International J.Math. Combin. Vol.2(2012), 52-58

On Folding of Groups

Mohamed Esmail Basher

(Department of Mathematics, Faculty of Science in Suez, Suez Canal University, Egypt)

E-mail: m_e_basher@yahoo.com

Abstract: The aim of our study is to give a definition of the folding of groups and study the folding of some types of groups such as cyclic groups and dihedral groups, also we discussed the folding of direct product of groups. Finally the folding of semigroups are investigated.

Key Words: Folding, multi-semigroup, multi-group, group, commutative semigroup.

AMS(2010): 54C05, 20K01

§1. Introduction

In the last two decades there has been tremendous progress in the theory of folding. The notion of isometric folding is introduced by S. A. Robertson who studied the stratification determined by the folds or singularities [10]. The conditional foldings of manifolds are defined by M. El-Ghoul in [8]. Some applications on the folding of a manifold into it self was introduced by P. Di. Francesco in [9]. Also a folding in the algebra's branch introduced by M.El-Ghoul in [7]. Then the theory of isometric foldings has been pushed and also different types of foldings are introduced by E. El-Kholy and others [1,2,5,6].

Definition 1.1([4]) *A non empty set G on which is defined $s \geq 1$ associative binary operations $*$ is called a multi-semigroup, if for all $a, b \in G, a * b \in G$, particularly, if $s = 1$, such a multi-semigroup is called a semigroup.*

Example 1 $\mathbb{Z}_p = \{0, 1, 2, ..., p-1\}$ is a semigroup under multiplication p, $p \in \mathbb{Z}^+$.

Definition 1.2([4]) *A subset H is a subsemigroup of G if H is closed under the operation of G ; that it if $a * b \in H$ for all $a, b \in H$.*

Definition 1.3 *A multi-group $(G, \mathcal{O})$ is a non empty set G together with a binary operation set $\mathcal{O}$ on G such that for $* \in \mathcal{O}$, the following conditions hold:*
(1) $\forall a, b \in G$ *then* $a * b \in G$.
(2) *There exists an element $e \in G$ such that $a * e = e * a = a$, for all $a \in G$.*
(3) *For $a \in G$ there is an element a^{-1} in G such that $a * a^{-1} = a^{-1} * a = e$.*
Particularly, if $|\mathcal{O}| = 1$, such a multi-group is called a group and denoted by G

[1]Received January 17, 2012. Accepted June 14, 2012.

[2]Current Address: Qassim University,College of Science, Dept. of Math., P. O. Box 6644, Buriedah 51452, Al Qassim, Saudi Arabia

A group G is called *Abelian* if $a * b = b * a$ for all $a, b \in G$. The order of a group is its cardinality, i.e., the number of its elements. We denote the order of a group G by $|G|$.

Definition 1.4([3]) *The group G is called a cyclic group of order n, if there exists an element $g \in G$, such that $G = \langle g \rangle = \{g \mid g^n = 1\}$. In this case g is called a generator of G.*

Definition 1.5([3]) *The dihedral group D_{2n} of order $2n$, is defined in the following equivalent ways: $D_{2n} = \big\{a, b \mid \; a^2 = b^n = 1, bab = a\big\}$.*

Definition 1.6([3]) *A subset H is a subgroup of G ,$H \leq G$, if H is closed under the operation of G ; that it if $a * b \in H$ for all $a, b \in H$.*

Definition 1.7([3]) *The trivial subgroup of any group is the subgroup $\{e\}$, consisting of just the identity element.*

Definition 1.8([3]) *A subgroup of a group G that does not include the entire group itself is known as a proper subgroup, denoted by $H < G$.*

Theorem 1.1 *Every subgroup of a cyclic group is cyclic.*

§2. Group Folding

In this section we give the notions of group folding and discuss the folding of some kinds of groups

Definition 2.1 *Let G_1, G_2 are two groups, The group folding **g.f** of G_1 into G_2 is the map $f : (G_1, *) \to (G_2, \circ)$ st.*

$$\forall a \in G_1,\; f(a) = b \;, f(a^{-1}) = b^{-1}$$

where $b \in G_2$ and $f(G_1)$ is subgroup of G_2.

Definition 2.2 *The set of singularities $\sum f$ is the set of elements $a_i \in G$ such that $f(a_i) = a_i$.*

Definition 2.3 *A group folding is called good group folding **g.g.f** if H is non trivial subgroup of G.*

Proposition 1.1 *The limit of group folding of any group is $\{e\}$.*

Proof Let G be any group and since any group has two trivial subgroups $(G, *)$, $(\{e\}, *)$,where $(\{e\}, *)$ is the minimum subgroup of G. Then if we define a sequence of group folding $f_i : (G, *) \to (G, *)$, such that $f(a_i) = b_i$, $f(a_i^{-1}) = b_i^{-1}$. We found that $\lim\limits_{i \longrightarrow \infty} f_i(G) = \{e\}$. □

Theorem 2.1 *Let G be cyclic group $G = \{g : g^p = 1\}$, where p is prime. Then there is no **g.g.f** map can be defined on G.*

Proof Given G is cyclic group of order p, since p is prime. Then there is no proper subgroup can be found in G. Hence we can not able to defined any **g.g.f** map on group G. □

Theorem 2.2 *Let G be cyclic group $G = \{g : g^p = 1\}$, where $p = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_n^{\alpha_n}$ and $q_1, q_2, \cdots, q_n$ are distinct prime number. Then the limit of* ***g.g.f*** *map of G is the proper subgroup , $H = \left\{ g^{q_i} : (g^{q_i})^{\frac{p}{q_i}} = 1 \right\}$.*

Proof Since G be cyclic group and $|G| = p = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_n^{\alpha_n}$, $q_1, q_2, \cdots, q_n$ are distinct prime number. Then there exist proper subgroup $H < G$ such that $|H| = k$. So we can defined **g.g.f** map $f_1 : G \longrightarrow G$, such that $f_1(G) = H$, after this there exist two cases.

Case 1. If k is prime number then we can not found a proper subgroup of H and so $\lim\limits_{i \longrightarrow \infty} f_i(G) = H$ and $k = q_i, 1 \leq i \leq n$, $f_i(G) = H = \left\{ g^{q_i} : (g^{q_i})^{\frac{p}{q_i}} = 1 \right\}$.

Case 2. If k is not prime. Then $k = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_m^{\alpha_m}$, $m < n$, hence there exist a proper subgroup $\bar{H} < H$, $|\bar{H}| = \bar{k}$. Thus we can defined a **g.g.f** map $f_2 : H \longrightarrow H$, such that $f_2(H) = \bar{H}$. Again if $|\bar{H}|$ is prime. Then $\lim\limits_{i \longrightarrow \infty} f_i(G) = \bar{H}$, $\bar{k} = q_i, 1 \leq i \leq m$, $f_i(G) = \bar{H} = \left\{ g^{q_i} : (g^{q_i})^{\frac{p}{q_i}} = 1 \right\}$ or, if $|\bar{H}| = \bar{k}$ is not prime. Then we can repeat the Case II again. Finally the limit of **g.g.f** of G is the subgroup $H = \left\{ g^{q_i} : (g^{q_i})^{\frac{p}{q_i}} = 1 \right\}$. □

Corollary 2.1 *If $p = q^\alpha$ then the limit of* ***g.g.f*** *map of G is a subgroup $H = \left\{ g^q : (g^q)^{\frac{p}{q}} = 1 \right\}$.*

Example 2 Let $G = \{g : g^{12} = 1\}$ be a cyclic group, $|G| = 12 = 2^2.3$, so we can defined a **g.g.f** map as the following $f_1 : G \longrightarrow G$, such that:

$$
\begin{aligned}
&f_1(1) = 1, f_1(g^2) = g^2, f_1(g^{10}) = g^{10},\\
&f_1(g^4) = g^4, f_1(g^6) = g^6, f_1(g^8) = g^8,\\
&f_1(g^5) = g^2, f_1(g^7) = g^{10},\\
&f_1(g^3) = g^4, f_1(g^9) = g^8,\\
&f_1(g) = g^6, f_1(g^{11}) = g^6
\end{aligned}
$$

and $f_1(G) = H = \{1, g^2, g^4, g^6, g^8, g^{10}\}$, since the order of H is not prime then we can defined a **g.g.f** map as the following $f_2 : H \longrightarrow H$ such that

$$
\begin{aligned}
&f_2(1) = 1, f_2(g^6) = 1\\
&f_2(g^2) = g^4, f_2(g^{10}) = g^8,\\
&f_2(g^4) = g^4, f_2(g^8) = g^8,
\end{aligned}
$$

and $f_2(H) = \bar{H} = \{1, g^4, g^8\}$ is proper subgroup of H. Since the order of $\bar{H}$ is prime, then $\lim\limits_{i \longrightarrow \infty} f_i(G) = \bar{H} = \{g^4 | (g^4)^3 = 1\}$.

Proposition 2.2 *For any dihedral group $D_{2n} = \{a, b \,| a = b^n = 1, bab = 1\}$, we can defined a* ***g.g.f*** *map.*

Theorem 2.3 *Let D_{2n} be a dihedral group ,where $n = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_m^{\alpha_m}$ and $q_1, q_2, \cdots, q_m$ are distinct prime number. Then the limit of* ***g.g.f*** *of D_{2n} is one of these $H_i = \{1, ab^i\}, i = 1, 2, \cdots, n$ or $H_i = \left\{ b^{q_i} | (b^{q_i})^{\frac{n}{q_i}} = 1 \right\}, i = 1, 2, \cdots, m$.*

Proof Let $D_{2n} = \{a, b \mid a = b^n = 1, bab = 1\}$ be a dihedral group of order $2n$. Then the group D_{2n} has n proper subgroups $H_i = \left\{1, ab^i\right\}, i = 1, 2, ..., n$ of order 2, and so we can defined a **g.g.f** map $f : D_{2n} \longrightarrow D_{2n}$, such that $f(D_{2n}) = H_i$.

Also since the dihedral group has proper cyclic subgroup $H = \{b|b^n = 1\}$. Then we can defined a **g.g.f** map $f : D_{2n} \longrightarrow D_{2n}$, such that $f(D_{2n}) = \{b|b^n = 1\}$, and since the subgroup $H = \{b|b^n = 1\}$ be cyclic group of order $n = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_m^{\alpha_m}$. Thus by applying the Theorem 2.2 we get that $\lim\limits_{i \longrightarrow \infty} f_i(D_{2n}) = H = \left\{b^{q_i} | (b^{q_i})^{\frac{n}{q_i}} = 1\right\}, i = 1, 2, \cdots, m$. □

§3. Group Folding of the Direct Product of Groups

In this section we discuss the group folding of direct product of groups. Let $f_1 : (G_1, *) \to (G_1, *)$, $f_2 : (G_2, *) \to (G_2, *)$ are two **g.g.f** maps. Then we define the direct product of the f_1, f_2 as the following:

$$f_1 \times f_2 : (G_1 \times G_2, *) \to (G_1 \times G_2, *)$$
$$\forall a \in G_1, b \in G_2, \quad (f_1 \times f_2)(a, b) = (f_1(a), f_2(b))$$

Theorem 3.1 *The direct product of two **g.g.f** maps are **g.g.f** map also, but the converse is not always true.*

Proof Since $f_1 : (G_1, *) \to (G_1, *)$, $f_2 : (G_2, *) \to (G_2, *)$ are two **g.g.f** maps then there exist two proper subgroups H_1, H_2 of G_1, G_2 respectively, such that $f_1(G_1) = H_1$, $f_2(G_2) = H_2$. As the direct product of the any two proper subgroups are proper subgroups. Hence $H_1 \times H_2 < G_1 \times G_2$. Now we will proof that the map, $f^* = f_1 \times f_2 : G_1 \times G_2 \to G_1 \times G_2$ is **g.g.f** map. Let $a, b \in G_1, G_2$ and $c, d \in H_1, H_2$ respectively, then we have:

$$f^*(a, b) = (f_1 \times f_2)(a, b) = (f_1(a), f_2(b)) = (c, d) \in H_1 \times H_2,$$
$$f^*\left(a^{-1}, b^{-1}\right) = (f_1 \times f_2)\left(a^{-1}, b^{-1}\right) = \left(c^{-1}, d^{-1}\right) \in H_1 \times H_2.$$

Hence the map f^* is **g.g.f** map. To proof the converse is not true, let $G_1 = \left\{g_1 \mid g_1^2 = 1\right\}$,$G_2 = \left\{g_2 \mid g_2^3 = 1\right\}$ be cyclic groups and since the order of them are prime then from Theorem 2.1, we can not able to define a **g.g.f** map of them. But the direct product of G_1, G_2 is

$$G_1 \times G_2 = \left\{(1, 1), (1, g_2), (1, g_2^2), (g_1, 1), (g_1, g_2), (g_1, g_2^2)\right\}.$$

and since the $|(G_1 \times G_2)| = 6$. Hence $G_1 \times G_2$ has a proper subgroup $H = \{(1, 1), (1, g_2)\}$ and so we can define the **g.g.f** map $f^* : G_1 \times G_2 \to G_1 \times G_2$ as the following:

$$f^*(1, 1) = (1, 1),$$
$$f^*(1, g_2) = (1, g_2), \quad f^*(g_1, 1) = (1, 1),$$
$$f^*(g_1, g_2) = (1, g_2),$$
$$f^*(g_1, g_2^2) = (1, g_2).$$

This completes the proof. □

§4. Folding of Semigroups

In this section we will be discuss the folding of semigroups into itself. Let $(G, *)$ be a commutative semigroup with identity 1, i.e. a monoid.

Definition 4.1([4]) *A nonzero element a of a semigroup G is a left zero divisor if there exists a nonzero b such that $a * b = 0$. Similarly, a is a right zero divisor if there exists a nonzero element $c \in G$ such that $c * a = 0$.*

The element a is said to be a *zero divisor* if it is both a left and right zero divisor. we will denote to the set of all zero-divisors by $Z(G)$, and the set of all elements which have the inverse by $I(G)$.

Definition 4.2 *The zero divisor folding of the semigroup G, **z.d.f**, is the map $f_z : (G, *) \to (G, *)$, st.*

$$f_z(x) = \begin{cases} 0 & \text{if } x = 0 \\ a & \text{if } x * a = 0,\ x, a \neq 0 \\ x & \text{if } x * a \neq 0,\ x, a \neq 0 \end{cases}$$

where $a \in Z(G)$.

Note that $f_z(G)$ may be semigroup or not. We will investigate the zero divisor folding for $\mathbb{Z}_p$ semigroups.

Definition 4.3 *Let $\mathbb{Z}_p$ be a semigroup under multiplication modulo p. The **z.d.f** map of $\mathbb{Z}_p$ is the map $f_z : (G, .) \to (G, .)$, st.*

$$f_z(x) = \begin{cases} 0 & \text{if } x = 0 \\ q & \text{if } xq = 0,\ x, q \neq 0 \\ x & \text{if } xq \neq 0,\ x, q \neq 0 \end{cases}$$

where $q \in Z(\mathbb{Z}_p)$, is the greatest divisor of p.

Proposition 4.1 *If the order of $\mathbb{Z}_p$ is prime. Then $f_z(G) = G$, i.e. f_z is identity map.*

Proof Since the order of semigroup $\mathbb{Z}_p$ is prime. Then the semigroup $\mathbb{Z}_p$ has not got any zero divisor, and so the **z.d.f** which can defined on $\mathbb{Z}_p$ the identity map $f_z(x) = x$, for all $x \in \mathbb{Z}_p$. □

Theorem 4.1 *Let $\mathbb{Z}_p$be semigroup of order p, then **z.d.f** of $\mathbb{Z}_p$ into itself is a subsemigroup under multiplication modulo p. Has one zero divisor if $4 \mid p$ or has not any zero divisor if $4 \nmid p$.*

Proof Let $\mathbb{Z}_p$ be semigroup under multiplication modulo p. Then $\mathbb{Z}_p$ consists of two subsets $Z(\mathbb{Z}_p),\ I(\mathbb{Z}_p)$.

Case 1. If p is even, then the **z.d.f** map defined as follows:

$$f_z(x) = \begin{cases} 0 & \text{if} x = 0; \\ \frac{p}{2} & \text{if } x \in Z(\mathbb{Z}_p),\ x \text{ is even;} \\ x & \text{if } x \in Z(\mathbb{Z}_p),\ x \text{ is odd or } x \in I(\mathbb{Z}_p), \end{cases}$$

where $\frac{p}{2}$ is the greatest divisor of p. Hence $f_z(\mathbb{Z}_p) = I(\mathbb{Z}_p) \cup \left\{0, \frac{p}{2}, x_1, \cdots, x_n\right\}$, where $x_1, \cdots, x_n$ are odd zero divisors and $x_i.x_j \neq 0$, for all $i, j = 1, \cdots, n$. Notice that $f_z(\mathbb{Z}_p) = I(\mathbb{Z}_p) \cup \left\{0, \frac{p}{2}, x_1, \cdots, x_n\right\}$ is subsemigroup under multiplication modulo p. If $4 \mid p$ then $\frac{p}{2}.\frac{p}{2} = 0$, hence the subsemigroup $f_z(\mathbb{Z}_p)$ has one zero divisor $\frac{p}{2}$. Otherwise, if $4 \nmid p$ then $\frac{p}{2}, \frac{p}{2} \neq 0$. And so $H = I(\mathbb{Z}_p) \cup \left\{0, \frac{p}{2}, x_1, \cdots, x_n\right\}$ is subsemigroup under multiplication modulo p without any zero divisor.

Case 2. If p is odd, then the **z.d.f** map defined as follows:

$$f_z(x) = \begin{cases} 0 & \text{if } x = 0; \\ q & \text{if } x \in Z(\mathbb{Z}_p),\ x \text{is odd;} \\ x & \text{if } x \in Z(\mathbb{Z}_p),\ x \text{ is even or } x \in I(\mathbb{Z}_p), \end{cases}$$

where q is the greatest divisor of p. Hence $f_z(\mathbb{Z}_p) = I(\mathbb{Z}_p) \cup \{0, q, x_1, \cdots, x_n\}$, where $x_1, \cdots, x_n$ are even zero divisors and $x_i.x_j \neq 0$ for all $i, j = 1, \cdots, n$. Notice that $f_z(\mathbb{Z}_p) = I(\mathbb{Z}_p) \cup \{0, q, x_1, \cdots, x_n\}$ is subsemigroup under multiplication modulo p, and since p is odd. Then $f_z(\mathbb{Z} - p)$ has not any zero divisor. □

Corollary 4.1 *If $\mathbb{Z}_p$ be semigroup under multiplication modulo p and $p = (q)^m, m \in \mathbb{N}$. Then $f_z(\mathbb{Z}_p) = I(\mathbb{Z}_p) \cup \left\{0, q^{m-1}\right\}$ is subsemigroup under multiplication modulo p.*

Example 3. (1) Let $\mathbb{Z}_{10} = \{0, 1, 2, 3, \cdots, 9\}$ be semigroup under multiplication modulo 10. Since $p = 10$ is even then the **z.d.f** map defined as follows:

$$f_z(x) = \begin{cases} 0 & \text{if } x = 0; \\ 5 & \text{if } x \in \{2, 4, 8, 6\}; \\ x & \text{if } x \in \{1, 3, 7, 9, 5\}. \end{cases}$$

Thus $f_z(\mathbb{Z}_{10}) = \{0, 1, 3, 7, 9, 5\}$. Obviously, $f_z(\mathbb{Z}_{10})$ is a subsemigroup under multiplication modulo 10. Since $4 \nmid 10$, then $f_z(\mathbb{Z}_{10})$ has not any zero divisor.

(2) Let $\mathbb{Z}_{12} = \{0, 1, 2, 3, cdots, 11\}$ be a semigroup under multiplication modulo 12. Since $p = 12$ is even then the **z.d.f** map defined as follows:

$$f_z(x) = \begin{cases} 0 & \text{if } x = 0; \\ 6 & \text{if } x \in \{2, 4, 6, 8, 10\}; \\ x & \text{if} x \in \{1, 3, 5, 7, 9\}. \end{cases}$$

Hence $f_z(\mathbb{Z}_{12}) = \{0, 1, 3, 7, 9, 5, 6\}$. Obviously, $f_z(\mathbb{Z}_{12})$ is a subsemigroup under multiplication modulo 12 and 6 is a zero divisor.

References

[1] E.EL-Kholy and M. EL-Ghoul, Simplicial foldings, *Journal of the Faculty of Education*, 18(1993), 443-455.

[2] H.R.Farran, E.EL-Kholy and S.A.Robertson, Folding a surface to a polygon, *Geometriae Dedicata*, 63(1996), 255-266.

[3] John A.Beachy, William D.Blair, *Abstract Algebra* (3th edition), Waveland Pr. Inc, 2006.

[4] J.M.Howie, *An Introduction to Semigroup Theory*, Academic Press, 1976.

[5] M.Basher, On the Folding of Finite Topological Space, *International Mathematical Forum*, Vol. 7, No.15(2012), 745 - 752.

[6] M.El-Ghoul, M.Basher, The invariant of immersions under isotwist folding, *KYUNGPOOK Math. J.*, 46(2006), 139-144.

[7] M.El-Ghoul, S.I.Nada and R.M.Abo Elanin, On the folding of rings, *International Journal of Algebra*, 3(2009), 475-482.

[8] M.El-Ghoul , *Folding of Manifolds*, Ph.D. Thesis, Tanta Univ., Egypt, 1985.

[9] P.DL-Francessco, Folding and coloring problem in mathematics and physics, *Bulletin of the American Mathematics Socitey*, 37(2000), 251-307.

[10] S.A.Robertson, Isometric folding of Riemannian manifolds, *Proceedings of the Royal Society of Edinburgh*, 79:3-4(1977), 275-284.