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On Generalized m-Power Matrices and Transformations

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Abstract: In this paper, generalized $m-$power matrices and generalized $m-$power transformations are defined and studied. First, we give two equivalent characterizations of generalized $m-$power matrices, and extend the corresponding results about $m-$ idempotent matrices and $m-$unit-ponent matrices. And then, we also generalize the relative results of generalized $m-$power matrices to the ones of generalized $m-$power transformations.

Key Words: Generalized $m-$power matrix, generalized $m-$power transformation, equivalent characterization.

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§1. Introduction

The m-idempotent matrices and $m-$unit-ponent matrices are two typical matrices and have many interesting properties (for example, see [1]-[5]).

A matrix $A \in \mathbb{C}^{n\times n}$ is called an $m-$idempotent ($m-$unit-ponent) matrix if there exists positive integer m such that $A^m = A(A^m = I)$. Notice that

$$A^m = A \text{ if and only if } \prod_{i=1}^{m}(A + \varepsilon_i I) = O,$$

where $\varepsilon_1 = 0$, $\varepsilon_2, \varepsilon_3, \cdots, \varepsilon_m$ are the $m-1$ power unit roots,

$$A^m = I \text{ if and only if } \prod_{i=1}^{m}(A + \varepsilon_i I) = O,$$

where ε_1, $\varepsilon_2, \cdots, \varepsilon_m$ are the $m-$power unit roots. Naturally, we will consider the class of matrices which satisfies that

$$\prod_{i=1}^{m}(A + \lambda_i I) = O,$$

where $\lambda_1, \lambda_2, \cdots, \lambda_m$ are the pairwise different complex numbers.

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For convenience, we call a matrix $A \in \mathbb{C}^{n\times n}$ to be a generalized $m-$power matrix if it satisfies that $\prod_{i=1}^{m}(A+\lambda_i I)=O$, where $\lambda_1,\lambda_2,\cdots,\lambda_m$ are the pairwise different complex numbers.

In this paper, we firstly study the generalized $m-$power matrices, and give two equivalent characterizations of such matrices. Consequently, the corresponding results about $m-$idempotent matrices and $m-$unit-ponent matrices are generalized. And then, we also define the generalized $m-$power transformations, and generalize the relative results of generalized $m-$power matrices to those of generalized $m-$power transformations.

For terminologies and notations occurred but not mentioned in this paper, the readers are referred to the reference [6].

§2. Generalized $m-$Power Matrices

In this section, we are going to study some equivalent characterizations of generalized $m-$power matrices. First, we introduce some lemmas following.

Lemma 2.1([4]) *Let $\lambda_1,\lambda_2,\cdots,\lambda_m$ be the pairwise different complex numbers and $A \in \mathbb{C}^{n\times n}$. Then*

$$r(\prod_{i=1}^{m}(A+\lambda_i I))=\sum_{i=1}^{m} r(A+\lambda_i I)-(m-1)n.$$

Lemma 2.2([4]) *Let $f_1(x),f_2(x),\cdots,f_m(x)\in\mathbb{C}[x]$ be pairwisely co-prime and $A \in \mathbb{C}^{n\times n}$. Then*

$$\sum_{i=1}^{m} r(f_i(A))=(m-1)n+r(\prod_{i=1}^{m}(f_i(A))).$$

Lemma 2.3([1]) *Assume that $f(x),g(x)\in\mathbb{C}[x]$, $d(x)=(f(x),g(x))$ and $m(A)=[f(x),g(x)]$. Then for any $A\in\mathbb{C}^{n\times n}$,*

$$r(f(A))+r(g(A))=r(d(A))+r(m(A)).$$

Theorem 2.4 *Let $\lambda_1,\lambda_2,\cdots,\lambda_m\in\mathbb{C}$ be the pairwise different complex numbers and $A \in \mathbb{C}^{n\times n}$. Then $\prod_{i=1}^{m}(A+\lambda_i I)=O$ if and only if $\sum_{i=1}^{m} r(A+\lambda_i I)=(m-1)n$.*

Proof Assume that $\prod_{i=1}^{m}(A+\lambda_i I)=O$, by Lemma 2.1, we can immediately get $\sum_{i=1}^{m} r(A+\lambda_i I)=(m-1)n$.

Assume that $\sum_{i=1}^{m} r(A+\lambda_i I)=(m-1)n$. Take $f_i(x)=x+\lambda_i (i=1,2,\cdots,m)$, where $\lambda_i\neq\lambda_j$ if $i\neq j$. Clearly, we have $(f_i(x),f_j(x))=1$ if $i\neq j$. Now, by Lemma 2.2,

$$\sum_{i=1}^{m} r(f_i(A))=(m-1)n+r(\prod_{i=1}^{m}(f_i(A))).$$

Also, since $\sum_{i=1}^{m} r(A+\lambda_i I)=(m-1)n$, we can get $r(\prod_{i=1}^{m}(f_i(A)))=0$, this implies that

$$\prod_{i=1}^{m}(A+\lambda_i I)=O. \qquad \square$$

By Theorem 2.4, we can obtain the following conclusions. Consequently, the corresponding result in [3] is generalized.

Corollary 2.5 *Let $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{m-1} \in \mathbb{C}$ be the $m-1$ power unit roots and $A \in \mathbb{C}^{n\times n}$. Then $A^m = A$ if and only if $r(A) + r(A-\varepsilon_1 I) + r(A-\varepsilon_2 I) + \cdots + r(A-\varepsilon_{m-1} I) = (m-1)n$.*

Corollary 2.6 *Let $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m \in \mathbb{C}$ be the m power unit roots and $A \in \mathbb{C}^{n\times n}$. Then $A^m = I$ if and only if $r(A-\varepsilon_1 I) + r(A-\varepsilon_2 I) + \cdots + r(A-\varepsilon_m I) = (m-1)n$.*

Now, we give another equivalent characterizations of the generalized $m-$power matrices.

Theorem 2.7 *Let $\lambda_1, \lambda_2, \cdots, \lambda_m \in \mathbb{C}$ be the pairwise different complex numbers and $A \in \mathbb{C}^{n\times n}$. Then*

$$\prod_{i=1}^{m}(A+\lambda_i I) = O \text{ if and only if } \sum_{1\le i<j\le m} r((A+\lambda_i I)(A+\lambda_j I)) = \frac{(m-2)(m-1)n}{2}.$$

Proof Assume that $\prod_{i=1}^{m}(A+\lambda_i I) = O$. Then $r(\prod_{i=1}^{m}(A+\lambda_i I)) = 0$. Notice that

$$\sum_{1\le i<j\le m} r((A+\lambda_i I)(A+\lambda_j I)) = \sum_{i=1}^{m-1}\sum_{j=i+1}^{m} r((A+\lambda_i I)(A+\lambda_j I))$$

and by Lemmas 2.2 and 2.3, it is not hard to get that

$$\begin{aligned}\sum_{j=2}^{m} r((A+\lambda_1 I)(A+\lambda_j I)) &= (m-2)\cdot r(A+\lambda_1 I) + r(\prod_{i=1}^{m}(A+\lambda_i I)) \\ &= (m-2)\cdot r(A+\lambda_1 I),\end{aligned}$$

$$\begin{aligned}&\sum_{j=3}^{m} r((A+\lambda_2 I)(A+\lambda_j I)) + r(A+\lambda_1 I) = n + (m-3)\cdot r(A+\lambda_2 I),\\ &\sum_{j=4}^{m} r((A+\lambda_3 I)(A+\lambda_j I)) + r(A+\lambda_1 I) + r(A+\lambda_2 I) = 2n + (m-4)\cdot r(A+\lambda_3 I),\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\ &\sum_{j=m-1}^{m} r((A+\lambda_{m-2} I)(A+\lambda_j I)) + \sum_{i=1}^{m-3} r(A+\lambda_i I) = (m-3)\cdot n + r(A+\lambda_{m-2} I),\\ &r((A+\lambda_{m-1} I)(A+\lambda_m I)) + \sum_{i=1}^{m-2} r(A+\lambda_i I) = (m-2)\cdot n.\end{aligned}$$

Thus, we have

$$\sum_{1\le i<j\le m} r((A+\lambda_i I)(A+\lambda_j I)) = \frac{(m-2)(m-1)n}{2}.$$

From the discussions above, we have

$$\sum_{1\le i<j\le m} r((A+\lambda_i I)(A+\lambda_j I)) = \frac{(m-2)(m-1)n}{2} + (m-1)\cdot r(\prod_{i=1}^{m}(A+\lambda_i I)).$$

Hence, if

$$\sum_{1\leq i<j\leq m} r((A+\lambda_i I)(A+\lambda_j I)) = \frac{(m-2)(m-1)n}{2},$$

then

$$r(\prod_{i=1}^{m}(A+\lambda_i I)) = 0,$$

i.e.,

$$\prod_{i=1}^{m}(A+\lambda_i I) = O. \qquad \square$$

By Theorem 2.7, we can get corollaries following. Also, the corresponding result in [3] is generalized.

Corollary 2.8 *Let $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{m-1} \in \mathbb{C}$ be the $m-1$ power unit roots and $A \in \mathbb{C}^{n\times n}$. Then $A^m = A$ if and only if $r(A(A-\varepsilon_1 I)) + \cdots + r(A(A-\varepsilon_{m-1} I)) + r((A-\varepsilon_1 I)(A-\varepsilon_2 I)) + \cdots + r((A-\varepsilon_1 I)(A-\varepsilon_{m-1} I)) + \cdots + r((A-\varepsilon_{m-2} I)(A-\varepsilon_{m-1} I)) = \frac{(m-2)(m-1)n}{2}$.*

Corollary 2.9 *Let $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m \in \mathbb{C}$ be the m power unit roots and $A \in \mathbb{C}^{n\times n}$. Then $A^m = I$ if and only if*

$$\sum_{1\leq i<j\leq m} r((A+\varepsilon_i I)(A+\varepsilon_j I)) = \frac{(m-2)(m-1)n}{2}.$$

§3. Generalized $m-$Power Transformations

In this section, analogous with the discussions of the generalized $m-$power matrices, we will firstly introduce the concepts of generalized $m-$power linear transformations, and then study some of their properties.

Let V be a n dimensional vector space over a field F and σ a linear transformation on V. We call σ to be a generalized $m-$power transformation if it satisfies that

$$\prod_{i=1}^{m}(\sigma+\lambda_i\epsilon) = \theta$$

for pairwise different complex numbers $\lambda_1, \lambda_2, \cdots, \lambda_m$, where ϵ is the identical transformation and θ is the null transformation. Especially, σ is called an $m-$idempotent ($m-$unit-ponent) transformation if it satisfies that $\sigma^m = \sigma (\sigma^m = \epsilon)$.

From [6], it is known that n dimensional vector space V over a field F is isomorphic to F^n and the linear transformation space $L(V)$ is isomorphic to $F^{n\times n}$. Thus, we can obtain the following results about generalized $m-$power transformations whose proofs are similar with the corresponding ones in Section 2. And we omit them here.

Theorem 3.1 *Let V be a n dimensional vector space over a field F and σ a linear transformation on V. Then σ is a generalized $m-$power transformation if and only if*

$$\sum_{i=1}^{m} \dim Im(\sigma+\lambda_i\epsilon) = (m-1)n.$$

By Theorem 3.1, we obtain the following conclusions.

Corollary 3.2 *Let V be a n dimensional vector space over a field F and σ a linear transformation on V. Then σ is an $m-$ idempotent transformation if and only if*

$$\dim Im(A) + \dim Im(A - \varepsilon_1 I) + \dim Im(A - \varepsilon_2 I) + \cdots + \dim Im(A - \varepsilon_{m-1} I) = (m-1)n.$$

Corollary 3.3 *Let V be a n dimensional vector space over a field F and σ a linear transformation on V. Then σ is an $m-$unit-ponent transformation if and only if*

$$\dim Im(A - \varepsilon_1 I) + \dim Im(A - \varepsilon_2 I) + \cdots + \dim Im(A - \varepsilon_m I) = (m-1)n.$$

Corollary 4.4 *Let V be a n dimensional vector space over a field F and σ a linear transformation on V. Then σ is a generalized $m-$power transformation if and only if*

$$\sum_{1 \le i < j \le m} \dim Im((\sigma + \lambda_i \epsilon)(\sigma + \lambda_j \epsilon)) = \frac{(m-2)(m-1)n}{2}.$$

Corollary 3.5 *Let V be a n dimensional vector space over a field F and σ a linear transformation on V. Then σ is an $m-$ idempotent transformation if and only if*

$$\begin{aligned}
&\dim Im(\sigma(\sigma - \varepsilon_1 \epsilon)) + \cdots + \dim Im(\sigma(\sigma - \varepsilon_{m-1} \epsilon)) \\
&+ \dim Im(\sigma - \varepsilon_1 \epsilon)(\sigma - \varepsilon_2 \epsilon)) + \cdots + \dim Im((\sigma - \varepsilon_1 \epsilon)(\sigma - \varepsilon_{m-1} \epsilon)) \\
&+ \cdots + \dim Im((\sigma - \varepsilon_{m-2} \epsilon)(\sigma - \varepsilon_{m-1} \epsilon)) = \frac{(m-2)(m-1)n}{2}.
\end{aligned}$$

Corollary 3.6 *Let V be a n dimensional vector space over a field F and σ a linear transformation on V. Then σ is an $m-$unit-ponent transformation if and only if*

$$\sum_{1 \le i < j \le m} \dim Im((\sigma + \varepsilon_i \epsilon)(\sigma + \varepsilon_j \epsilon)) = \frac{(m-2)(m-1)n}{2}.$$

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