines. The pathos total semitotal block graph of a tree T denoted by is defined as the graph whose point set is the union of set of points and set of blocks of T and the path of pathos of T in which two points are adjacent if and only if the corresponding members of T are incident and the lines lie on the corresponding path P_i of pathos. The pathos entire total block graph of a tree denoted by is defined as the graph whose set of points is the union of set of points, set of blocks and the path of pathos of T in which two points are adjacent if and only if the corresponding members of T are adjacent or incident and the lines lie on the corresponding path P_i of pathos. Since the system of pathos for T is not unique, the corresponding pathos total semitotal block graph and pathos entire total block graphs are also not unique.

In Figure 1, a tree T and its semi total block graph $T_b(T)$ and their pathos total semitotal block graph are shown. In Figure 2, a tree T and its total block graph $T_B(T)$ and their pathos entire total block graphs are shown.

The line degree of a line uv in T , pathos length in T , pathos point in T was defined by Muddebihal [9]. If G is planar, the inner point number $i(G)$ of G is the minimum number of points not belonging to the boundary of the exterior region in any embedding of G in the plane. A graph G is said to be minimally nonouterplanar if $i(G) = 1$ as was given by Kulli [4].

We need the following results for our further results.

Theorem A([10]) *For any non-trivial tree T , the pathos semitotal block graph $P_{T_b}(T)$ of a tree T , whose points have degree d_i , then the number of points are $(2q + k + 1)$ and the number of lines are $\left(2q + 2 + \frac{1}{2} \sum_{i=1}^p d_i^2\right)$, where k is the path number.*

Theorem B([10]) *For any non-trivial tree whose points have degree d_i , the number of points and lines in total block graph $T_B(T)$ of a tree T are $(2q + 1)$ and $\left(2q + \frac{1}{2} \sum_{i=1}^p d_i^2\right)$.*

Theorem C([10]) *For any non-trivial tree T , the pathos total block graph $P_{T_B}(T)$ of a tree*

T , whose points have degree d_i , then the number of points in $P_{TB}(T)$ are $(2q + k + 1)$ and the number of lines are $\left(q + 2 + \sum_{i=1}^p d_i^2\right)$, where k is the path number.

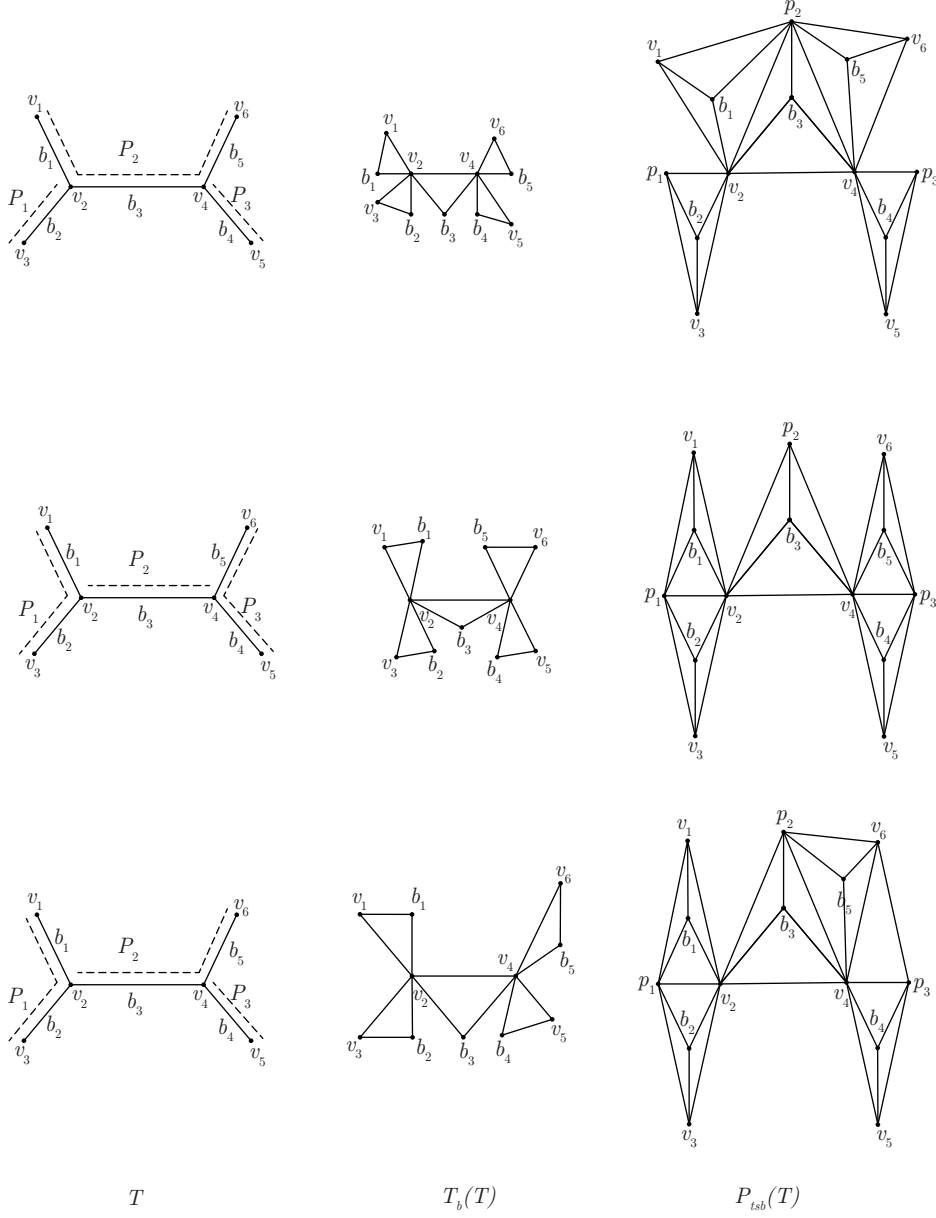


Figure 1

Theorem D ([7]) *The total block graph $T_B(G)$ of a graph G is planar if and only if G is outerplanar and every cut point of G lies on at most three blocks.*

Theorem E ([6]) *The total block graph $T_B(G)$ of a connected graph G is minimally nonouter-*

planar if and only if,

- (1) G is a cycle, or
- (2) G is a path of length $n \geq 2$, together with a point which is adjacent to any two adjacent points of P .

Theorem $F([8])$ The total block graph $T_B(G)$ of a graph G has crossing number one if and only if,

- (1) G is outerplanar and every cut point in G lies on at most 4 blocks and G has a unique cut point which lies on 4 blocks, or
- (2) G is minimally nonouterplanar, every cut point of G lies on at most 3 blocks and exactly one block of G is theta-minimally nonouterplanar.

Corollary $A([1])$ Every non-trivial tree T contains at least two end points.

§2. Pathos Total Semitotal Block Graph of a Tree

We start with a few preliminary results.

Remark 2.1 The number of blocks in pathos total semitotal block graph $P_{tsb}(T)$ of a tree T is equal to the number of pathos in T .

Remark 2.2 If the pathos length of the path P_i of pathos in T is n , then the degree of the corresponding pathos point in $P_{etb}(T)$ is $2n + 1$.

In the following theorem we obtain the number of points and lines in pathos total semitotal block graph $P_{tsb}(T)$ of a tree T .

Theorem 2.1 For any non-trivial tree T , the pathos total semi total block graph $P_{tsb}(T)$ of a tree T , whose points have degree d_i , then the number of points in $P_{tsb}(T)$ are $(2q + k + 1)$ and the number of lines are

$$\left(3q + 2 + \frac{1}{2} \sum_{i=1}^p d_i^2 \right)$$

where k is the path number.

Proof By Theorem A, the number of points in $P_{T_b}(T)$ are $(2q + k + 1)$, and by definition of $P_{tsb}(T)$, the number of points in $P_{tsb}(T)$ are $(2q + k + 1)$, where k is the path number. Also by Theorem A, the number of lines in $P_{T_b}(T)$ are $\left(2q + 2 + \frac{1}{2} \sum_{i=1}^p d_i^2 \right)$. The number of lines in $P_{tsb}(T)$ is equal to the sum of lines in $P_{T_b}(T)$ and the number of lines which lie on the lines (or blocks) of pathos, which are equal to q , since the number of lines are equal to the number of blocks in a tree T . Thus the number of lines in $P_{tsb}(T)$ is equal to

$$\left[q + (2q + 2 + \frac{1}{2} \sum_{i=1}^p d_i^2) \right] = 3q + 2 + \frac{1}{2} \sum_{i=1}^p d_i^2. \quad \square$$

§3. Planar Pathos Total Semitotal Block Graphs

A criterion for pathos total semitotal block graph $P_{tsb}(T)$ of a tree T to be planar is presented in our next theorem.

Theorem 3.1 *For any non-trivial tree T , the pathos total semi total block graph $P_{tsb}(T)$ of a tree T is planar.*

Proof Let T be a non-trivial tree, then in $T_b(T)$ each block is a triangle. We have the following cases.

Case 1 Suppose G is a path, $G = P_n : u_1, u_2, u_3, \dots, u_n, n > 1$. Further, $V[T_b(T)] = \{u_1, u_2, u_3, \dots, u_n, b_1, b_2, b_3, \dots, b_{n-1}\}$, where $b_1, b_2, b_3, \dots, b_{n-1}$ are the corresponding block points. In pathos total semi total block graph $P_{tsb}(T)$ of a tree T , the pathos point w is adjacent to, $\{u_1, u_2, u_3, \dots, u_n, b_1, b_2, b_3, \dots, b_{n-1}\}$. For the pathos total semitotal block graph $P_{tsb}(T)$ of a tree T , $\{u_1b_1u_2w, u_2b_2u_3w, u_3b_3u_4w, \dots, u_{n-1}b_{n-1}u_nw\} \in V[P_{tsb}(T)]$, in which each set $\{u_{n-1}b_{n-1}u_nw\}$ forms an induced subgraph as K_4 . Hence one can easily verify that each induced subgraphs of corresponding set $\{u_{n-1}b_{n-1}u_nw\}$ is planar. Hence $P_{tsb}(T)$ is planar.

Case 2 Suppose G is not a path. Then $V[T_b(G)] = \{u_1, u_2, u_3, \dots, u_n, b_1, b_2, b_3, \dots, b_{n-1}\}$ and $w_1, w_2, w_3, \dots, w_k$ be the pathos points. Since $u_{n-1}u_n$ is a line and $u_{n-1}u_n = b_{n-1} \in V[T_b(G)]$. Then in $P_{tsb}(G)$ the set $\{u_{n-1}, b_{n-1}, u_n, w\} \forall n > 1$, forms K_4 as an induced subgraphs. Hence $P_{tsb}(T)$ is planar. $\square$

The next theorem gives a minimally nonouterplanar $P_{tsb}(T)$.

Theorem 3.2 *For any non-trivial tree T , the pathos total semitotal block graph $P_{tsb}(T)$ of a tree T is minimally nonouterplanar if and only if $T = K_2$.*

Proof Suppose $T = K_3$, and $P_{tsb}(T)$ is minimally nonouterplanar, then $T_b(T) = K_4$ and one can easily verify that $i(P_{tsb}(T)) > 1$, a contradiction.

Suppose $T \neq K_2$. Now assume $T = K_{1,2}$ and $P_{tsb}(T)$ is minimally nonouterplanar. Then $T_b(T) = k_3 \cdot k_3$. Since $K_{1,2}$ has exactly one pathos and let v be a pathos point which is adjacent to all the points of $k_3 \cdot k_3$ in $P_{tsb}(T)$. Then one can easily see that, $i(P_{tsb}(T)) > 1$ a contradiction.

For converse, suppose $T = K_2$, then $T_b(T) = K_3$ and $P_{tsb}(T) = K_4$. Hence $P_{tsb}(T)$ is minimally nonouterplanar. $\square$

From Theorem 3.2, we developed the inner point number of a tree as shown in the following corollary.

Corollary 3.1 *For any non-trivial tree T with q lines, $i(P_{tsb}(T)) = q$.*

Proof The result is obvious for a tree with $q = 1$ and 2. Next we show that the result is true for $q \geq 3$. Now we consider the following cases.

Case 1 Suppose T is a path, $P : v_1, v_2, \dots, v_n$ such that $v_1v_2 = e_1, v_2v_3 = e_2 \dots, v_{n-1}v_n =$

e_{n-1} be the lines of P . Since each $e_i, 1 \leq i \leq n-1$, be a block of P , then in $T_b(P)$, each e_i is a point such that $V[T_b(P)] = V(P) \cup E(P)$. In $T_b(P)$ each $v_1e_1v_2, v_3e_2v_3, \dots, v_{n-1}e_{n-1}v_n$ forms a block in which each block is k_3 . Since each line is a block in P , then the number of k_3 's in $T_b(P)$ is equal to the numbers of lines in P . In $P_{tsb}(P)$, it has exactly one pathos. Then $V[P_{tsb}(P)] = V[T_b(P)] \cup \{P\}$ and P together with each block of $T_b(P)$ forms a block as $P_{tsb}(P)$. Now the points p, v_1, e_1, v_2 forms k_4 as a subgraph of a block $P_{tsb}(P)$. Similarly each $\{v_2, e_2, v_3, p\}, \{v_3, e_3, v_4, p\}, \dots, \{v_{n-1}, e_{n-1}, v_n, p\}$ forms k_4 as a subgraph of a block $P_{tsb}(P)$. One can easily find that each point $e_i, 1 \leq i \leq n-1$ lie in the interior region of k_4 , which implies that $i(P_{tsb}(P)) = q$.

Case 2 Suppose T is not a path, then T has at least one point of degree greater than two. Now assume T has exactly one point v , $\deg v \geq 3$. Then $T = K_{1,n}$. If $P_{tsb}(T)$ has inner point number two which is equal to $n = q$. Similarly if n is odd then $P_{tsb}(T)$ has $n-1$ blocks with inner point number two and exactly one block which is isomorphic to k_4 . Hence $i[P_{tsb}(K_{1,n})] = q$. Further this argument can be extended to a tree with at least two or more points of degree greater two. In each case we have $i[P_{tsb}(T)] = q$. $\square$

In the next theorem, we characterize the noneulerian $P_{tsb}(T)$.

Theorem 3.3 *For any non-trivial tree T , the pathos total semitotal block graph $P_{tsb}(T)$ of a tree T is noneulerian.*

Proof We have the following cases.

Case 1 Suppose $\Delta(T) \leq 2$ and if $p = 2$ points, then $P_{tsb}(T) = K_4$, which is noneulerian. If T is a path with $p > 2$ points. Then in $T_b(T)$ each block is a triangle such that they are in sequence with the vertices of $T_b(T)$ as $\{v_1, b_1, v_2, v_1\}$ an induced subgraph as a triangle in $T_b(T)$. Further $\{v_2, b_2, v_3, v_2\}, \{v_3, b_3, v_4, v_3\}, \dots, \{v_{n-1}, b_n, v_n, v_{n-1}\}$, in which each set form a triangle as an induced subgraph of $T_b(T)$. Clearly one can easily verify that $T_b(T)$ is eulerian. Now this path has exactly one pathos point say k_1 , which is adjacent to $v_1, v_2, v_3, \dots, v_n, b_1, b_2, b_3, \dots, b_{n-1}$ in $P_{tsb}(T)$ in which all the points $v_1, v_2, v_3, \dots, v_n, b_1, b_2, b_3, \dots, b_{n-1} \in P_{tsb}(T)$ are of odd degree. Hence $P_{tsb}(T)$ is noneulerian.

Case 2 Suppose $\Delta(T) \geq 3$. Assume T has a unique point of degree ≥ 3 and also assume that $T = K_{1,n}$. Then in $T_b(T)$ each block is a triangle, such that there are n number of blocks which are K_3 with a common cut point k . Since the degree of a vertex $k = 2n$. One can easily verify that $T_b(K_{1,3})$ is eulerian. To form $P_{tsb}(T)$, $T = K_{1,n}$, the points of degree 2 and the point k are joined by the corresponding pathos point which gives points of odd degree in $P_{tsb}(T)$. Hence $P_{tsb}(T)$ is noneulerian. $\square$

In the next theorem we characterize the hamiltonian $P_{tsb}(T)$.

Theorem 3.4 *For any non-trivial tree T , the pathos semitotal block graph $P_{tsb}(T)$ of a tree T is hamiltonian if and only if T is a path.*

Proof For the necessity, suppose T is a path and has exactly one path of pathos.

Let $V[T_b(T)] = \{u_1, u_2, u_3, \dots, u_n\} \{b_1, b_2, b_3, \dots, b_{n-1}\}$, where $b_1, b_2, b_3, \dots, b_{n-1}$ are

block points of T . Since each block is a triangle and each block consists of points as $B_1 = \{u_1, b_1, u_2\}, B_2 = \{u_2, b_2, u_3\}, \dots, B_m = \{u_m, b_m, u_{m+1}\}$. In $P_{tsb}(T)$ the pathos point w is adjacent to $\{u_1, u_2, u_3, \dots, u_n, b_1, b_2, b_3, \dots, b_{n-1}\}$. Hence $V[P_{tsb}(T)] = \{u_1, u_2, u_3, \dots, u_n\} \cup \{b_1, b_2, b_3, \dots, b_{n-1}\} \cup w$ form a spanning cycle as $w, u_1, b_1, u_2, b_2, u_3, \dots, u_{n-1}, b_{n-1}, u_n, w$ of $P_{tsb}(T)$. Clearly $P_{tsb}(T)$ is hamiltonian. Thus the necessity is proved.

For the sufficiency, suppose $P_{tsb}(T)$ is hamiltonian. Now we consider the following cases.

Case 1 Assume T is a path. Then T has at least one point with $\deg v \geq 3, \forall v \in V(T)$, suppose T has exactly one point u such that degree $u > 2$ and assume $G = T = K_{1,n}$. Now we consider the following subcases of case 1.

Subcase 1.1 For $K_{1,n}, n > 2$ and if n is even, then in $T_b(T)$ each block is k_3 . The number of path of pathos are $\frac{n}{2}$. Since n is even we get $\frac{n}{2}$ blocks in $P_{tsb}(T)$, each block contains two times of $\langle K_4 \rangle$ with some edges common. Since $P_{tsb}(T)$ has a cut points, one can easily verify that there does not exist any hamiltonian cycle, a contradiction.

Subcase 1.2 For $K_{1,n}, n > 2$ and n is odd, then the number of path of pathos are $\frac{n+1}{2}$, since n is odd we get $\frac{n-1}{2} + 1$ blocks in which $\frac{n-1}{2}$ blocks contains two times of $\langle K_4 \rangle$ which is nonline disjoint subgraph of $P_{tsb}(T)$ and remaining blocks is $\langle K_4 \rangle$. Since $P_{tsb}(T)$ contain a cut point, clearly $P_{tsb}(T)$ does not contain a hamiltonian cycle, a contradiction. Hence the sufficient condition. $\square$

§4. Pathos Entire Total Block Graph of a Tree

A tree T , its total block graph $T_B(T)$, and their pathos entire total block graphs $P_{etb}(T)$ are shown in Figure 2. We start with a few preliminary results.

Remark 4.1 If the pathos length of path P_i of pathos in T is n , then the degree of the corresponding pathos point in $P_{etb}(T)$ is $2n + 1$.

Remark 4.2 For any nontrivial tree T , the pathos entire total block graph $P_{etb}(T)$ is a block.

Theorem 4.1 For any non-trivial tree T , the pathos total block graph $P_{etb}(T)$ of a tree T , whose points have degree d_i , then the number of points in $P_{etb}(T)$ are $(2q + k + 1)$ and the number of lines are $\left(2q + 2 + \sum_{i=1}^p d_i^2\right)$, where k is the path number.

Proof By Theorem C, the number of points in $P_{TB}(T)$ are $(2q + k + 1)$, by definition of $P_{etb}(T)$, the number of points in $P_{etb}(T)$ are $(2q + k + 1)$, where k is the path number in T . Also by Theorem B, the number of lines in $T_B(T)$ are $\left(2q + \frac{1}{2} \sum_{i=1}^p d_i^2\right)$. By Theorem C, The number of lines in $P_{TB}(T)$ are $\left(q + 2 + \sum_{i=1}^p d_i^2\right)$. By definition of pathos entire total block graph $P_{etb}(T)$ of a tree equal to the sum of lines in $P_{TB}(T)$ and the number of lines which lie on block points b_i of $T_B(T)$ from the pathos points P_i , which are equal to q . Thus the number

of lines in $P_{etb}(T) = \left(q + 2 + \sum_{i=1}^p d_i^2 \right) = \left(2q + 2 + \sum_{i=1}^p d_i^2 \right)$. □

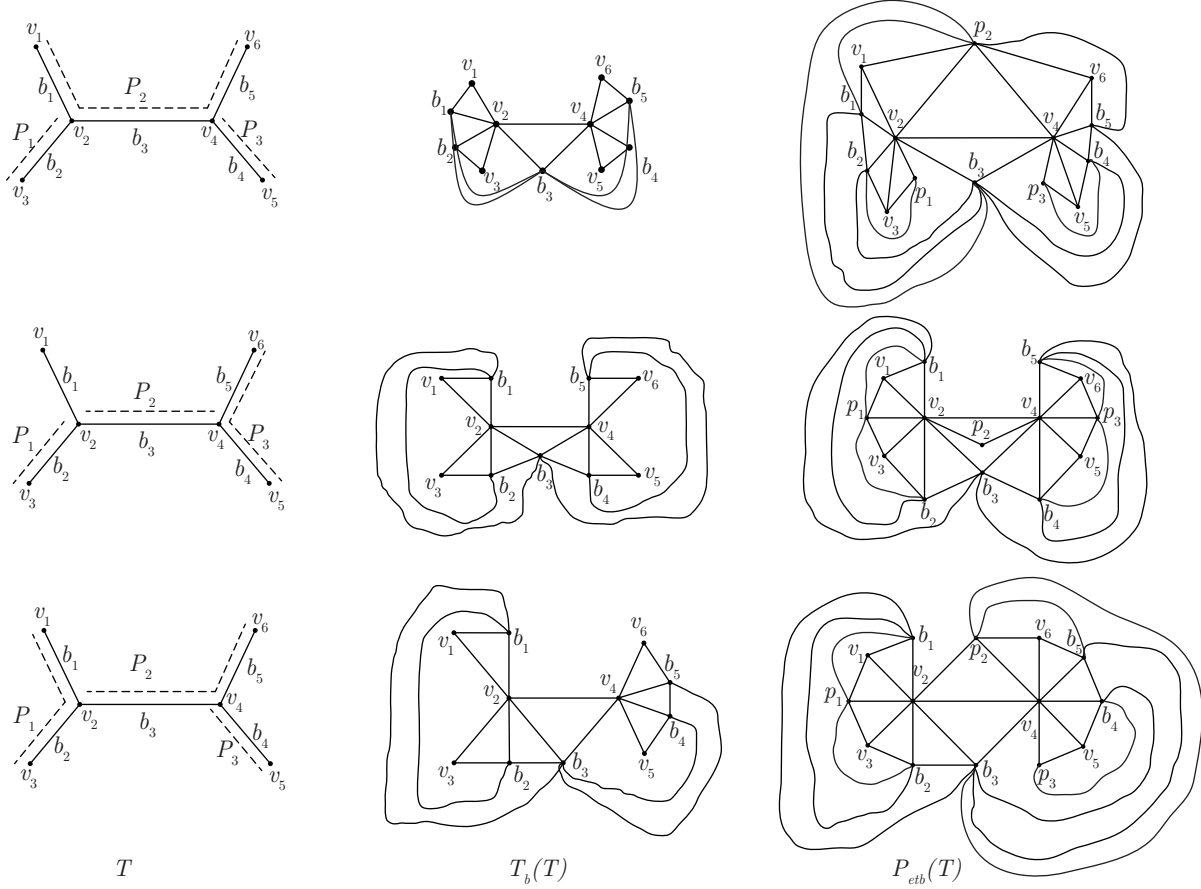


Figure 2

§5. Planar Pathos Entire Total Block Graphs

A criterion for pathos entire total block graph to be planar is presented in our next theorem.

Theorem 5.1 *For any non-trivial tree T , the pathos entire total block graph $P_{etb}(T)$ of a tree T is planar if and only if $\Delta(T) \leq 3$.*

Proof Suppose $P_{etb}(T)$ is planar. Then by Theorem D, each cut point of T lie on at most 3 blocks. Since each block is a line in a tree, now we can consider the degree of cut points instead of number of blocks incident with the cut points. Now suppose if $\Delta(T) \leq 3$, then by Theorem D, $T_b(T)$ is planar. Let $\{b_1, b_2, b_3, \dots, b_{p-1}\}$ be the blocks of T with p points such that $b_1 = e_1, b_2 = e_2, \dots, b_{p-1} = e_{p-1}$ and P_i be the number of pathos of T . Now $V[P_{etb}(T)] = V(G) \cup b_1, b_2, b_3, \dots, b_{p-1} \cup \{P_i\}$. By Theorem D, and by the definition of pathos,

the embedding of $P_{etb}(T)$ in any plane gives a planar $P_{etb}(T)$.

Conversely, Suppose $\Delta(T) \geq 4$ and assume that $P_{etb}(T)$ is planar. Then there exists at least one point of degree 4, assume that there exists a vertex v such that $\deg v = 4$. Then in $T_B(T)$, this point together with the block points form k_5 as an induced subgraph. Further the corresponding pathos point which is adjacent to the $V(T)$ in $T_B(T)$ which gives $P_{etb}(T)$ in which again k_5 as an induced subgraph, a contradiction to the planarity of $P_{etb}(T)$. This completes the proof. $\square$

The following theorem results the minimally nonouterplanar $P_{etb}(T)$.

Theorem 5.2 *For any non-trivial tree T , the pathos entire total block graph $P_{etb}(T)$ of a tree T is minimally nonouterplanar if and only if $T = k_2$.*

Proof Suppose $T = k_3$ and $P_{etb}(T)$ is minimally nonouterplanar. Then $T_B(T) = k_4$ and one can easily verify that, $i(P_{etb}(T)) > 1$, a contradiction. Further we assume that $T = K_{1,2}$ and $P_{etb}(T)$ is minimally outerplanar, then $T_B(T)$ is $W_p - x$, where x is outer line of W_p . Since $K_{1,2}$ has exactly one pathos, this point together with $W_p - x$ gives W_{p+1} . Also in $P_{etb}(T)$ and by definition of $P_{etb}(T)$ there are two more lines joining the pathos points there by giving W_{p+3} . Clearly, $P_{etb}(T)$ is nonminimally nonouterplanar, a contradiction.

For the converse, if $T = k_2$, $T_B(T) = k_3$ and $P_{etb}(T) = K_4$ which is a minimally nonouterplanar. This completes the proof of the theorem. $\square$

Now we have a pathos entire total block graph of a path $p \geq 2$ point as shown in the below remark.

Remark 5.1 For any non-trivial path with $p \geq 2$ points, $i[P_{etb}(T)] = 2p - 3$. The next theorem gives a nonminimally nonouterplanar $P_{etb}(T)$.

Theorem 5.3 *For any non-trivial tree T , the pathos entire total block graph $P_{etb}(T)$ of a tree T is nonminimally nonouterplanar except for $T = k_2$.*

Proof Assume T is not a path. We consider the following cases.

Case 1 Suppose T is a tree with $\Delta(T) \geq 3$. Then there exists at least one point of degree at least 3. Assume v be a point of degree 3. Clearly, $T = K_{1,3}$. Then by the Theorem F, $i[T_B(T)] > 1$. Since $T_B(T)$ is a subgraph of $P_{etb}(T)$. Clearly, $i(P_{etb}(T)) \geq 2$. Hence $P_{etb}(T)$ is nonminimally nonouterplanar.

Case 2 Suppose T is a path with p points and for $p > 2$ points. Then by Remark 5.1, $i[P_{etb}(T)] > 1$. Hence $P_{etb}(T)$ is nonminimally nonouterplanar. $\square$

In the following theorem we characterize the noneulerian $P_{etb}(T)$.

Theorem 5.4 *For any non-trivial tree T , the pathos entire total block graph $P_{etb}(T)$ of a tree T is noneulerian.*

Proof We consider the following cases.

Case 1 Suppose T is a path P_n with n points. Now for $n = 2$ and 3 points as follows. For $p = 2$ points, then $P_{etb}(T) = K_4$, which is noneulerian. For $p = 3$ points, then $P_{etb}(T)$ is a wheel W_6 together with two lines joining the non adjacent points in which one point is common for these two lines as shown in the Figure 3, which is noneulerian.

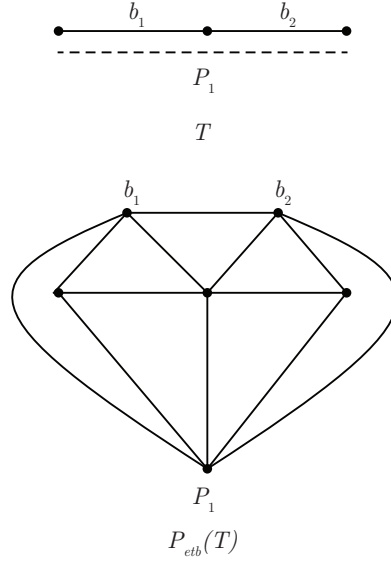


Figure 3

For $p \geq 4$ points, we have a path $P : v_1, v_2, v_3, \dots, v_p$. Now in path each line is a block. Then $v_1v_2 = e_1 = b_1, v_2v_3 = e_2 = b_2, \dots, v_{p-1}v_p = e_{p-1} = b_{p-1}$, $\forall e_{p-1} \in E(G)$, and $\forall b_{p-1} \in V[T_B(P)]$. In $T_B(P)$, the degree of each point is even except b_1 and b_{p-1} . Since the path P has exactly one pathos which is a point of $P_{etb}(P)$ and is adjacent to the points $v_1, v_2, v_3, \dots, v_p$, of $T_B(P)$ which are of even degree, gives as an odd degree points in $P_{etb}(P)$ including odd degree points b_1 and b_{p-1} . Clearly $P_{etb}(P)$ is noneulerian.

Case 2 Suppose T is not a path. We consider the following subcases.

Subcase 2.1 Assume T has a unique point degree ≥ 3 and $T = K_{1,n}$, with n is odd. Then in $T_B(T)$ each block is a triangle such that there are n number of triangles with a common cut points k which has a degree $2n$. Since the degree of each point in $T_B(K_{1,n})$ is odd other than the cut point k which are of degrees either 2 or $n + 1$. Then $P_{etb}(T)$ eulerian. To form $P_{etb}(T)$ where $T = K_{1,n}$, the points of degree 2 and 4 the point k are joined by the corresponding pathos point which gives $(2n + 2)$ points of odd degree in $P_{etb}(K_{1,n})$. $P_{etb}(T)$ has n points of odd degree. Hence $P_{etb}(T)$ noneulerian.

Assume that $T = K_{1,n}$, where n is even, Then in $T_B(T)$ each block is a triangle, which are $2n$ in number with a common cut point k . Since the degree of each point other than k is either 2 or $(n + 1)$ and the degree of the point k is $2n$. One can easily verify that $T_B(K_{1,n})$ is noneulerian. To form $P_{etb}(T)$ where $T = K_{1,n}$, the points of degree 2 and 5 the point k are joined by the corresponding pathos point which gives $(n + 2)$ points of odd degree in $P_{etb}(T)$.

Hence $P_{etb}(T)$ noneulerian.

Subcase 2.2 Assume T has at east two points of degree ≥ 3 . Then $V[T_B(T)] = V(G) \cup b_1, b_2, b_3, \dots, b_p, \forall b_p \in E(G)$. In $T_B(T)$, each endpoint has degree 2 and these points are adjacent to the corresponding pathos points in $P_{etb}(T)$ gives degree 3, From case 1, Tree T has at least 4 points and by Corollary [A], $P_{etb}(T)$ has at least two points of degree 3. Hence $P_{etb}(T)$ is noneulerian. $\square$

In the next theorem we characterize the hamiltonian $P_{etb}(T)$.

Theorem 5.5 *For any non-trivial tree T , the pathos entire total block graph $P_{etb}(T)$ of a tree T is hamiltonian.*

Proof we consider the following cases.

Case 1 Suppose T is a path with $\{u_1, u_2, u_3, \dots, u_n\} \in V(T)$ and $b_1, b_2, b_3, \dots, b_m$ be the number of blocks of T such that $m = n - 1$. Then it has exactly one path of pathos. Now point set of $T_B(T)$, $V[T_B(T)] = \{u_1, u_2, \dots, u_n\} \cup \{b_1, b_2, \dots, b_m\}$. Since given graph is a path then in $T_B(T)$, $b_1 = e_1, b_2 = e_2, \dots, b_m = e_m$, such that $b_1, b_2, b_3, \dots, b_m \subset V[T_B(T)]$. Then by the definition of total block graph, $\{u_1, u_2, \dots, u_{m-1}, u_m\} \cup \{b_1, b_2, \dots, b_{m-1}, b_m\} \cup \{b_1 u_1, b_2 u_2, \dots, b_m u_{n-1}, b_m u_n\}$ form line set of $T_B(T)$ (see Figure 4).

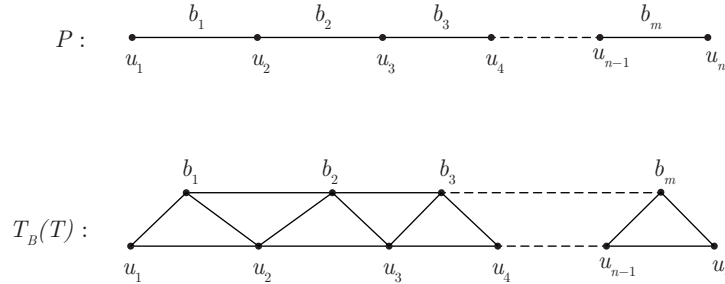


Figure 4

Now this path has exactly one pathos say w . In forming pathos entire total block graph of a path, the pathos w becomes a point, then $V[P_{etb}(T)] = \{u_1, u_2, \dots, u_n\} \cup \{b_1, b_2, \dots, b_m\} \cup \{w\}$ and w is adjacent to all the points $\{u_1, u_2, \dots, u_n\}$ shown in the Figure 5.

In $P_{etb}(T)$, the hamiltonian cycle $w, u_1, b_1, u_2, b_2, u_3, b_3, \dots, u_{n-1}, b_m, u_n, w$ exist. Clearly the pathos entire total block graph of a path is hamiltonian graph.

Case 2 Suppose T is not a path. Then T has at least one point with degree at least 3. Assume that T has exactly one point u such that degree > 2 . Now we consider the following subcases of Case 2.

Subcase 2.1 Assume $T = K_{1,n}, n > 2$ and is odd. Then the number of paths of pathos are $\frac{n+1}{2}$. Let $V[T_B(T)] = \{u_1, u_2, \dots, u_n, b_1, b_2, \dots, b_{m-1}\}$. By the definition of pathos total block graph. By the definition $P_{etb}(T)$ $V[P_{etb}(T)] = \{u_1, u_2, \dots, u_n, b_1, b_2, \dots, b_{n-1}\} \cup \{p_1, p_2, \dots, p_{n+1/2}\}$. Then there exists a cycle containing the points of By the definition of $P_{etb}(T)$ as

$p_1, u_1, b_1, b_2, u_3, p_2, u_2, b_3, u_4, \dots, p_1$ and is a hamiltonian cycle. Hence $P_{etb}(T)$ is a hamiltonian.

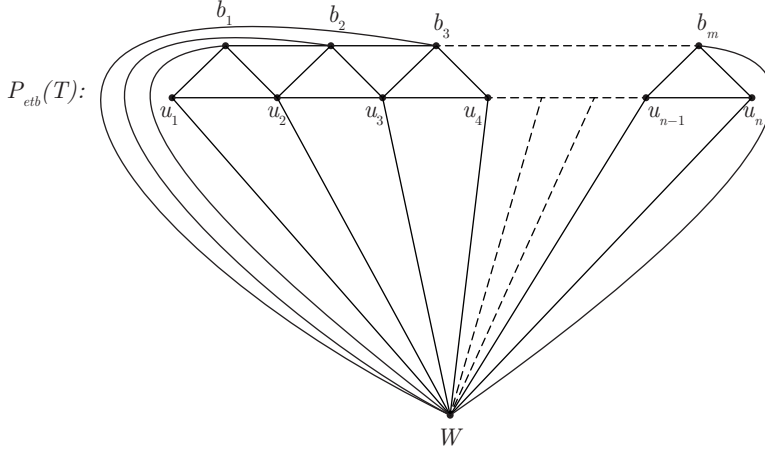


Figure 5

Subcase 2.2 Assume $T = K_{1,n}, n > 2$ and is even. Then the number of path of pathos are $\frac{n}{2}$, then $V[T_B(T)] = \{u_1, u_2, \dots, u_n, b_1, b_2, \dots, b_{n-1}\}$. By the definition of pathos entire total block graph $P_{etb}(T)$ of a tree T . $V[P_{etb}(T)] = \{u_1, u_2, \dots, u_n, b_1, b_2, \dots, b_{n-1}\} \cup \{p_1, p_2, \dots, p_{n/2}\}$. Then there exist a cycle containing the points of $P_{etb}(T)$ as $p_1, u_1, b_1, b_2, u_3, p_2, u_4, b_3, b_4, \dots, p_1$ and is a hamiltonian cycle. Hence $P_{etb}(T)$ is a hamiltonian.

Suppose T is neither a path nor a star, then T contains at least two points of degree > 2 . Let $u_1, u_2, u_3, \dots, u_n$ be the points of degree ≥ 2 and $v_1, v_2, v_3, \dots, v_m$ be the end points of T . Since end block is a line in T , and denoted as $b_1, b_2, \dots, b_k$, then $V[T_B(T)] = \{u_1, u_2, \dots, u_n\} \cup \{v_1, v_2, \dots, v_m\} \cup \{b_1, b_2, \dots, b_k\}$, and tree T has p_i pathos points, $i > 1$ and each pathos point is adjacent to the point of T where the corresponding pathos lie on the points of T . Let $\{p_1, p_2, \dots, p_i\}$ be the pathos points of T . Then there exists a cycle C containing all the points of $P_{etb}(T)$, as $p_1, v_1, b_1, v_2, p_2, u_1, b_3, u_2, p_3, v_3, b_4, v_{m-1}, b_{n-1}, b_n, v_m, \dots, p_1$. Hence $P_{etb}(T)$ is a hamiltonian cycle. Clearly, $P_{etb}(T)$ is a hamiltonian graph. $\square$

In the next theorem we characterize $P_{etb}(T)$ in terms of crossing number one.

Theorem 5.6 For any non-trivial tree T , the pathos entire total block graph $P_{etb}(T)$ of a tree T has crossing number one if and only if $\Delta(T) \leq 4$, and there exist a unique point in T of degree 4.

Proof Suppose $P_{etb}(T)$ has crossing number one. Then it is nonplanar. Then by Theorem 5.1, we have $\Delta(T) \leq 4$. We now consider the following cases.

Case 1 Assume $\Delta(T) = 5$. Then by Theorem [F], $T_B(T)$ is nonplanar with crossing number more than one. Since $T_B(T)$ is a subgraph of $P_{etb}(T)$. Clearly $cr(P_{etb}(T)) > 1$, a contradiction.

Case 2 Assume $\Delta(T) = 4$. Suppose T has two points of degree 4. Then by Theorem F, $T_B(T)$ has crossing number at least two. But $T_B(T)$ is a subgraph of $P_{etb}(T)$. Hence $cr(P_{etb}(T)) > 1$,

a contradiction.

Conversely, suppose T satisfies the given condition and assume T has a unique point v of degree 4. The lines which are blocks in T such that they are the points in $T_B(T)$. In $T_B(T)$, these block points and a point v together forms an induced subgraph as K_5 . In forming $P_{etb}(T)$, the pathos points are adjacent to at least two points of this induced subgraph. Hence in all these cases the $cr(P_{etb}(T)) = 1$. This completes the proof. $\square$

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