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On (k, d)-Maximum Indexable Graphs and (k, d)-Maximum Arithmetic Graphs

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Abstract: A (n, m) graph G is said to be (k, d) maximum indexable graph, if its vertices can be assigned distinct integers $0, 1, 2, \cdots, n-1$, so that the values of the edges, obtained as the sum of the numbers assigned to their end vertices and maximum of them can be arranged in the arithmetic progression $k, k+1, k+2d, \cdots, k+(m-1)d$ and also a (n, m) graph G is said to be (k, d) maximum arithmetic graph, if its vertices can be assigned distinct non negative integers so that the values of the edges, obtained as the sum of the numbers assigned to their end vertices and maximum of them can be arranged in the arithmetic progression $k, k+1, k+2d, \cdots, k+(m-1)d$. The energy $E(G)$ of a graph G is equal to the sum of the absolute values of the eigenvalues of G. In this paper we introduce some families of graphs which are (k, d)- maximum indexable and (k, d)-maximum arithmetic and also compute energies of some of them.

Key Words: Graph labeling, indexable graphs, (k, d)-Maximum indexable graphs, (k, d)-maximum arithmetic graphs.

AMS(2010): 05C78, 05C50, 58C40

§1. Introduction

Let $G = (V, E)$ be a (n, m) graph and let its vertex set be $V(G) = \{v_1, v_2, \cdots, v_n\}$. We assume that G is a finite, undirected, connected graph without loops or multiple edges. Graph labelings where the vertices are assigned values subject to certain conditions are interesting problems and have been motivated by practical problems. Applications of graph labeling have been found in $X-$ ray, crystallography, Coding theory, Radar, Circuit design, Astronomy and communication design.

Given a graph $G = (V, E)$, the set N of non-negative integers, a subset A of N of non-negative integers, a set A of N and a Commutative binary operation $* : N \times N \longrightarrow N$ every vertex function $f : V(G) \longrightarrow A$ induces an edge function $f^* : E(G) \longrightarrow N$ such that $f^*(uv) = *(f(u), f(v)) = f(u) * f(v)$, $\forall uv \in E(G)$. We denote such induced map f^* of f by f^{max}.

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Acharya and Hedge [1,2] have introduced the concept of indexable and arithmetic graph labelings. Recently present author [7] has introduced the concept of maximum indexable graphs. The adjacency matrix $A(G)$ of the graph G is a square matrix of order n whose (i,j)-entry is equal to unity if the vertices v_i and v_j are adjacent, and is equal to zero otherwise. The eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of $A(G)$ are said to be the eigenvalues of the graph G, and are studied within the Spectral Graph Theory [3]. The energy of the graph G is defined as $E = E(G) = \sum_{i=1}^{n} |\lambda_i|$. The graph energy is an invariant much such studied in mathematical and mathema-tico-chemical literature; for details see [4,5,6], In this paper we introduce some families of graphs which are (k,d)- maximum indexable and (k,d)-maximum arithmetic and also compute energies of some of them.

Definition 1.1 *A graph $G = (V,E)$ is said to be $(k,d)-$ maximum indexable graph if it admits a $(k,d)-$ indexer, namely an injective function $f : V(G) \longrightarrow \{0,1,\cdots,n-1\}$ such that $f(u) + f(v) + \max\{f(u), f(v)\} = f^{max}(uv) \in f^{max}(G) = \{f^{max}(uv) : \forall uv \in E(G)\} = \{k, k+d, k+2d, \cdots, k+(m-1)d\}$, for every $uv \in E$.*

Example 1.2 The graph $K_{2,3}$ is a $(4,1)-$maximum indexable graph.

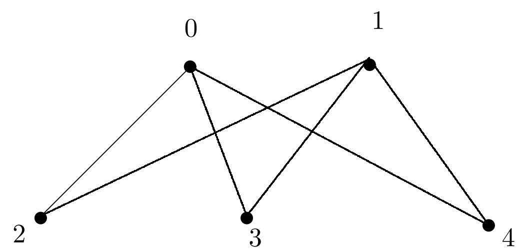

Figure 1

In this example,

$$f^{\max}(G) = \{4,5,6,7,8,9\}.$$

Lemma 1.3 *Let f be any $(k,d)-$maximum indexable labeling of G. Then*

$$2 \le k \le 3n - md + (d-4).$$

Proof Since $f^{\max}(G) \subseteq \{2,3,\cdots,3n-4\}$, the largest edge label is at most $3n-4$. Hence k must be less than or equal to $3n-4-(m-1)d$. Since the edge values are in the set $\{k, k+d, k+2d, \cdots, k+(m-1)d\}$, we have

$$2 \le k \le 3n - md + (d-4). \qquad \square$$

Theorem 1.4 *The star $K_{1,n}$ is $(k,d)-$maximum indexable if and only if $(k,d)=(2,2)$ or $(2n,1)$. Furthermore, there are exactly $n+1$ maximum indexable labelings of $K_{1,n}$ of which only two are $(k,d)-$maximum indexable up to isomorphism .*

Proof By assigning the value 0 to the central vertex and $1,2,3,\cdots,n$ to the pendent vertices we get $(2,2)-$maximum indexable graph, since $f^{\max}(G)=\{2,4,6,\cdots,2n\}$. By assigning the value n to the root vertex and $0,1,2,\cdots,n-1$ to the pendent vertices, one can see that $f^{\max}(G)=\{2n,2n+1,2n+3,\cdots,3n-1\}$, which shows that G is $(2n,1)$-maximum indexable graph.

Conversely, if we assign the value c $(0<c<n)$ to the central vertex and $0,1,2,\cdots,c-1,c+1,\cdots,n$ to the pendent vertices, we obtain $f^{\max}(K_{1,n})=\{2c,2c+1,\cdots,3c-1,3c+2,\cdots,2n+c\}$ which is not an arithmetic progressive, i.e., $K_{1,n}$ is not a $(k,d)-$maximum indexable.

For the second part, note that $K_{1,n}$ is of order $n+1$ and size n. Since there are $n+1$ vertices and $n+1$ numbers (from 0 to n), each vertex of $K_{1,n}$ can be labeled in $n+1$ different ways. Observe that the root vertex is adjacent with all the pendent vertices. So if the root vertex is labeled using $n+1$ numbers and the pendent vertices by the remaining numbers, definitely the sum of the labels of each pendent vertex with the label of root vertex, will be distinct in all $n+1$ Maximum Indexable labelings of $K_{1,n}$. It follows from first part that, out of $n+1$ maximum indexable labelings of $K_{1,n}$, only two are $(k,d)-$maximum indexable. This completes the proof. □

Corollary 1.5 *The graph $G=K_{1,n}\cup K_{1,n}$, $n\geq 1$ is $(k,d)-$maximum indexable graph and its energy is equal to $4\sqrt{n-1}$.*

Proof Let the graph G be $K_{1,n}\cup K_{1,n}$, $n\geq 1$. Denote

$$V(G)=\{u_1,v_{1j}:1\leq j\leq n\}\cup\{u_2,v_{2j}:1\leq j\leq n\},$$

$$E(G)=\{u_1v_{1j}:1\leq j\leq n\}\cup\{u_2v_{2j}:1\leq j\leq n\}.$$

Define a function $f:V(G)\longrightarrow\{0,1,\cdots,2n+1\}$ by

$$f(u_1)=2n,\qquad f(u_2)=2n-1$$

$$f(v_{1j})=(2j-1)\quad\text{and}\quad f(v_{2j})=2(j-1),$$

for $j=1,2,\cdots,n$. Thus

$$f^{\max}(K_{1,n}\cup K_{1,n})=\{4n+1,4n+2,4n+3,\cdots,6n-2,6n-1,6n\}.$$

Thus $K_{1,n}\cup K_{1,n}$ is $(4n+1,1)$-maximum indexable graph. The eigenvalues of $K_{1,n}\cup K_{1,n}$ are $\pm\sqrt{n-1},\pm\sqrt{n-1},\underbrace{0,0,\cdots,0}_{2n-4\ times}$. Hence $E(K_{1,n}\cup K_{1,n})=4\sqrt{n-1}$. □

Theorem 1.6 *For any integer $m\geq 2$, the linear forest $F=nP_3\cup mP_2$ is a $(2m+2n,3)$ maximum indexable graph and its energy is $2(n\sqrt{2}+m)$.*

Proof Let $F = nP_3 \cup mP_2$ be a linear forest and $V(F) = \{u_i, v_i, w_i : 1 \leq i \leq n\} \cup \{x_j, y_j : 1 \leq j \leq m\}$. Then $|V(F)| = 3n + 2m$ and $|E(F)| = 2n + m$. Define $f : V(F) \longrightarrow \{0, 1, \cdots, 2m + 3n - 1\}$ by

$$f(y) = \begin{cases} i-1, & y = u_i, \quad 1 \leq i \leq n, \\ m+n+(i-1), & y = v_i, \quad 1 \leq i \leq n, \\ j+n-1, & y = x_j, \quad 1 \leq j \leq m, \\ 2n+m+(j-1), & y = y_j, \quad 1 \leq j \leq m, \\ 2(n+m)+(i-1), & y = w_i, \quad 1 \leq i \leq n. \end{cases}$$

Then

$$\begin{aligned} f^{\max}(F) &= \{2(n+m), 2(n+m)+3, 2(m+n)+6 \cdots, 2(m+n)+ \\ &\quad +3(n-1), 5n+2m, \cdots, 5n+5m-3, 5n+5m, 8n+5m-3\}. \end{aligned}$$

Therefore $nP_3 \cup mP_2$ is a $(2(n+m), 3)-$maximum indexable graph.

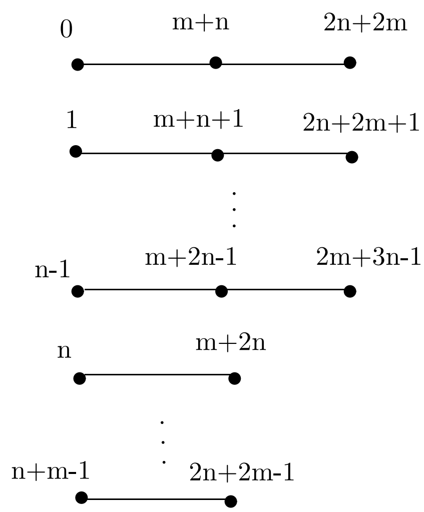

Figure 2

The eigenvalues of $nP_3 \cup mP_2$ are $\underbrace{1, -1}_{m\ times}, \underbrace{0, \sqrt{2}, -\sqrt{2}}_{n\ times}$. Hence its energy is equal to $E(nP_3 \cup mP_2)) = 2(n\sqrt{2} + m)$. □

Corollary 1.7 *For any integer* $m \geq 2$*, the linear forest* $F = P_3 \cup mP_2$ *is a* $(2m+2, 3)-$*maximum indexable graph.*

Proof Putting $n = 1$ in the above theorem we get the result. □

§2. $(k,d)-$ Maximum Arithmetic Graphs

We begin with the definition of $(k,d)-$maximum arithmetic graphs.

Definition 2.1 *Let N be the set of all non negative integers. For a non negative integer k and positive integer d, a (n,m) graph $G=(V,E)$, a $(k,d)-$maximum arithmetic labeling is an injective mapping $f: V(G)\longrightarrow N$, where the induced edge function $f^{max}: E(G)\longrightarrow \{k,k+d,k+2d,\cdots,k+(m-1)d\}$ is also injective. If a graph G admits such a labeling then the graph G is called $(k,d)-$maximum arithmetic.*

Example 2.2 Let S_n^3 be the graph obtained from the star graph with n vertices by adding an edge. S_6^3 is an example of $(4,2)-$maximum arithmetic graph.

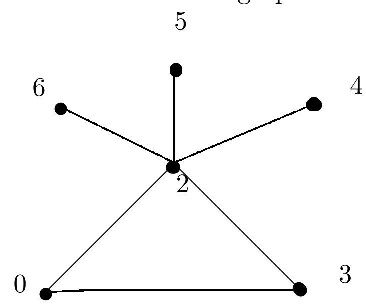

Figure 3 S_6^3

Here

$$f^{\max}(S_6^3)=\{4,6,8,10,12,14\}.$$

Definition 2.3 *Let P_n be the path on n vertices and its vertices be ordered successively as $x_1,x_2,\cdots,x_n$. P_n^l is the graph obtained from P_n by attaching exactly one pendent edge to each of the vertices $x_1,x_2,\cdots,x_l$.*

Theorem 2.4 *P_n^l is $(k,d)-$maximum indexable graph.*

Proof We consider two cases.

Case 1 If n is odd. In this case we label the vertices of P_n^l as shown in the Figure . The value of edges can be written as arithmetic sequence $\{5,8,\cdots,(3n-1),(3n+2),(3n+5),\cdots,(3n+3l-1)\}$. It is clear that P_n^l is $(5,3)-$maximum arithmetic for any odd n.

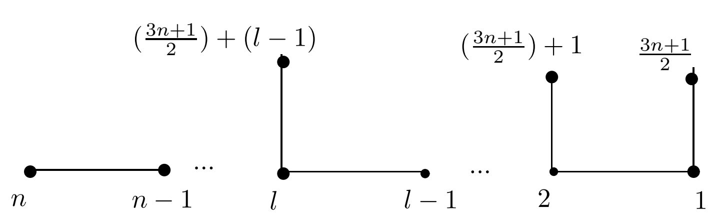

Figure 4

Case 2 If n is even. In this case we label the vertices of P_n^l as shown in the Figure 5. The value of edges can be written as arithmetic sequence $\{7, 13, \cdots, (6n-5), (6n+1), (6n+7), (6n+13), \cdots, (6n+6l-5)\}$. It is clear that P_n^l is $(7,6)-$ maximum arithmetic for any even n. □

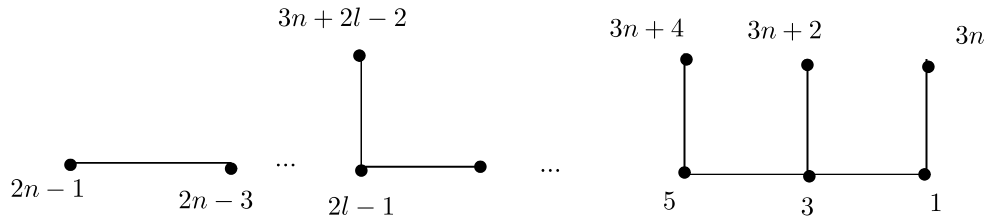

Figure 5

Theorem 2.5 *The graph $G = K_2\nabla\overline{K_{n-2}}$ for $n \geq 3$ is a $(10,2)-$ maximum arithmetic graph.*

Proof Let the vertices of K_2 be v_1 and v_2 and those of $\overline{K_{n-2}}$ be $v_3, v_4, \cdots, v_n$. By letting $f(v_i) = 2i$ for $i = 1,2$ and $f(v_i) = 2i - 1$, $3 \leq i \leq n$, we can easily arrange the values of edges of $G = K_2\nabla\overline{K_{n-2}}$ in an increasing sequence $\{10, 12, 14, \cdots, 4n, 4n+2\}$. □

Lemma 2.6 *The complete tripartite graph $G = K_{1,2,n}$ is a $(7,1)-$maximum arithmetic graph.*

Proof Let the tripartite A, B, C be $A = \{w\}$, $B = \{v_1, v_2\}$ and $C = \{v_3, v_4, \cdots, v_n\}$. Define the map $f : A \cup B \cup C \longrightarrow N$ by

$$f(v_i) = i, \quad 1 \leq i \leq n+2$$
$$f(w) = n+3.$$

Then one can see that $G = K_{1,2,n}$ is $(7,1)-$maximum arithmetic. In fact $f^{\max}(K_{1,2,n}) = \{7, 8, 9, 10, \cdots, 2n+5, 2n+6, 2n+7, \cdots, 3n+8\}$. □

Theorem 2.7 *For positive integers m and n, the graph $G = mK_{1,n}$ is a $(2mn + m + 2, 1)-$maximum arithmetic graph and its energy is $2m\sqrt{n-1}$.*

Proof Let the graph G be the disjoint union of m stars. Denote $V(G) = \{v_i : 1 \leq i \leq m\} \cup \{u_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(G) = \{v_i u_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$. Define $f : V(G) \longrightarrow N$ by

$$f(x) = \begin{cases} m(n+1) - (i-1) & \text{if } x = v_i \quad 1 \leq i \leq m \\ (j-1)m + i & \text{if } x = u_{i,j} \quad 1 \leq i \leq m,\ 1 \leq j \leq n. \end{cases}$$

Then

$$f^{\max}(G) = \{2mn + m + 2, 2mn + m + 3, \cdots, 3mn + m + 1\}.$$

Therefore $mK_{1,n}$ is a $(2mn + m + 2, 1)-$maximum arithmetic graph.

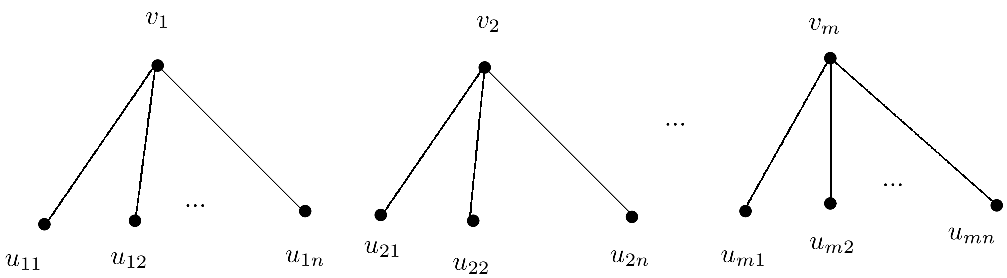

Figure 6

Also its eigenvalues are $\pm\sqrt{n-1}$ (m *times*). Hence the energy of $mK_{1,n}$ is

$$E(mK_{1,n}) = 2m\sqrt{n-1}. \qquad \square$$

§3. Algorithm

In this section we present an algorithm which gives a method to constructs a maximum (k,d)-maximum indexable graphs.

```
MATRIX-LABELING (Vlist,Vsize)
      //Vlist is list of the vertices, Vsize is number of vertices
   Elist= Empty();
      for j <-0 to (Vsize -1)
       do for i <-0 to (Vlist - 1) && i! = j
        X=VList[i] + Vlist[j] + Max {Vlist[i], Vlist[j]};
         if (Search (Elist, X)=False)      //X does not exist in Elist
          then ADD (Elist, X);
         end if
       end for
      end for
     SORT-INCREASINGLY (Elist);
      Flag= 0;
        For j <-0 to (Esize - 1)
          do if (Elist[j + 1]! = Elist[j] + d)
            then Flag= 1;
          end if
        end for
     if (Flag== 0)
```

```
    then PRINT ("This graph is a (k,d)−Max indexable graph");
     else if (Flag= 1)
      then PRINT ("This graph is not (k,d)−Max indexable graph");
     end if
end MATRIX-LABELING(Vlist, Vsize)
```

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